The Average Errors for Linear Combinations of Bernstein Operators on the Wiener Space*

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ABSTRACT

In this paper, we discuss the average errors of function approximation by linear combinations of Bernstein operators. The strongly asymptotic orders for the average errors of the combinations of Bernstein operators sequence are determined on the Wiener space.

Keywords: Linear Combinations; Bernstein Operators; Weighted $L_p$-Norm; Average Error; Wiener Space

1. Introduction

Let $F$ be a real separable Banach space equipped with a probability measure $\mu$ on the Borel sets of $F$. Let $X$ be another normed space such that $F$ is continuously embedded in $X$. By $\|\cdot\|_X$ we denote the norm in $X$. Any $T: F \to X$ such that $f \mapsto \|f - T(f)\|_X$ is a measurable mapping is called an approximation operator.

The $p$-average error of $T$ is defined as

$$e_p(T, F; \|\cdot\|_X, \mu) := \left( \frac{1}{\mu(F)} \int_{F} \|f - T(f)\|_X^p \, \mu(\text{d}f) \right)^{1/p}.$$ 

Let

$$F_0 := \{f \in C[0,1] : f(0) = 0\}.$$ 

For every $f \in F_0$ set

$$\|f\| := \max_{0 \leq t \leq 1} |f(t)|.$$ 

Then $(F_0, \|\cdot\|_C)$ becomes a separable Banach space. Denote by $\mathcal{B}(F_0)$ the Borel class of $(F_0, \|\cdot\|_C)$ and by $\omega_k$ the Wiener measure on $\mathcal{B}(F_0)$ (see [1]). From [1, p. 70] we know

$$\int_{F_0} f(s) f(t) \omega_k(\text{d}f) = \min \{s, t\} = \frac{1}{2} (s + t - |s - t|), \quad \forall s, t \in [0, 1].$$

The Bernstein operator on $C[0,1]$ defined by

$$B_n(f, x) := \sum_{k=0}^{n} \binom{n}{k} p_{k,n}(x),$$

where

$$p_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, k = 0, 1, \ldots, n.$$ 

This operator turned out to be a very interesting operator, easy to deal with and having many applications in approximation theory and practice.

Since Bernstein operators cannot be used in the investigation of higher orders of smoothness, Butzer [2] introduced combinations of Bernstein operators. Ditzian and Totik [3, p. 116] extended this method and defined the combinations as

$$L_{n,m}(f, x) := \sum_{i=0}^{n-m} C_i(n) B_n(f, x),$$

where $n_i$ and $C_i(n)$ satisfy the following conditions:

(a) $n = n_0 < \cdots < n_{m-1} \leq Cn$;

(b) $\sum_{i=0}^{m-1} C_i(n) \leq C$;

(c) $\sum_{i=0}^{m-1} C_i(n) = 1$;

(d) $\sum_{i=0}^{m-1} C_i(n) n_i^\rho = 0, \quad \rho = 1, 2, \ldots, m - 1$.

Throughout this paper, $C$ denotes a positive constant.

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independent of \( n \) and \( x \), which may be a different constant in different cases.

For \( \rho \in L_1[0,1] \), \( \rho \geq 0, \rho \leq p < \infty \),
the weighted \( L_\rho \)-norm of \( f \in C[0,1] \) is defined by
\[
\| f \|_{L_\rho} := \left( \int_0^1 |f(t)|^\rho \cdot \rho(t) \, dt \right)^{\frac{1}{\rho}}.
\]

2. Main Result

Recently G. Q. Xu [4] studied the average errors of Bernstein operators approximation on the Wiener space. Motivated by [4], we considered the average errors of function approximation by linear combinations of Bernstein operators. The strongly asymptotic orders for the average errors of the linear combinations of Bernstein operators sequence are determined on the Wiener space. We obtain:

