

Solutions for Series of Exponential Equations in Terms of Lambert-W Function and Fundamental Constants

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Abstract

Series of exponential equations in the form of $x = y^n y^{\left(\frac{x}{y^{n+1}}\right)}$ were solved graphically, numerically and analytically. The analytical solution was derived in terms of Lambert-W function. A general numerical solution for any y is

found in terms of n or in base y . A solution $\frac{10^2 W\left(\frac{-\ln 10}{10}\right)}{\left(\frac{-\ln 10}{10}\right)} = 137.129$ is

close to the fine structure constant. The equation which provided the solution as the fine structure constant was derived in terms of the fundamental constants.

Keywords

Exponential Equation, Lambert-W Function, Fine Structure Constant, Logarithmic Equation, Numerical Analysis, Fundamental Constants

1. Introduction

Exponential equations are widely used in natural and social sciences. In this paper, we considered series of exponential equations and solved them graphically, numerically, and analytically in terms of Lambert-W function. One equation connected to the fine structure constant, was derived in terms of the fundamental constants and led to a new equation. The Lambert-W function for real variables is defined by the equation $W(x)\exp[W(x)] = x$ [1] [2] [3] [4] and it has applications in Planks spectral distribution law [5] [6], QCD renormalization [7], solar cells [8], bio-chemical kinetics [9], optics [10], population growth

and water movement in soil [11].

Considering the series of exponential equations defined by the following equation

$$x = y^n y^{\left(\frac{x}{y^{n+1}}\right)} \quad (1.1)$$

where x, y, n are real variables.

Taking \log_y on both sides of the Equation (1.1)

$$\log_y x = n + \frac{x}{y^{n+1}} \quad (1.2)$$

Converting the Equation (1.2) to natural logarithm

$$\frac{\ln x}{\ln y} = \frac{x}{y^{n+1}} + n \quad (1.3)$$

The trivial solution of the Equations (1.1) to (1.3) is

$$x = y^{n+1} \quad (1.4)$$

In this paper, we are focusing on the non-trivial solutions.

For $n = 2, 1, 0, -1, -2$, the Equations (1.1) and (1.3) become:

$$x = y^2 y^{\frac{x}{y^3}} \text{ or } \frac{\ln x}{\ln y} = \frac{x}{y^3} + 2 \quad (1.5)$$

$$x = y y^{\frac{x}{y^2}} \text{ or } \frac{\ln x}{\ln y} = \frac{x}{y^2} + 1 \quad (1.6)$$

$$x = y^y \text{ or } \frac{\ln x}{\ln y} = \frac{x}{y} \text{ or } x^y = y^x \quad (1.7)$$

$$x = y^{-1} y^x \text{ or } \frac{\ln x}{\ln y} = x - 1 \quad (1.8)$$

$$x = y^{-2} y^{xy} \text{ or } \frac{\ln x}{\ln y} = xy - 2 \quad (1.9)$$

2. Graphical Solutions

If $y=10$, the Equations (1.5) to (1.9) become

$$x = 10^2 \times 10^{\frac{x}{10^3}} \quad (2.1)$$

$$x = 10 \times 10^{\frac{x}{10^2}} \quad (2.2)$$

$$x = 10^{10} \quad (2.3)$$

$$x = 10^{-1} \times 10^x \quad (2.4)$$

$$x = 10^{-2} \times 10^{10x} \quad (2.5)$$

The curves and the straight line to obtain the graphical solutions of the Equations (2.1) to (2.5) are shown in **Figure 1**. The intercepts of the curves and the straight line indicate the solutions.

