Energy Decay for a Von Karman Equation of Memory Type with a Delay Term

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Abstract

We consider a von Karman equation of memory type with a delay term

\[\begin{align*}
  &\rho \frac{\partial^2 u}{\partial t^2} - \alpha \Delta u + \Delta^2 u - \int_0^t g(t-s) \Delta^2 u(s) \, ds + a_0 u_t + a_1 u_t(x,t-\tau \right) = [u,v].
  
\end{align*}\]

By introducing suitable energy and Lyapunov functional, we establish a general decay estimate for the energy, which depends on the behavior of g.

Keywords

Von Karman Equation, Memory Type, Delay Damping Term, General Decay Estimate

1. Introduction

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with sufficiently smooth boundary \( \Gamma := \partial \Omega \), \( \Gamma_0 \cup \Gamma_1 = \Gamma \), \( \Gamma_0 \cap \Gamma_1 \neq \emptyset \), \( \Gamma_0 \) and \( \Gamma_1 \) have positive measures and \( \nu = (\nu_1, \nu_2) \) be the outward unit normal vector on \( \partial \Omega \). We denote \( u_t = \frac{\partial u}{\partial t} \), \( \Delta u = \sum_{i=1}^2 \frac{\partial^2 u}{\partial x_i^2} \), where \( x = (x_1, x_2) \in \Omega \).

In this paper, we investigate the decay of energy of solutions for a von Karman system with memory and a delay term

\[\begin{align*}
  &\rho \frac{\partial^2 u}{\partial t^2} - \alpha \Delta u + \Delta^2 u - \int_0^t g(t-s) \Delta u(s) \, ds + a_0 u_t + a_1 u_t(x,t-\tau \right) = [u,v], \text{ in } \Omega \times (0, \infty), \\
  &u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_0 \times (0, \infty), B_1 u - B_2 \left( \int_0^t g(t-s) u(s) \, ds \right) = 0 \text{ on } \Gamma_1 \times (0, \infty), \\
  &B_1 u - \alpha \frac{\partial u}{\partial \nu} - B_2 \left( \int_0^t g(t-s) u(s) \, ds \right) = 0 \text{ on } \Gamma_1 \times (0, \infty), \\
  &u(x,0) = u_0(x), u_t(x,0) = u_t(x), \quad x \in \Omega, \quad u_t(x,t-\tau) = f_0(x,t-\tau), \quad (x,t) \in \Omega \times (0, \tau),
  
\end{align*}\]

This work was supported by the national research foundation of Korea (Grant NRF-2016R1D1A1B03930361).

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where \( \rho \) is assumed to satisfy \( \frac{2}{N-2} < \rho \leq 2 \) if \( N \geq 3 \) or \( \rho > 0 \) if \( N = 1, 2 \), \( \alpha > 0 \); \( a_0 \) is a positive constant; \( a_i \) is a real number; \( g \) is the kernel of the memory term; \( \tau > 0 \) represents the time delay; \( u_0, u, f \) are given functions belonging to suitable spaces; and the Airy stress function \( \nu \) satisfies the following elliptic problem

\[
\begin{align*}
\Delta^2 v &= -[u, u] \text{ in } \Omega \times (0, \infty), \\
v &= \frac{\partial}{\partial v} v = 0 \text{ on } \Gamma \times (0, \infty).
\end{align*}
\]

(2)

The von Karman bracket \([u, \phi]\) is given by

\[
[u, \phi] = u_{x_1} \phi_{x_2} + u_{x_2} \phi_{x_1} - 2u_{x_1} \phi_{x_2},
\]

and

\[
B_1 u = \Delta u + (1 - \mu) B_2 u, \quad B_2 u = \frac{\partial}{\partial v} \Delta u + (1 - \mu) B_2 u,
\]

here \( \mu \in \left( 0, \frac{1}{2} \right) \) is Poisson's ratio,

\[
B_1 u = 2v_1 v_2 u_{x_1 x_2} - v_1^2 u_{x_2 x_2} - v_2^2 u_{x_1 x_1},
\]

\[
B_2 u = \frac{\partial}{\partial v} \left[ (v_1^2 - v_2^2) u_{x_1 x_2} + v_1 v_2 (u_{x_2 x_2} - u_{x_1 x_1}) \right].
\]