**Theorem 1.** Let \( 1 \leq p < \infty \), \( L_{m,n}(f,x) \) be given by (2), \( \rho \in L_1[0,1], \rho(x) > 0 \) and \( \rho(x) \) is continuous on \( (0,1) \). Then we have
\[
e_p \left( L_{m,n}, F_0, \| \cdot \|_{L_\rho}, \omega_h \right) = \left[ \sum_{j=0}^{m-1} \sum_{j=0}^{m-1} C_i(n) C_j(n) \left( \int_0^1 \frac{2}{\pi n_i} - \frac{n_i}{\sqrt{2\pi(n_i+n_j)}} \right) \right]^\frac{1}{2} \cdot \left( \int_0^1 \rho(x) \, dx \right)^\frac{1}{2} + o(n^\frac{1}{2}).
\]

where
\[
v_p = \frac{1}{\sqrt{2\pi}} \int_\infty^\infty |x|^\alpha e^{-\frac{x^2}{2}} \, dx.
\]

Here and in the following the notation \( a_n = o(b_n) \) for sequences \( \{a_n\} \) and \( \{b_n\} \) means that
\[
\lim_{n \to \infty} a_n/b_n = 0.
\]

3. Proof of Theorem 1

To prove Theorem 1 we need the following two lemmas.

**Lemma 1 ([5, p. 15]).** If
\[
0 < x < 1, \delta > \frac{1}{3},
\]
then
\[
\sum_{k=0}^n p_{n,k}(x) \leq Cn^k
\]
for each \( k > 0 \), the constant \( C \) depending only on \( \alpha \) and \( k \).

**Lemma 2 ([5, p. 15]).** For fixed
\[
0 < x < 1, \delta > \frac{1}{3},
\]
the asymptotic relation
\[
p_{n,k}(x) = \left( \frac{n}{k} \right)^{\frac{k}{2}} (1-x)^{\frac{n-k}{2}} \cdot \left( 2\pi(1-x) \right)^{\frac{1}{2}} \exp \left\{ \frac{n-k}{2\pi(1-x)} \left( \frac{k}{n} \right)^2 \right\}
\]
holds uniformly for all values of \( k \) satisfying the inequality
\[
\left| \frac{k}{n} - x \right| \leq n^{-\delta}.
\]

In other words,
\[
\lim_{n \to \infty} \frac{p_{n,k}(x)}{p_{n,k}(x)} = 1
\]
uniformly for all \( k \) satisfying (4).

**Proof of Theorem 1.** From [1, p.107] we have
\[
e_p \left( L_{m,n}, F_0, \| \cdot \|_{L_\rho}, \omega_h \right) = v_p \left( L_{m,n}, F_0, \omega_h \right) \left( \int_0^1 \rho(x) \, dx \right)^\frac{1}{2} \cdot \left( \int_0^1 \rho(x) \, dx \right)^\frac{1}{2} + o(n^\frac{1}{2}).
\]

By (2),
\[
\left( \int_0^1 \rho(x) \, dx \right)^\frac{1}{2} = \frac{1}{\sqrt{\pi}} \int_0^1 f(x) \, df + \sum_{k=0}^{m-1} C_i(n) \sum_{j=0}^{m-1} C_j(n) \left( \int_0^1 \frac{k}{n} \rho(x) \, dx \right)^\frac{1}{2}
\]
by (1), we have
\[
A_1(x) = \int_0^1 f^2(x) \, df = x.
\]

Note that
\[
\sum_{k=0}^n p_{n,k}(x) = 1, \quad \sum_{k=0}^n k p_{n,k}(x) = nx,
\]
by (1), we have
\[ A_2(x) = \sum_{k=0}^{n-1} C_i(n) \sum_{k=0}^{n-1} p_{n,k}(x) \int_{[a,x]} f(x) f\left(\frac{k}{n}\right) \omega_b(\text{df}) \]
\[ = \sum_{k=0}^{n-1} C_i(n) \sum_{k=0}^{n-1} p_{n,k}(x) \left(1 - \frac{x}{n} \right) \left(1 - \frac{k}{n} \right) \left(1 - \frac{x - k}{n} \right) \]  
\[ = x - \frac{1}{2} \sum_{i=0}^{n-1} C_i(n) \sum_{k=0}^{n-1} p_{n,k}(x) \left| x - \frac{k}{n} \right| \]  
(9)

