Optimal Investment Strategy for Defined Contribution Pension Scheme under the Heston Volatility Model

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Abstract

In this paper, the optimal investment strategy for a defined contribution (DC) pension scheme was modeled with the assumption that the fund is invested partly in riskless assets and partly in risky assets. The market has a constant interest rate, a stochastic volatility that follows the Heston model, the salary is assumed constant over the entire career of the Pension Plan Participant (PPP) and the contribution is a constant proportion of the salary. The CRRA utility function was utilized to obtain a Hamilton-Jacobi-Bellman (HJB) equation. The resulting HJB equation was solved using the Prandtl Asymptotic Matching Method following the works in the literature.

Keywords

Defined Contributory Pension Scheme, Stochastic Volatility, CRRA, Prandtl Asymptotic Matching, Optimal Investment Strategy, HJB

1. Introduction

The defined contribution (DC) model is a pension model that has predetermined contribution from the Pension Plan Participant (PPP) and the benefit to the PPP depends on the return on investment of the pension assets. The DC plan is fully funded, privately managed, and there is a third party management of the funds and assets. In this paper, the optimal investment strategies derived with Constant Relative Risk Aversion (CRRA) utility function was considered, volatility was assumed to follow the Heston model with constant interest rate, while
salary is stochastic. The PPP in the DC pension plan seeks to maximize certain utility function based on his attitude to risk. The most commonly used utility functions are constant relative risk aversion (CRRA), that is, the power or logarithmic utility function, constant absolute risk aversion (CARA), that is, the exponential utility function and hyperbolic absolute risk aversion (HARA) which is a combo utility function since under different assumptions it can transform to the other types. In this work, the CRRA utility function was used.

2. Theoretical Background

It is a well known fact that the conditional variance of asset returns, especially stock market returns, is not constant over time [1] [2]. Stock return volatility is serially correlated, and shocks to volatility are negatively correlated with unexpected stock returns. Changes in volatility are persistent [3] [4]. Large negative stock returns tend to be associated with increases in volatility that persist over long periods of time. Stock return volatility appears to be correlated across markets over the world [5].

The groundbreaking papers by Merton [6] opened the floodgates of research on continuous-time portfolio optimization with a constant investment scenario, others have looked at different stochastic scenarios such as stochastic interest rates or stochastic volatility. The investment-consumption problems with the Heston model was considered by [7].

In [8] and [9] optimal control of a defined contribution pension plan where the guarantee depends on the level of interest rates at the fixed retirement date were considered. [10] measured the effect of a minimum interest rate guarantee constraint through the wealth equivalent in case of no constraints and show numerically the guarantees may induce a significant utility loss for relative risk tolerant investors. [11] the most widely used utility function exhibits constant relative risk aversion (CRRA), that is, the power or logarithmic utility function [12] [13] [14] and [15].

3. The Model

It is assumed that a PPP who is seeking to maximize the expected utility of his terminal wealth invests in a market made up of risk-free assets and a risky assets. The risky asset is volatile with which can indeed represent the index of the stock market. The dynamics of the underlying assets is given by

$$\frac{dB(t)}{B(t)} = rd\tau$$

(1)

for the riskless asset and

$$\frac{dS(t)}{S(t)} = \mu dt + \sqrt{\nu} dW(t)$$

(2)

for the risky asset Where $r$ is a constant risk free interest rate, $\mu(t)$ is the expected growth rate of the risky asset, $\sqrt{\nu}$ is the volatility of stock with respect
The salary process \( L(t) \) is assumed to be constant over the entire working career of the PPP. Let \( c(t) \) be the contribution rate of the PPP at time \( t \), then the contribution process \( \psi \) satisfy
\[
\psi(t) = c(t)L(t)dt
\]
(3)

Let \( X(t,V) \) be the wealth process of a PPP at time \( t \). We assume that the fund administrator invests the contribution in \( \eta \) riskless assets and \( \pi \) risky assets, then we have the dynamics of the wealth process as
\[
dX(t,V) = \left( \frac{\eta dB(t)}{B(t)} + \frac{\pi dS(t)}{S(t)} + \psi(t) \right) dt
\]
(4)