From the physical point of view, problem (1) describes small vibrations of a thin homogeneous isotropic plate of uniform thickness of \( \alpha \); \( u = u(x,t) \) denotes the transversal displacement of the plate; the Airy stress function \( \nu = \nu(x,t) \) is a vibrating plate.

When \( a_0 = 0 \) and \( \rho = 0 \), problem (1) was studied by many authors \cite{1}-\cite{8}. The authors in \cite{1} \cite{3} \cite{4} proved uniform decay rates for the von Karman system with frictional dissipative effects in the boundary. The stability for a von Karman system with memory and boundary memory conditions was treated in \cite{5} \cite{6} \cite{7} \cite{9}. They proved the exponential or polynomial decay rate when the relaxation function decay is at the same rate. The aim of this work is to prove a general decay result for a nonlinear von Karman equation of memory type with a delay term in the first equation of (1), when the relaxation function does not necessarily decay exponentially or polynomially. As for the works about general decay for viscoelastic system, we refer \cite{10}-\cite{15} and references therein. Considering delay term \( a_0 u(t - \tau) \), the problem is different from existing literature. Time delays arise in many applications depending not only on the present state but also on some past occurrences. And the presence of delay may be a source of instability (see e.g. \cite{16} \cite{17}). Thus, recently, the control of partial differential equations with time delay effects has become an active area of research (see \cite{18} \cite{17} \cite{19} \cite{20} and references therein). Nicaise and Pignotti \cite{17} examined a wave equation with a time-delay of the form

\[
u(t) - \Delta u(t) + a_0 u(t - \tau) + a_1 u(t) = 0.
\]

(3)
They proved that the energy of the problem decays exponentially under the condition

\[ 0 < a_i < a_0, \]

and there exists a sequence of delays such that instability occurs in the case \( a_i \geq a_0 \). Kirane and Said-Houari [21] considered a viscoelastic wave equation with a delay

\[
u_t(x,t) - \Delta u(x,t) + \int_0^t g(t-s) \Delta u(x,s) ds + a_i u(x,t) + a_i u(x,t-\tau) = 0. \tag{4}\]

The authors proved the existence of a solution and a general decay result under the condition

\[ 0 < a_i \leq a_0. \tag{5} \]

They showed that the energy of solutions is still asymptotically stable even if \( a_i = a_0 \) owing to the presence of the viscoelastic damping. Recently, Wu [20] obtained similar decay results as in [21] for problem (1) without von Karman bracket \([u,v]\) under the condition (5). Motivated by these results, we prove a general decay result for a nonlinear viscoelastic von Karman Equation (1) with a time-delay under the condition

\[ |a_i| \leq a_0, \tag{6} \]

which is an extension and improvement of the previous result from [20] to a nonlinear viscoelastic von Karman equation without the assumption \( a_i > 0 \).

The plan of this paper is as follows. In Section 2, we give some notations and materials needed for our work. In Section 3, we derive general decay estimate of the energy.

### 2. Statement of Main Results

Throughout this paper, we denote

\[
V = \left\{ u \in H^3(\Omega) : u = 0 \text{ on } \Gamma_0 \right\},
\]

\[
W = \left\{ u \in H^2(\Omega) : u = \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_0 \right\},
\]

\[
(u, \phi) = \int_{\Omega} u(x) \phi(x) dx \quad \text{and} \quad (u, \phi)_{11} = \int_{\Gamma_1} u(x) \phi(x) d\Gamma.
\]

For a Banach space \( X \), \( \| \cdot \|_X \) denotes the norm of \( X \). For simplicity, we denote \( \| u \|_{L^p(\Omega)} \) by \( \| u \|_p \) and \( \| u \|_{L^p(\Gamma_1)} \) by \( \| u \|_p^{\Gamma_1} \), respectively. We define for all \( 1 \leq p < \infty \)

\[
\| u \|_p^{\Gamma_1} = \int_{\Gamma_1} |u(x)|^p dx.
\]

From now on, we shall omit \( x \) and \( t \) in all functions of \( x \) and \( t \) if there is no ambiguity, and \( c \) denotes a generic positive constant different from line to line or even in the same line.

