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Researches on Six Lattice-Valued Logic

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Abstract

Based on the direct product of Boolean algebra and Lukasiewicz algebra, six lattice-valued logic is put forward in this paper. The algebraic structure and properties of the lattice are analyzed profoundly and the tautologies of six-valued logic system $L_6P(X)$ are discussed deeply. The researches of this paper can be used in lattice-valued logic systems and can be helpful to automated reasoning systems.

Keywords

Six Lattice-Valued Logic, Lattice Implication Algebra, Filter, Tautology

1. Introduction

Lattice-valued logic is an important case of multi-valued logic, and it plays more and more important roles in artificial intelligence and automated reasoning. Six lattice-valued is a kind of common lattice, which can express logic in real world, such as language values, and evaluation values. It can deal with not only comparable information but also non-comparable information. Therefore, theoretical researches and logic and reasoning systems based on six lattice-valued logic are of great significance.

2. The Structure of Lattice L_6

The set of $L=\{O,a,b,c,d,I\}$ is a lattice, and the order relation of L is shown in **Figure 1**. The complement operator "′"and implication operation "→" are defined in **Table 1** respectively.

L means an lattice implication algebra.

Then set $A=\{O,I\}$, $B=\{O,m,I\}$. As A is the true set of classical binary logic, the operation rules of the complement operation and the implication operation are the same with the classical two-valued logic systems. B is the true-value set of Lukasiewicz system with three-valued logic, and complement operations and implication operations are defined in **Table 2**.

Let $L^*=A\times B$, the order relations, disjunctive, conjunctive, complement operation and implication operation

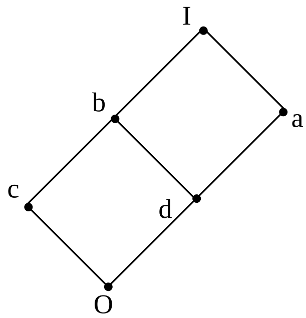

Figure 1. Structure of the six-valued lattice.

Table 1. Computing of the six-valued lattice L_6.

x	x'	$\to$	O	a	b	c	d	I
O	I	O	I	I	I	I	I	I
a	c	a	c	I	b	c	b	I
b	d	b	d	a	I	b	a	I
c	a	c	a	a	I	I	a	I
d	b	d	b	I	I	b	I	I
I	O	I	O	a	b	c	d	I

Table 2. Computing of L_3.

x	x'	$\to$	O	m	I
O	I	O	I	I	I
m	m	m	m	I	I
I	O	I	O	m	I

on L are defined as follows:

For any $(a,b)\in L^*$, $(c,d)\in L^*$:

(1) $(x,y)\le(z,r)$, if and only if $x\le z$ and $y\le r$.

(2) $(x,y)=(z,r)$, if and only if $x=z$ and $y=r$.

(3) Under other circumstances, (x, y) cannot be compared with (z, r).

(4) $(x,y)\wedge(z,r)=(x\wedge z,y\wedge r)$, $(x,y)\vee(z,r)=(x\vee z,y\vee r)$.

(5) $(x, y)\to(z, r)=(x\to z,y\to r)$.

(6) $(x,y)'=(x',y')$.

The L^* constitute a six element lattice and its operation diagram is shown in Hasse **Figure 2**.

Theorem 1. L is isomorphic lattice implication of L^*.

Proof:

Obviously, we can construct a upward one-to-one mapping from L to L^*: $f:L\to L^*$, making

$$f(O)=(O,O),\quad f(a)=(O,I),\quad f(b)=(I,m)$$
$$f(c)=(I,O),\quad f(d)=(O,m),\quad f(I)=(I,I)$$

Clearly f is conjunctive homomorphic mapping and disjunctive homomorphism mapping.

Here is the proof that f is complement homomorphic mapping and implication homomorphism mapping.

According to the definition of implication operations and complement operations, it can be easily obtained in **Table 3**.

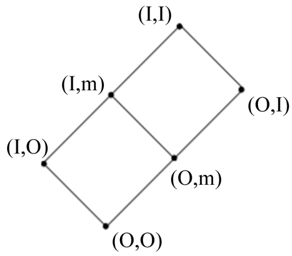

Figure 2. Six-valued lattice generated by the direct product.