Substituting (1), (2), and (3) in (4) we have
\[
dX = (X - \pi)r dt + \pi \left( \mu dt + \sqrt{V} dW(t) \right) + c(t)L(t)dt
\]
(5)

Simplifying, we have
\[
dX = (cL(t) + \pi(\mu - r) + XV)dt + \pi\sqrt{V} dW(t)
\]

Dividing through by \( X \) we have
\[
\frac{dX}{X} = (c\theta + \Pi(-r + \mu) + rv)dt + \Pi\sqrt{V} dW(t)
\]
(6)

Where \( (L(t), V)X = \theta \) and \( \Pi X = \Pi \). In addition \( V(t) \) satisfies the Heston model
\[
dV_i = k(\theta - V)dt + \xi\sqrt{V} dW^{i*}(t)
\]
(7)

Where:
- \( \theta \) is the long variance at \( t \to \infty \),
- \( k \) is the rate at which \( V \) reverts to \( \theta \), \( \xi \) is the volatility of volatility and determines the variance of \( V \).
- If \( 2k\theta > \xi^2 \) then, \( V \) is strictly positive and is said to satisfy the Feller condition.
- \( W \) and \( W^{i*} \) are correlated with correlation coefficient \( \rho \).

4. The Optimal Control and Value Function

4.1. The Admissible Portfolio Strategy

The investor will choose the optimal investment and optimal contribution rate that will maximize the expected utility of terminal wealth of the operation. For an arbitrary admissible strategy \( f(.) = \{\Pi(t,v), c(t,v): t = 0\} \). The objective function \( G(.) \) follows:
\[
G(x,v,u) = E\left[ F(T,X,v) | X(t) = x, V(t) = v \right]
\]

For admissible portfolio strategy in stocks \( \Pi(t,V), t \in [0,T] \): we have
\[
E\int_0^T \Pi(t,V)\Pi(t,V) dt < \infty
\]

For admissible contribution rate strategy \( c(t,V), t \in [0,T] \), we have
Definition. A strategy \( f(t) = \{\Pi(t,v), c(t,v) : t = 0\} \) which is progressively measurable with respect to \( \{W(t): 0 \leq s < t\} \) is referred to as an admissible strategy.

Let the collection of all admissible strategies be denoted by \( A \). It then follows that the set \( A \) can be defined as follows:

\[
A = f(t,V) = \{\Pi(t,V), C(t,V)\} \in R^d \times R^d
\]

\[
E \int_0^T \Pi(t,V) \Pi(t,V) dt < \infty; E \int_0^T c(t,V)^2 dt < \infty
\]

4.2. The Value Function

Let the value function be defined as

\[
\max \prod_{t=0}^{T} \left[ \alpha e^{-\beta V_{t-1}} + \frac{1 - \alpha}{1 - \gamma} \right]
\]

\[
X_t \quad \text{is the value at time } T \text{ of a trading strategy that finances } \{C_t\}_{t=0}^T.
\]

\[ \gamma \] is the risk aversion coefficients.

\[ \beta \] is the subjective discount rate.

\[ \alpha \] determines the relative importance of the intermediate consumption when \( \alpha = 0 \), expected utility only depends on the terminal wealth.

The problem confronting the PPP at time \( t \) is to select the portfolio weights and contribution rate processes \( \{\Pi(t,v), c(t,v)\}_{t=0}^T \) that maximize the expected utility of terminal wealth of the investor subject to:

\[
\begin{aligned}
\frac{dV_t}{V_t} &= (c\theta + \Pi(-r + \mu + r))dt + \Pi\sqrt{V_t}dW_t(t) \\
\frac{dV_t}{V_t} &= k(\theta - V_t)dt + \xi\sqrt{V_t}dW_t\nu
\end{aligned}
\]