For \( 0 < \mu < \frac{1}{2} \), the bilinear form \( a(\cdot, \cdot) \) is defined by

\[
a(u, \phi) = \int_{\Omega} \left\{ u_{x_1} \phi_{x_1} + u_{x_2} \phi_{x_2} + \mu \left( u_{x_1} \phi_{x_2} + u_{x_2} \phi_{x_1} \right) + 2(1-\mu) u_{x_1} \phi_{x_2} \right\} dx. \tag{7}\]
A simple calculation, based on the integration by parts formula, yields

\[ \int_{\Omega} \Delta^2 u \phi d\Omega = a(u, \phi) - \left( B_u, \frac{\partial \phi}{\partial V} \right)_G + (B_u, \phi)_G. \]

Thus, for \((u, \phi) \in (H^4(\Omega) \cap W) \times W\) it holds

\[ \int_{\Omega} \Delta^2 u \phi d\Omega = a((u, \phi) - \left( B_u, \frac{\partial \phi}{\partial V} \right)_G + (B_u, \phi)_G. \]

Since \(\Gamma_0 \neq \emptyset\), we know (see e.g. [1]) that \(\sqrt{a(u, u)}\) is equivalent to the \(H^5(\Omega)\) norm on \(W\), i.e.

\[ c_1 \|u\|_{H^5(\Omega)} \leq a(u, u) \leq c_2 \|u\|_{H^5(\Omega)} \]

for some \(c_1, c_2 > 0\). (8)

This and Sobolev imbedding theorem imply that for some positive constants \(C_\rho, \tilde{C}_\rho\) and \(C_s\)

\[ \|u\|_V \leq C_\rho a(u, u), \|u\|_{\omega_1} \leq \tilde{C}_\rho a(u, u) \] and

\[ \|\nabla u\|_V \leq C_s a(u, u), \forall u \in W. \] (9)

By (7) and Young’s inequality, we see that

\[ a(u, \phi) \leq \delta \|\phi\|_{H^{-1}(\Omega)}^2 + \frac{5}{8\delta} \|\phi\|_{H_{\omega_1}(\Omega)}^2 \] for all \(\delta > 0\).

From this and (8), it holds that

\[ a(u, \phi) \leq \delta a(u, u) + \frac{5}{8c_1^2 \delta} \rho(\phi, \phi) \] for all \(\delta > 0\). (10)

We introduce the relative results of the Airy stress function and von Karman bracket \([\cdot, \cdot]\).

**Lemma 2.1.** ([4]) If \(u, \phi\) and \(\psi\) belong in \(H^5(\Omega)\) and at least one of them belongs in \(H^5(\Omega)\), then \(\{u, \phi, \psi\} = \{u, \phi, \psi\}\).

**Lemma 2.2.** ([1]) Let \(u \in H^5(\Omega)\) and \(v\) be the Airy stress function satisfying (2). Then, the following relations hold:

\[ [u, v] \in L^2(\Omega) \text{ and } \|u, v\|_V \leq C \|v\|_{H^2(\Omega)} \|u\|_{H^2(\Omega)} \leq \tilde{C} \|v\|_{H^2(\Omega)} \|u\|_{H^2(\Omega)}. \]

Now, we state the assumptions for problem (1).