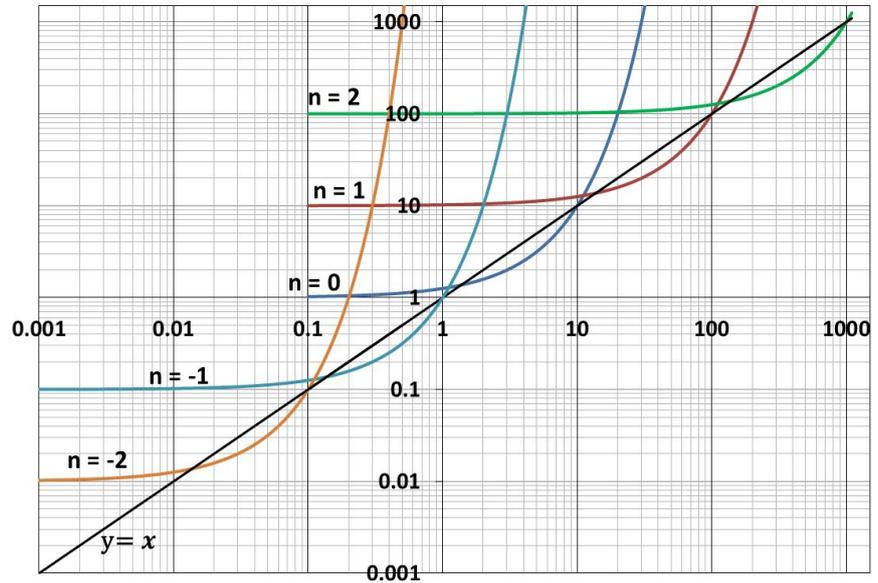


Figure 1. Plots of the functions to obtain the graphical solutions for the Equations (2.1) to (2.5).

The intersecting points of 0.1, 1, 10, 100 and 1000 are the trivial solutions and the intersecting points at around 0.0137, 0.137, 1.37, 13.7 and 137 are the non-trivial solutions.

The non-trivial solutions imply the following equations:

$$10^{0.1371} = 1.371 \quad (2.6)$$

$$\frac{\ln 1.371}{1.371} = \frac{\ln 10}{10} = 0.2302 \quad (2.7)$$

$$10^{1.371} = 1.371^{10} = 23.5 \quad (2.8)$$

3. Numerical Solutions

Higher precision non-trivial numerical solutions were obtained for the series of equations $x = y^n y^{\left(\frac{x}{y^{n+1}}\right)}$ using the iterative technique for $n = 2, 1.5, 1, 0.5, 0, -0.5, -1, -2$ and $1 \leq y \leq 15$ (Table 1). The iterations do not converge on non-trivial solutions for $y < e$, and solutions in this range were obtained by trial and error.

The solutions in Table 1 for $n = -2, -1, 0, 0.5, 1, 2$ are plotted as x vs y with x axis in log scale (Figure 2). Sharp turning points in the plots are observed for y values in the range of 1 to 2.

4. Analytical Solution

Consider the Equation (1.3)

$$\frac{\ln x}{\ln y} = \frac{x}{y^{n+1}} + n$$

Table 1. Non-trivial numerical solutions for the series of equations $x = y^n y^{\left(\frac{x}{y^{n+1}}\right)}$.

y	Solutions x for different n values							
	-2	-1	-0.5	0	0.5	1	1.5	2
1.1	36	39.6	41.53	43.56	45.68	47.92	50.25	52.71
1.3	7.41	9.63	10.99	12.53	14.28	16.28	18.56	21.17
1.5	3.29	4.94	6.05	7.41	9.07	11.11	13.61	16.67
2	1	2	2.83	4	5.66	8	11.31	16
e	0.366937	1.00000	1.644494	2.718282	4.46877	7.3890	12.13412	20.0855
3	0.275339	0.82601	1.430704	2.478052	4.292113	7.4341	12.87634	22.3024
4	0.125	0.500000	1	2.000000	4	8.0000	16	32.0000
5	0.070597	0.352984	0.789297	1.764922	3.946485	8.8246	19.73243	44.1230
6	0.045118	0.270707	0.663095	1.624244	3.978569	9.7454	23.87141	58.4727
7	0.031227	0.218591	0.578339	1.530140	4.04837	10.7109	28.33859	74.9768
8	0.022852	0.182813	0.517072	1.462501	4.136579	11.7000	33.09263	93.6000
9	0.017424	0.156820	0.470461	1.411382	4.234145	12.7024	38.10731	114.321
10	0.013713	0.137129	0.43364	1.371299	4.336395	13.7129	43.36395	137.129
11	0.011066	0.121721	0.403704	1.338936	4.440749	14.7282	48.84823	162.011
12	0.009113	0.109353	0.37881	1.312235	4.545715	15.7468	54.54858	188.961
13	0.007632	0.099215	0.357724	1.289792	4.650411	16.7672	60.45534	217.974
14	0.006483	0.090760	0.339593	1.270640	4.7543	17.7889	66.56021	249.045
15	0.005574	0.083606	0.323804	1.254088	4.857064	18.8113	72.85595	282.169