We now consider an investor that chooses power utility function. By applying stochastic dynamic programming approach and Ito Lemma our Hamilton-Jacobi-Bellman equation characterized by the optimal solutions to the problem of the investor becomes [5]

\[
\frac{\partial}{\partial t} H(t,X,V) + \frac{1}{2} X^2 \Pi^2 \sqrt{V} \frac{\partial^2}{\partial X^2} H(t,X,V) + X\left(\Pi(-r + \mu + r) \frac{\partial}{\partial X} H(t,X,V) + C \frac{\partial}{\partial X} H(t,X,V) \right)
\]

\[
+ X \Pi \sqrt{V} \rho \dot{\xi} \sqrt{V} \frac{\partial^2}{\partial X \partial V} H(t,X,V) + \frac{1}{2} \dot{\xi}^2 \sqrt{V} \frac{\partial^2}{\partial V^2} H(t,X,V)
\]

\[
+ k(\theta - V) \frac{\partial}{\partial V} H(t,X,V) + \frac{\alpha e^{-\beta V_{t-1}}}{1 - \gamma} = 0
\]

With boundary condition

\[
H(t,X,V) = (1 - \alpha) \frac{X^{1-\gamma}}{1 - \gamma}
\]

\( H \) is conjectured to have a solution of the form
\( H(t,X,V) = e^{-B_t} \frac{X^\gamma}{\gamma} F(X,t) \)

where \( \frac{\partial}{\partial t} H(t,X,V), \frac{\partial}{\partial X} H(t,X,V), \frac{\partial^2}{\partial X^2} H(t,X,V) \) and \( \frac{\partial^2}{\partial V^2} H(t,X,V) \) are the partial derivatives with respect to \( t \), \( X \) and \( V \).

With the above assumptions and the fact that contribution cannot be negative, then the optimal values of the portfolio weight and contributions are:

\[
\Pi^* = \frac{V \rho \gamma \frac{\partial^2}{\partial V} F(t,V) - F(t,V)(r-\mu)}{V \gamma F(t,V)}
\]

and

\[
C^* = -\frac{\frac{1}{X \alpha}}{F(t,V)}
\]

Substituting the derivatives as well as the optimal values in (10) results in the HJB Equation (7)

\[
-\frac{\partial}{\partial t} F(t,V) + \frac{\partial}{\partial V} F(t,V) + \left( \frac{\partial}{\partial V} + \frac{\partial}{\partial V} \right) F(t,V) + \left( \frac{\partial}{\partial V} + \frac{\partial}{\partial V} \right) F(t,V) + \frac{1}{\alpha} = 0
\]

with boundary conditions

\[
F(0,V) = (1-\alpha)^{-1}
\]

5. Solving the PDE

The HJB PDE so derived has no closed form solution, this is mainly as a result of the presence the non homogeneous terms \( \frac{1}{\alpha} F_{V} \) and \( \frac{F_2}{F} \) in (13).

A closed-form approximate solution for the PDE was derived by Zhang and Ge [7]. This they did by using the Prandtl’s assymptotic matching method. the trick is, first remove the constant term \( \frac{1}{\alpha} \) solve the resulting PDE and call the solution \( F_1 \). Thus \( F_1 \) satisfy the PDE

\[
-\frac{\partial}{\partial t} F(t,V) + \frac{\partial}{\partial V} F(t,V) + \left( \frac{\partial}{\partial V} + \frac{\partial}{\partial V} \right) F(t,V) + \left( \frac{\partial}{\partial V} + \frac{\partial}{\partial V} \right) F(t,V) + \frac{1}{\alpha} = 0
\]

The second step is to remove the two non linear terms while retaining the non homogeneous term, solve the resulting PDE and call the solution \( F_2 \), thus \( F_2 \) satisfy the PDE

\[
-\frac{\partial}{\partial t} F(t,V) + \left( \frac{\partial}{\partial V} + \frac{\partial}{\partial V} \right) F(t,V) + \left( \frac{\partial}{\partial V} + \frac{\partial}{\partial V} \right) F(t,V) + \alpha^{-1} = 0
\]
The third step is to factor out the common terms in $F_1$ and $F_2$ and call that $F_3$. The solution of the original HJB equation will then be given by

$$F = F_1(t,V) + F_2(t,V) - F_3(t,V)$$ (16)