From [4,(3.24)], we know
\[ \sum_{k=0}^{n-1} p_{n,k}(x) \left| x - \frac{k}{n} \right| \leq \sqrt{\frac{2x(1-x)}{\pi n^2}} + o\left(n^{-\frac{1}{2}}\right). \]
Combining (3) and (9) we get
\[ A_2(x) = x - \frac{1}{2} \sum_{i=0}^{n-1} C_i(n) \left(\sqrt{2x(1-x)} + o\left(n^{-\frac{1}{2}}\right)\right) \]
\[ = x - \frac{\sqrt{x(1-x)}}{\sqrt{2\pi}} \sum_{i=0}^{n-1} C_i(n) + o\left(n^{-\frac{1}{2}}\right) \]  
(10)

Now, we estimate the term \( A_1(x) \). From (2) and (8),
\[ A_1(x) = \sum_{i=0}^{n-1} C_i(n) \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} p_{n,k}(x) \sum_{s=0}^{n-1} p_{n,s}(x) \]
\[ \cdot \int_{[a,x]} f \left(\frac{k}{n}\right) f \left(\frac{s}{n}\right) \omega_b(\text{df}) \]
\[ = \sum_{i=0}^{n-1} C_i(n) \sum_{j=0}^{n-1} C_j(n) \sum_{k=0}^{n-1} p_{n,k}(x) \sum_{s=0}^{n-1} p_{n,s}(x) \]
\[ \cdot \left(1 - \frac{k}{n} \right) \left(1 - \frac{s}{n} \right) \left(1 - \frac{k + s}{n} \right) \]
\[ = x - \frac{1}{2} \sum_{i=0}^{n-1} C_i(n) \sum_{j=0}^{n-1} C_j(n) \sum_{k=0}^{n-1} p_{n,k}(x) \]
\[ \sum_{s=0}^{n-1} p_{n,s}(x) \left| x - \frac{k}{n} \right| \left| x - \frac{s}{n} \right| \]
(11)

Using Lemma 1 and (3), we have
\[ \sum_{s=0}^{n-1} p_{n,s}(x) \leq Cn^{-2}. \]

Note that
\[ 0 \leq \frac{k}{n}, \frac{s}{n} \leq 1, \]
we get
\[ \sum_{s=0}^{n-1} p_{n,s}(x) \left| x - \frac{k}{n} \right| \left| x - \frac{s}{n} \right| \leq Cn^{-2}. \]  
(12)

By (8) and (12), we obtain
\[ \sum_{s=0}^{n-1} p_{n,s}(x) \left| x - \frac{k}{n} \right| \left| x - \frac{s}{n} \right| \leq Cn^{-2}. \]  
(13)
By a simple computation we know
\[ F'_1(\xi, \eta) \leq C, \quad F'_2(\xi, \eta) \leq C. \]

From (15),
\[
\begin{align*}
\left| k \frac{s - \eta}{n_j} \right| \exp \left\{ \frac{-n_i}{2x(1-x)} \left( x - \frac{k}{n_i} \right)^2 + \frac{-n_j}{2x(1-x)} \left( x - \frac{s}{n_j} \right)^2 \right\} \\
= \left\| (x - v_1) - (x - v_2) \right\| \exp \left\{ \frac{-n_i}{2x(1-x)} (x - v_1)^2 + \frac{-n_j}{2x(1-x)} (x - v_2)^2 \right\} + \left( n_{11}^{1/2} \right).
\end{align*}
\]