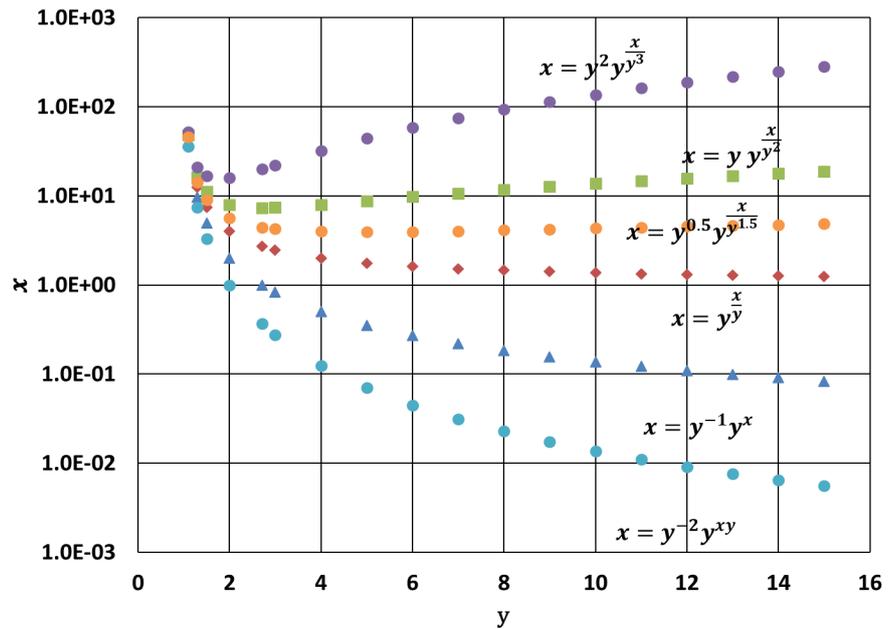


Figure 2. Plots of x vs y for the series of equations, $x = y^n y^{\left(\frac{x}{y^{n+1}}\right)}$ for $n = 2, 1, 0, 0.5, -1$ and -2 .

Let

$$t = -\ln x$$

Then (1.3) becomes

$$\frac{-t}{\ln y} = \frac{e^{-t}}{y^{n+1}} + n$$

$$(t + n \ln y) e^t = \frac{-\ln y}{y^{n+1}}$$

$$(t + n \ln y) e^{t+n \ln y} = \frac{-e^{n \ln y} \ln y}{y^{n+1}}$$

$$(t + n \ln y) e^{t+n \ln y} = \frac{-\ln y}{y}$$

$$(t + n \ln y) = W\left(\frac{-\ln y}{y}\right)$$

Substituting $-\ln x$ for t

$$(-\ln x + n \ln y) = W\left(\frac{-\ln y}{y}\right)$$

Using the Equation (1.3)

$$-\frac{x \ln y}{y^{n+1}} = W\left(\frac{-\ln y}{y}\right)$$

Hence the solution to Equation (1.3) is

$$x = \frac{W\left(\frac{-\ln y}{y}\right)}{\left(\frac{-\ln y}{y^{n+1}}\right)} = \frac{y^n W\left(\frac{-\ln y}{y}\right)}{\left(\frac{-\ln y}{y}\right)} \quad (3.1)$$

If $n = 0$, the Equation (1.1) $x = y^n y^{\left(\frac{x}{y^{n+1}}\right)}$ becomes Equation (1.7) $x = y^{\left(\frac{x}{y}\right)}$.

Using the solution in the Equation (3.1), the analytical solution in terms of Lambert-W function is

$$x = \frac{W\left(\frac{-\ln(y)}{y}\right)}{\left(\frac{-\ln(y)}{y}\right)} \quad (3.2) \quad [12] \quad [13]$$

In Equation (3.2), if $y = e$, $x = \frac{W\left(\frac{-1}{e}\right)}{\left(\frac{-1}{e}\right)}$.