Following [5] and [16], $F_1$ is

$$F_1(t,V) = (1-\alpha)^{-1} e^{\psi_1(t)V + \psi_2(t)}$$ (17)

where

$$\psi_1 = \begin{cases} (1-\alpha)^{-1} \zeta_1 + \zeta_2 & a_2 \neq 0 \\ (1-\alpha)^{-1} \zeta_3 + \zeta_4 & a_2 = 0 \end{cases}$$

and

$$\psi_2 = \begin{cases} \frac{1}{2} a_1 \ln \left( \left( a_2 + \sqrt{\Phi} \right) e^{-2 \sqrt{\Phi} t} - a_2 + \sqrt{\Phi} \right) \left( m - \frac{1}{2} a_7 \right) + m t & \Phi > 0 \\ a_1 \ln \left( \frac{1}{2} a_2 t - 1 \right) - a_7 t + m t & \Phi = 0 \\ \frac{1}{2} a_1 \ln \left( - \cos \left( \frac{1}{2} \sqrt{-\Phi} t \right) + \frac{\sin \left( \frac{1}{2} \sqrt{-\Phi} t \right) a_2}{\sqrt{-\Phi}} \right) + m t & \Phi < 0 \end{cases}$$

$$= \frac{1}{2} \frac{\zeta_2 a_2}{\gamma} = \frac{\beta}{\gamma} + \frac{(1-\gamma) t}{\gamma}$$

with

$$\zeta_1 = e^{2 \beta \psi_1(t)V e^{\psi_1(t)}}$$,  \quad \zeta_2 = \frac{\alpha^r \left( \Gamma \left( b_1, b_1(V) \right) - \Gamma \left( b_1, b_1 \left( V e^{\psi_1(t)} \right) \right) \right)}{a_1 \left( b_1(V) \right)^{b_1 \left( V e^{\psi_1(t)} \right)}}$$

$$\zeta_3 = e^{2 \beta \psi_2(t) V e^{\psi_2(t)}}$$ and $\zeta_4 = \frac{1}{2} \frac{\alpha^r \sqrt{\pi} \left( \text{erfi} \left( b_2 t + b_2 \left( V e^{\psi_1(t)} \right) \right) - \text{erfi} \left( b_2 \left( V e^{\psi_1(t)} \right) \right) \right)}{b_2 \left( V e^{\psi_1(t)} \right)^{1/2}}$

$F_2$ is given by

$$F_2 = \begin{cases} (1-\alpha)^{-1} \psi_1 + \alpha^{-1} \psi_2 & a_2 \neq 0 \\ (1-\alpha)^{-1} \psi_3 + \alpha^{-1} \psi_4 & a_2 = 0 \end{cases}$$ (18)

where

$$\psi_3 = e^{2 \beta \psi_1(t)V e^{\psi_1(t)}}$$ and $\psi_4 = \frac{1}{2} \frac{\sqrt{\pi} \left( \text{erfi} \left( b_2 t + b_2 \left( V e^{\psi_1(t)} \right) \right) - \text{erfi} \left( b_2 \left( V e^{\psi_1(t)} \right) \right) \right)}{b_2 \left( V e^{\psi_1(t)} \right)^{1/2}}$$
$F_3$ then becomes

$$F_3 = \begin{cases} (1-\alpha)^{-1} a_2 e^{a_{2b_1} e^{a_{2b_2} e^{a_{2b_3} e^{-1}}}} & a_2 \neq 0 \\ (1-\alpha)^{-1} e^{b_2} & a_2 = 0 \end{cases}$$  \hspace{1cm} (19)$$

with

$$b_1 = \frac{a_a - a_a}{a_2} \quad \text{and} \quad b_2 (V) = \frac{a_a - a_a}{a_2}$$

$$b_3 = \frac{\sqrt{2}}{2} \sqrt{a_3} \quad \text{and} \quad b_4 = \frac{\sqrt{2}}{2} \sqrt{a_3} a_3 m = a_a - \frac{1}{2} a_2 a_2$$.