Integrating two side of (16) about \((v_1, v_2)\) in
\[
\left[ \frac{k}{n_i}, \frac{k + 1}{n_i} \right] \times \left[ \frac{s}{n_j}, \frac{s + 1}{n_j} \right]
\]
we get
\[
\begin{align*}
\left| \sum_{1 \leq k \leq n_{11}} \sum_{1 \leq s \leq n_{11}} p_{n_i, k} (x) \frac{k - s}{n_i n_j} \right| p_{n_j, s} (x)
\end{align*}
\]
From (14)-(17), we have
\[
\begin{align*}
&\sum_{1 \leq k \leq n_{11}} \sum_{1 \leq s \leq n_{11}} \sum_{1 \leq j \leq n_{11}} \sum_{1 \leq l \leq n_{11}} \left( 1 + o(1) \right) \sqrt{n_{11} n_j n_i} \left\{ \frac{k + 1}{n_i} \int_{u_{n_{11}j}}^{v_{n_{11}j}} \int_{u_{n_{11}i}}^{v_{n_{11}i}} \left( x - v_1 \right) - \left( x - v_2 \right) \left\| \exp \left\{ \frac{-n_i}{2x(1-x)} (x - v_1)^2 + \frac{-n_j}{2x(1-x)} (x - v_2)^2 \right\} \right\| dv_2 + o \left( n_{11}^{1/2} \right) \right.
\end{align*}
\]

Let
\[
w_1 = \sqrt{\frac{n_i}{2x(1-x)} (x - v_1)}, \quad w_2 = \sqrt{\frac{n_j}{2x(1-x)} (x - v_2)},
\]
by (18), we get
\[
\sum_{k=0}^{n} p_{n,k}(x) \sum_{j=0}^{n} p_{n,j}(x) \left| \frac{k}{n_j} - \frac{s}{n_j} \right|
= \frac{\sqrt{2x(1-x)}}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{w_1 - w_2}{\sqrt{n_i} \sqrt{n_j}} \exp \left\{ -w_1^2 - w_2^2 \right\} dw_1 + o\left(n^{-\frac{1}{2}}\right).
\]

By (3), suppose that \( \frac{n}{n_j} = c_i^2 \), from (19) and the convergence of the improper integral
\[
\int_{-\infty}^{\infty} dw_1 \int_{-\infty}^{\infty} \left| c_j w_1 - c_j w_2 \right| \exp \left\{ -w_1^2 - w_2^2 \right\} dw_2,
\]
we have
\[
\sum_{k=0}^{n} p_{n,k}(x) \sum_{j=0}^{n} p_{n,j}(x) \left| \frac{k}{n_j} - \frac{s}{n_j} \right| = \frac{\sqrt{2x(1-x)}}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{w_1 - w_2}{\sqrt{n_i} \sqrt{n_j}} \exp \left\{ -w_1^2 - w_2^2 \right\} dw_1 + o\left(n^{-\frac{1}{2}}\right)
\]
\[
= \frac{\sqrt{2x(1-x)}}{\pi} \int_{-\infty}^{\infty} \exp \left\{ -\frac{1}{n_j} \left( \frac{n_i}{n_j} \right) w_1^2 + \frac{\sqrt{2x(1-x)}}{n_j} \exp \left\{ -\frac{1}{n_j} \left( \frac{n_i}{n_j} \right) w_2^2 \right\} dw_2 + o\left(n^{-\frac{1}{2}}\right)
\]
\[
= \frac{\sqrt{2x(1-x)}}{\pi} \left( \frac{1}{n_j} \sqrt{n_i + n_j} + \frac{1}{n_j} \sqrt{n_i + n_j} \right) + o\left(n^{-\frac{1}{2}}\right)
\]
Combining (11) and (20), we obtain
\[
A_0(x) = x - \frac{1}{2} \sum_{i=0}^{m-1} C_i(n) \sum_{j=0}^{m-1} C_j(n) \sum_{i=0}^{n_i} \sum_{j=0}^{n_j} p_{n,i}(x) p_{n,j}(x) \left| \frac{k}{n_j} - \frac{s}{n_j} \right|
\]
\[
= x - \frac{\sqrt{2x(1-x)}}{\pi} \frac{1}{n_j} \sqrt{n_i + n_j} + o\left(n^{-\frac{1}{2}}\right).
\]

From (5)-(7), (10), and (21), we complete the proof of Theorem 1.

REFERENCES


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