But $W\left(\frac{1}{e}\right) = -1$ [6].

Hence $x = e$, the result in **Table 1**.

If $n=0$ and $y=2$ in Equation (3.2), the solutions in **Table 1** and Equation (1.7) gives

$$2^4 = 4^2 \quad \text{and} \quad \frac{\ln 2}{2} = \frac{\ln 4}{4} = -0.346. \quad (3.3)$$

Equation (3.1) gives

$$4 = \frac{W\left(-\frac{\ln(2)}{2}\right)}{\left(-\frac{\ln(2)}{2}\right)} = \frac{W(-0.346)}{-0.346}$$

$$2 = \frac{W\left(-\frac{\ln(4)}{4}\right)}{\left(-\frac{\ln(4)}{4}\right)} = \frac{W(-0.346)}{-0.346}$$

$W(-0.346)$ is double valued with -0.693 and -1.386 .

If we substitute the solutions for $n = 0$ and $y = 10$ from **Table 1** to Equation (3.2);

$$W\left(-\frac{\ln(1.37129)}{1.37129}\right) = 10 \left(-\frac{\ln(1.37129)}{1.37129}\right) \quad (3.4)$$

$$W(-0.2302) = -2.302 \quad (3.5)$$

Since x and y are symmetric in Equation (1.7)

$$W\left(-\frac{\ln(10)}{10}\right) = 1.371289 \left(-\frac{\ln(10)}{10}\right) \quad (3.6)$$

$$W(-0.2302) = -0.3157 \quad (3.7)$$

The $W(x)$ has two real values for $-1/e \leq x < 0$ [1].

If $n = -1$, the Equation (1.1) $x = y^n y^{\left(\frac{x}{y^{n+1}}\right)}$ becomes Equation (1.8) $x = y^{-1} y^x$ or $xy = y^x$.

Using the solution in the Equation (3.1), the analytical solution in terms of the Lambert-W function is

$$x = \frac{W\left(-\frac{\ln y}{y}\right)}{-\ln y} \quad (3.8) \quad [12]$$

$$\text{If } y = e, x = \frac{W\left(-\frac{1}{e}\right)}{-\ln e}.$$

But $W\left(-\frac{1}{e}\right) = -1$, Hence $x = 1$, the result in **Table 1**.

In **Table 1**, for any value of n $y = e, x = e \times e^n$, the trivial and nontrivial solutions coincide.

$$y = 10, x = 1.371289 \times 10^n$$

Using the solution in Equation (3.1), for any y the solution x can be written as

$$x(n=0) \times y^n \quad (3.9)$$

Plots of $\ln x$ vs n shown in **Figure 3** are linear as expected from Equation (3.1).

The $\ln x$ vs n lines for different y values are crossing near the point (0.5, 1.4). This indicates the solutions for $n = 0.5$ have little dependency on y for $y \geq e$. This is also evident in the numerical results for $n = 0.5$ in **Table 1** and in the plot of $x = y^{0.5} y^{\frac{x}{1.5}}$ in **Figure 2**.

5. Solutions x in Base y

The solutions x in **Table 1** can be written in base y , (x_y) to indicate the general pattern.

For any valued of n , x_y can be written as

$$x_y (n=0) \times 10^n \quad (5.1)$$

For $n = 2$, the solutions x written in base y , x_y shown in **Table 2**.

For $y > 11$, the x_y are written using the hex notation.

There is a sharp change in the value of the x_y at $y = 4$.

For $n = 2$, plot x_y vs y , for $5 \leq y \leq 11$ is shown in **Figure 4**.