Table 3. Six-valued lattice generated by the direct product.

x	x'	$\to$	(O,O)	(O,I)	(I,m)	(I,O)	(O,m)	(I,I)
(O,O)	(I,I)	(O,O)	(I,I)	(I,I)	(I,I)	(I,I)	(I,I)	(I,I)
(O,I)	(I,O)	(O,I)	(I,O)	(I,I)	(I,m)	(I,O)	(I,m)	(I,I)
(I,m)	(O,m)	(I,m)	(O,m)	(O,I)	(I,I)	(I,m)	(O,I)	(I,I)
(I,O)	(O,I)	(I,O)	(O,I)	(O,I)	(I,I)	(I,I)	(O,I)	(I,I)
(O,m)	(I,m)	(O,m)	(I,m)	(I,I)	(I,I)	(I,m)	(I,I)	(I,I)
(I,I)	(O,O)	(I,I)	(O,O)	(O,I)	(I,m)	(I,O)	(O,m)	(I,I)

It can be seen from the **Table 3**, f is the implication operations and the complement operations homomorphic.

In summary, we proofed that:

For any $x, y \in L$, $f(x') = (f(x))'$, $f(x * y) = f(x) * f(y)$, where $*$ is one of disjunctive, conjunctive, complement operation.

Thus L and L^* is isomorphic lattice implication.

3. The Property and Language of Lattice L_6

Due to L_6 is a lattice implication algebra, it not only has all the properties of lattice implication algebra but also properties as follows.

Theorem 2. As shown the six-valued lattice L_6 in **Figure 1**, the implication operation satisfies the following properties: For any $x, y, z \in L_6$:

(1) $z \le y \to x$ iff $y \le z \to x$.

(2) $z \to (y \to x) \ge (z \to y) \to (z \to x)$.

(3) $(x \to y) \vee ((x \to y) \to (x' \vee y)) = I$.

(4) $y \to z \le (x \to y) \to (x \to z)$.

(5) $x \to (y \to z) = y \to (x \to z)$.

(6) $x \to y \le (x \vee z) \to (y \vee z)$.

Theorem 3. As the true subset of L_6, $L_0 = \{O, I, a, c\}$ is a sub lattice implication algebra. What's more, L_0 is a Boolean algebra, and the implication arithmetic of it meets that: for any x, $y \in L_0$, $x \to y = x' \vee y$.

Proof: It is clearly that L_0 is a sub lattice of L_6. For any $x, y \in L_0$, $x' \in L_0$, $x \to y \in L_0$, therefore when regarding L_6, the operation of L_0 is closed, that is to say, L_0 is a sub lattice implication algebras of L_6.

It can be verified easily: for any $x, y \in L_0$, $x \to y = x' \vee y$. Meeting the of Boolean algebra axiom, L_0 is a Boolean algebra.

Any sub-set of power set lattice in a collection is called the set lattice for the collection. The isomorphism from a lattice L to a set lattice $B(X)$ in collection X is named as a isomorphic representation L by $B(X)$, which can be denoted as L for abbreviation. Through establishing the lattice representation, lattice language can be simplified, which is very important for studying the structure and properties of the lattice.

Definition 1 [1]. Let L is a lattice, an element $x \in L$ is called as an join-irreducible element, if

(1) $x \neq O$ (when there is a minimum of O when L);

(2) For any $a, b \in L$, if $x = a \vee b$, then $x = a$ or $x = b$.

Assume L is a finite distributive lattice, $\Im(L)$ denotes the set of all join-irreducible element in the collection, and all the join-irreducible element in L can form under set lattice (*i.e.* ideal Lattice) according to the order relation which can be indicated as $O(\Im(L))$. Then we have the following conclusions:

Theorem 4 [2]. Let L is a finite distributive lattice, and mapping can be constructed as follows:

$$\eta : L \to O(\Im(L))$$

$$\eta(a) = \{x \in \Im(L) \mid x \leq a\}$$

The η is the lattice isomorphism from L to $O(\Im(L))$.

Theorem 5 [2]. Let L is a finite distributive lattice, then the following equivalent hold:

1) L is a distributive lattice;

2) $L \cong O(\Im(L))$;

3) L is isomorphic to a set lattice;

4) For any $n \geq 0$, L is isomorphic to 2^n sub lattice.

According to Theorem 5, theorem representation of six lattice-valued L_6 can be got easily.