The derivatives are given by

$$F_3 (t, V) = (1-\alpha)^{-1} e^{a_{2b_1} e^{a_{2b_2} e^{a_{2b_3} e^{-1}}}}$$ \hspace{1cm} (20)$$

with

$$\eta_1 = a_2 e^{a_{2b_1} e^{a_{2b_2} e^{a_{2b_3} e^{-1}}}} \quad \eta_2 = \frac{a_a \left( \Gamma (b_1, 1) - \Gamma (b_1, e^{a_{2b_2} e^{a_{2b_3} e^{-1}}}) \right)}{a_2^2 \left( -b_2 (V) \right)} e^{b_2 (V)} \left( \Gamma (b_2, 1) - \Gamma (b_2, e^{b_2 (V)}) \right)$$

$$\eta_3 = e^{b_2} \quad \text{and} \quad \eta_4 = a_2 \left( \Gamma (b_2, 1) - \Gamma (b_2, e^{b_2 (V)}) \right)$$

$$F_2 = \begin{cases} (1-\alpha)^{-1} \eta_1 - \alpha^{-1} \eta_2 & a_2 \neq 0 \\ (1-\alpha)^{-1} \eta_1 + \alpha^{-1} \eta_4 & a_2 = 0 \end{cases}$$  \hspace{1cm} (21)$$

$$F_3 = \begin{cases} (1-\alpha)^{-1} a_2 \left( e(a, t) - 1 \right) e^{a_{2b_1} e^{a_{2b_2} e^{a_{2b_3} e^{-1}}}} a_2 & a_2 \neq 0 \\ (1-\alpha)^{-1} e^{b_2} & a_2 = 0 \end{cases}$$  \hspace{1cm} (22)$$

$$F_V (t, V) = F_3 (t, V) + F_2 (t, V) - F_3 (t, V)$$  \hspace{1cm} (23)$$

$$\Pi^* = \rho F_\gamma F - \frac{r - \mu}{V \gamma}$$  \hspace{1cm} (24)$$

and

$$C^* = \frac{\chi \alpha^2}{F_1 + F_2 - F_3}$$  \hspace{1cm} (25)$$

$$\Pi^* = \rho F_\gamma F + F_2 F_3 - F_3 - \frac{r - \mu}{V \gamma}$$  \hspace{1cm} (26)$$

6. Optimal Investment Policy

**Proposition 1.** The optimal stock investment policy with the portfolio weight $\Pi$ is given as (17), (18) and (19)
Proposition 2. The optimal contribution rate of a PPP strategy is

\[ C' = \frac{X\alpha^T}{F_1 + F_2 - F_3} \]  

with \( F \) and \( F_r \) given by (17), (18), (19), (20), (21), and (22).

From proposition (1) and (2) it follows that for optimality of the stock investment and contribution to exist, the following must hold:

1) \( F_1 + F_2 \neq F_3 \)
2) \( \gamma \neq 0 \). In addition, for \( \Pi' \) to be positive, then, \( \rho, \xi \) and \( F_1 + F_2 - F_3 \) must not be 0, else \( \mu > r \).

The 3D plot for the optimal contribution and optimal stock allocations are given in Figure 1(a) and Figure 1(b) respectively using the following parameters; \( \alpha = 0, \beta = 1, \rho = 0.5, \xi = 0.5, k = 10, r = 0.05, \theta = 0.05 \).

Figure 1. (a) Optimal contribution; (b) Optimal stock allocation.
7. Conclusion

This paper considered the optimal investment problem for a pension plan participants in a defined contribution (DC) plan. It was assumed that the interest rate and salary are constant over time and stock price follows a Heston volatility model. The HJB equation was obtained and solved assuming that the CRRA utility function and following the works of [5] and [16].

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References