6. Connection to the Fine Structure Constant

In Equation (1.1), when $n = 2$ and $y = 10$, the equation becomes $x = 10^2 \times 10^{\frac{x}{10^3}}$ and the solution is

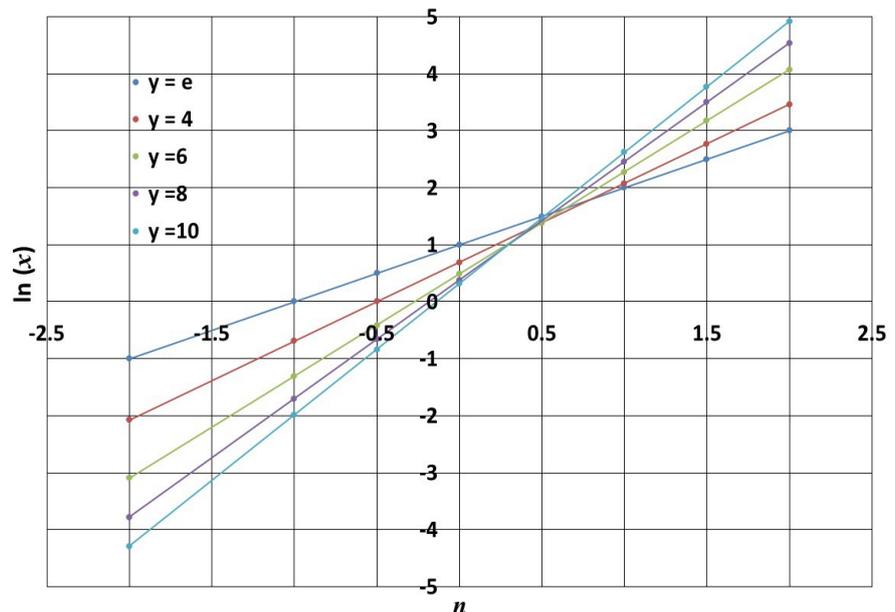


Figure 3. Plots of $\ln x$ vs n .

Table 2. Solutions x in base $y(x_y)$.

y	x written in base y for $n = 2$ x_y
2	10000.000
e	1000.000
3	211.022
4	200.000
5	134.030
6	134.250
7	134.656
8	135.463
9	136.281
10	137.129
11	138.014
12	138.B60
13	139.C67
14	13B.090
15	13C.280
16	13D.590
17	13E.9A0
18	13F.F00

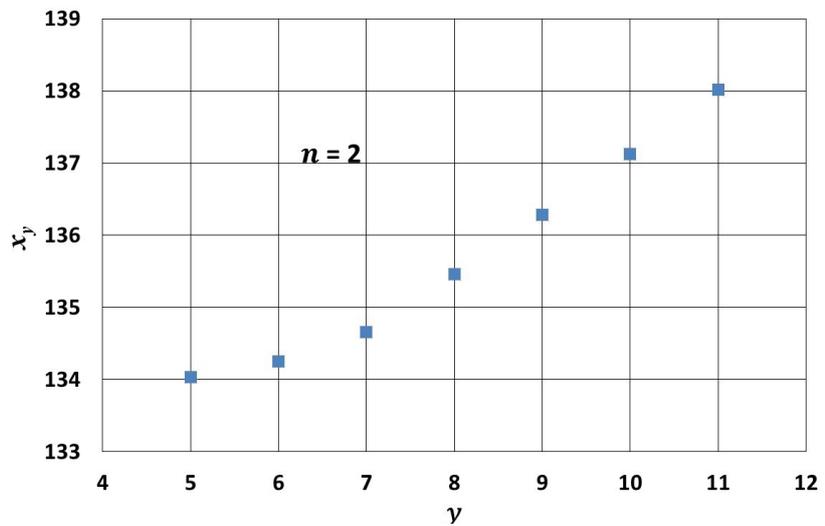


Figure 4. Plot of x_y vs y for $n = 2$.

$$x = \frac{10^2 W\left(\frac{-\ln 10}{10}\right)}{\left(\frac{-\ln 10}{10}\right)} = 137.129 \tag{6.1}$$

The solution 137.129 is close to the inverse of the fine structure constant 137.036 [14]-[21] which is dimensionless.

The inverse of the fine structure constant α^{-1} is given by the expression

$$\alpha^{-1} = \frac{4\pi\epsilon_0\hbar c}{e^2} = 137.036 \quad (6.2)$$

where;

$\hbar = 1.0545718 \times 10^{-34}$ J·s, reduced Planck constant;

$c = 2.99792458 \times 10^8$ m·s⁻¹, speed of light in vacuum;

$\epsilon_0 = 8.854187817 \times 10^{-12}$ F·m⁻¹, electric constant;

$e = 1.6021766208 \times 10^{-19}$ C, elementary charge;

α^{-1} , dimensionless constant [22].