Theorem 6. As shown the six-valued lattice L_6 in **Figure 1**, conclusions as follows can be got:

(1) The set of join-irreducible element in L_6 is $\Im(L_6) = \{a, b, c\}$, and its order relation are shown in **Figure 3**.

(2) The under set lattice (*i.e.* ideal lattice), which is the set of all the join-irreducible element and forms according to its order relation, is $O(\Im(L_6)) = \{\phi, \{c\}, \{a\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$.

(3) The Hasse diagram $O(\Im(L_6))$ of the ideal lattice of L_6, which forms through inclusion relation, is shown in **Figure 4**. Form the figure, we can see that L_6 is isomorphic of lattice implication to its ideal lattice $O(\Im(L_6))$. Lattice implication isomorphism η is defined as follows:

$$\eta : L \to O(\Im(L))$$

$$h(O) = \phi,\ \eta(a) = \{b, c\},\ \eta(b) = \{a, c\},\ \eta(c) = \{a\}$$

$$\eta(d) = \{c\},\ \eta(I) = \{a, b, c\}$$

4. The Filter of Lattice L_6

Since all Lukasiewicz algebras are lattice implication algebra [1], it can be proved that Lukasiewicz algebra filters are trivial.

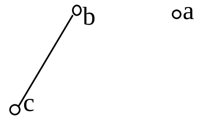

Figure 3. The order of $\Im(L_6)$.

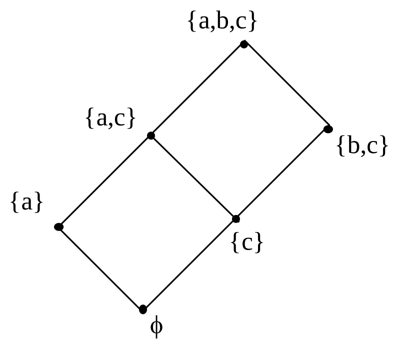

Figure 4. The ideal lattice of $O(\Im(L_6))$.

Theorem 7.
(1) The finite chain of Lukasiewicz only contains trivial filters.
(2) Lukasiewicz algebra [0,1] only contains trivial filters.

Proof: (1) Let's set $L = L_n = \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \cdots, \frac{n-2}{n-1}, 1\right\}$ $(n = 2,3,\cdots)$. Specific operations are as follows:

For any $x, y \in L$,

$$x \vee y = \max\{x, y\}, \quad x \wedge y = \min\{x, y\}$$

$$x' = 1 - x, \quad x \to y = \min\{1, 1 - x + y\}$$

It is clearly that set {1} and L are trivial filters in L. we can proof that L don't contain any other trivial filters.

From Theorem 6 we can see that filters in L are ideal dual filters of L, and the set of ideal dual filters of L are upper set of L.

If $F = \left\{\frac{k}{n-1}, \frac{k+1}{n-1}, \cdots, 1\right\}$ (where $k \geq 1$) is a filter of L, then

$$\frac{k}{n-1} \to \frac{k-1}{n-1} = \min\left\{1 - \frac{k}{n-1} + \frac{k-1}{n-1}, 1\right\} = \frac{n-2}{n-1} \in F,$$

And $\frac{k}{n-1} \in F$, so it can be seen that the definition of filters: $\frac{k-1}{n-1} \in F$.

This shows that $F = L$, so it demonstrated that L don't contain any other trivial filters.

(2) Let $L = [0,1]$, its upper operation is the same as defined C_2.

It is clearly that set {1} and L are trivial filters in L. we can proof that L don't contain any other trivial filters.

We can see that filters in L are ideal dual filters of L, and the set of ideal dual filters of L are upper set of L. So the filter of L must be an interval containing greatest element 1.

Firstly, we can proof that the filter of L must be a closed interval.

Let us set $F = (u, 1)$ is a filter of L, where $0 < u < 1$, for any x, satisfies $u < x < 1$, then $x \in F$, and $x \to u = \min\{1, 1 - u + x\} = 1 - u + x > u \in F$, conclusion can get $u \in F$.

This shows that F is a closed interval.

Secondly, assume $F = [u, 1]$ is a filter of L, where $0 < u < 1$.

For any x, making $u < x < 1$ and $x + u \geq 1$, then

$$u \to (x + u - 1) = \min\{1 - u + (x + u - 1), 1\} = x \in F$$

thereby $x + u - 1 \in F$, that is contradictory, because $0 \leq x + u - 1 < u$.

So F is an interval.

This proves that Lukasiewicz interval only have trivial filters.

As a special case of Theorem 7, we have the following corollary.

Corollary 1. $C_2 = \{O, I\}$ and $L_3 = \{O, m, I\}$ only contain trivial filters.

Theorem 8. The six element lattice only contains the following four filters:

$\{I\}$, L_6, $F_a = \{I, a\}$, $F_{bc} = \{I, b, c\}$.

Proof: According to Theorem 1, L_6 can be seen as the direct product of C_2 and L_3. According to Corollary 1, $C_2 = \{O, I\}$ and $L_3 = \{O, m, I\}$ only contain trivial filters. As followed:

The filters of $C_2 = \{O, I\}$ are $\{I\}$ and $\{O, I\}$.

The filters of $L_3 = \{O, m, I\}$ are $\{I\}$ and $\{O, m, I\}$.

It is easy to know, the filters of L_6 are the direct products of the filters of C_2 and the filters of L_3. So the filters of L_6 are as followed:

$\{(I, I)\}$, $\{(I, I), (O, I)\}$, $\{(I, I), (I, m), (I, O)\}$ and L_6 itself.

In other words: The six element lattice L_6 only contains the following four filters:

$\{I\}$, L_6, $\{I, a\}$, $\{I, b, c\}$.

5. The Tautologies of Lattice-Valued Logic System $L_6P(X)$

Here we take the lattice-valued logic system $L_6P(X)$ into consideration, and discuss its tautologies and F-tautologies, the true value domain is L_6.

It is easy to verify:

$$L_6 = C_2 \times L_3$$

where C_2 is a Boolean algebra $\{O, I\}$, L_3 is a Lukasiewicz algebra $\{O, m, I\}$.

Theorem 9. (The definition of tautologies in $L_6P(X)$ [3]) The tautologies in six lattice-valued logic system $L_6P(X)$ process the following relationship:

(1) $T^{L_6} = T^{C_2} \cap T^{L_3} = T^{L_3}$.

(2) $T_a^{L_6} = T^{L_3}$.

(3) $T_b^{L_6} = T^{C_2} \cap T_m^{L_3} = T^{C_2}$.

(4) $T_c^{L_6} = T^{C_2}$.

(5) $T_d^{L_6} = T_m^{L_3} = T^{C_2}$.

Proof: It is noticed that the tautologies in Lukasiewicz three-valued logic system process the following relationship:

$$T^{L_3} \subset T_m^{L_3} = T^{C_2}.$$

Proof of this theorem can be obtained.

From Theorem 7, the six element lattice L_6 only contains four filters as followed:

$\{I\}$, L_6, $F_a = \{I, a\}$, $F_{bc} = \{I, b, c\}$.

Therefore, its non-trivial filters are $F_a = \{I, a\}$, $F_{bc} = \{I, b, c\}$.

We can get the definition of F-tautologies in six lattice-valued logic system $L_6P(X)$ as Theorem 8 similarly.

Theorem 10. (The definition of F-tautologies in $L_6P(X)$ [4]) The F-tautologies in six lattice-valued logic system $L_6P(X)$ process the following relationship:

(1) $T_{F_a}^{L_6} = T^{L_3}$.

(2) $T_{F_{bc}}^{L_6} = T^{C_2}$.

Proof:

Since $T(f)(A) \neq T(g)(A)$, so T is an injection.

Clearly T is a surjection. For any $(U_1, \Psi_L(U_1)), (U_2, \Psi_L(U_2)) \in |\aleph(L)|$,

$\forall \hat{\Re}_f \in H_{S\aleph(L)}((U_1, \Psi_L(U_1)), (U_2, \Psi_L(U_2)))$, G has the inverse image.

Thus G is an isomorphic functor of $\aleph(L)$.

As isomorphic relationship means an equivalence relation, so $S\aleph(L)$ and $\Im(L)$ are isomorphic.

6. Conclusion

In this paper, the six element lattice is built by the direct product of Boolean algebra and Lukasiewicz algebra; the operation of the lattice is defined; the structures, properties and filters are studied; finally the tautologies and F-tautologies of the six lattice-valued logic system are discussed. The results of this paper can be applied to lattice-valued logic systems and automated reasoning applications.

Acknowledgements

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