In a recent publication Eaves [23] suggested an equation relating G and α ;

$$\frac{\alpha q^2}{8\pi^2 G m_e^2} \approx \exp\left(\frac{2}{3\alpha}\right) \quad (6.3)$$

where;

$G = 6.67408 \times 10^{-11}$ m³·kg⁻¹·s⁻², gravitational constant;

$m_e = 9.10938356 \times 10^{-31}$ kg, electron mass.

$$q^2 = \frac{e^2}{4\pi\epsilon_0}$$

By substituting the expression for α in Equation (6.3) we get

$$\frac{e^4}{32\pi^3 \epsilon_0^2 G m_e^2 \hbar c} \approx \exp\left(\frac{2}{3\alpha}\right) \quad (6.4)$$

Using Equation (6.2), the Equation (6.4) becomes

$$\alpha^{-1} \approx \frac{64\pi^4 \epsilon_0^3 \hbar^2 c^2 G m_e^2}{e^6} \exp\left(\frac{\alpha^{-1}}{1.5}\right) \quad (6.5)$$

Substituting numerical values for the pre-exponent,

$$\alpha^{-1} \approx 1.59947 \times 10^{-27} \exp\left(\frac{\alpha^{-1}}{1.5}\right) \quad (6.6)$$

$$\alpha^{-1} \approx 1.59947 \times 10^{-27} \times 10^{\left(\frac{\alpha^{-1}}{3.4538}\right)} \quad (6.7)$$

By taking the power of (1/289.5) on both sides of the Equation (6.8) and writing the equation for α^{-1} yields

$$\alpha^{-1} \approx 106.6 \times 10^{\left(\frac{\alpha^{-1}}{1000}\right)} \quad (6.8)$$

The Equation (6.8) is approximately the same as the equation $x = 10^2 \times 10^{\frac{x}{10^3}}$. The only difference is the 10² is 106.6 in Equation (6.8). But the Equation (6.8) based on the Equation (6.3) is only an approximate equation.

The value 1.59947×10^{-27} in Equation (6.6) is approximately equal to the $\alpha_G^{1/1.5}$, α_G defined by Jentschura [20].

$$\alpha_G = \frac{Gm_e m_p}{\hbar c} = 3.21 \times 10^{-42} \quad (6.9)$$

Hence the Equation (6.7) can be written as

$$\alpha^{-1} \approx \alpha_G^{1/1.5} \exp\left(\frac{\alpha^{-1}}{1.5}\right) \quad (6.10)$$

7. Conclusions

An equation in the form of $x = y^n y^{\left(\frac{x}{y^{n+1}}\right)}$ was solved graphically, numerically and analytically.

The plots of numerical solution x vs y indicate sharp turning points for y values in-between 1 to 2.

The analytical solution was found in terms of Lambert-W function as

$$x = \frac{y^n W\left(\frac{-\ln y}{y}\right)}{\left(\frac{-\ln y}{y}\right)}$$

The numerical solutions can be written as $x(n=0) \times y^n$.

The numerical solutions can also be written in base y as $x_y(n=0) \times 10^n$. For $y \geq 5x_y(n=0)$ is a universal number approximately equal to 1.37.

$$\text{If } n = 2 \text{ and } y = 10, \text{ the solution } x = \frac{10^2 W\left(\frac{-\ln 10}{10}\right)}{\left(\frac{-\ln 10}{10}\right)} = 137.129 \text{ (rounded) is}$$

close to the inverse of the fine structure constant value, 137.036.

The equation $x = 10^2 \times 10^{\frac{x}{10^3}}$ which gives the solution close to the fine structure constant can be derived from the equation $\frac{\alpha q^2}{8\pi^2 Gm_e^2} \approx \exp\left(\frac{2}{3\alpha}\right)$ suggested by Eaves.

$$\text{The derivation resulted in an equation } \alpha^{-1} \approx \alpha_G^{1/1.5} \exp\left(\frac{\alpha^{-1}}{1.5}\right).$$

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