**(H1)** For the relaxation function \(g\), as in [11] [15], we assume that \(g: \mathbb{R} \rightarrow \mathbb{R}^+\) is a nonincreasing differentiable function satisfying \(g(0) > 0\), \(l_0 := \int_0^\infty g(s)ds < \frac{1}{2}\) and

\[ g'(t) \leq -\zeta(t) g(t) \text{ for } t \geq 0, \] (11)

where \(\zeta: \mathbb{R} \rightarrow \mathbb{R}^+\) is a nonincreasing differentiable function.

**Theorem 2.1.** Assume that (H1) is hold. Then, for the initial data \((u_0, u_1, f_0) \in (H^4(\Omega) \cap W) \times \{H^2(\Omega) \cap V\} \times L^2(\Omega \times (0, 1))\), problem (1) has a unique weak solution \(u\) in the class

\[ u \in C(0, T; H^4(\Omega) \cap W) \cap C^1(0, T; H^2(\Omega) \cap V). \]

**Proof.** This can be proved by Faedo-Galerkin method (see e.g. [7] [21]).
3. General Decay of the Energy

In this section we shall prove a general decay rate of the solution for problem (1). For simplicity of notations, we denote

\[ g \star u = \int_0^t g(t-s)u(s)ds, \]
\[ g\Box u = \int_0^t g(t-s)[u(t)-u(s)]^2 ds, \]

and

\[ g\Box \partial^2 u = \int_0^t g(t-s)a(u(t)-u(s),u(t) - u(s))ds. \]

From (9), we see that

\[ g\Box u \leq C g\Box \partial^2 u. \]  \hspace{1cm} (12)

From now on, we shall omit \( t \) in all functions of \( t \) if there is no ambiguity, and \( c \) denotes a generic positive constant different in various occurrences. Multiplying the first equation of (1) by \( u_i \), we have

\[ E'(t) = -a_{i0} [u_i] - a_i (u_i(t-\tau),u_i) + a(g \star u, u_i), \]  \hspace{1cm} (13)

where

\[ E(t) = \frac{1}{\rho + 2} [u_i]^{\rho + 2} + \frac{\alpha}{2} [\nabla u_i]^2 + \frac{1}{2} a(u,u) + \frac{1}{4} \|\Delta u\|^2. \]

From the symmetry of \( a(\cdot,\cdot) \), we see that for any \( u \in C^1(0,T;H^2(\Omega)) \)

\[ a(g \star u, u_i) = -\frac{1}{2} g(t)a(u,u) + \frac{1}{2} g\Box \partial^2 u - \frac{1}{2} \frac{d}{dt} \left[ g\Box \partial^2 u - \left( \int_0^t g(s)ds \right) a(u,u) \right]. \]  \hspace{1cm} (14)

Moreover, (10) gives

\[ a(g \star u, u_i) \leq \int_0^t g(t-s)a(u(s)-u(t),u(t))ds + \left( \int_0^t g(s)ds \right)a(u,u) \]
\[ \leq 2 \left( \int_0^t g(s)ds \right)a(u,u) + \frac{5}{8c_i^2} g\Box \partial^2 u. \]  \hspace{1cm} (15)

Now, we define a modified energy by

\[ \mathcal{E}(t) = \frac{1}{\rho + 2} [u_i]^{\rho + 2} + \frac{\alpha}{2} [\nabla u_i]^2 + \frac{1}{2} \int_0^t \left[ \frac{d}{dt} - \left( \int_0^t g(s)ds \right) a(u,u) \right] \]
\[ + \frac{1}{2} g\Box \partial^2 u + \frac{1}{4} \|\Delta u\|^2 + \frac{p}{2} \int_{\Omega} \|u_i(s)\|^2 ds, \]

where \( p \) is a positive constant satisfying

\[ |a_i| \leq p \leq 2a_0 - |a_i|. \]  \hspace{1cm} (16)

It is noted that \( E(t) \leq \frac{1}{1-l_0} \mathcal{E}(t) \). Therefore, it is enough to obtain the desired decay for the modified energy \( \mathcal{E}(t) \) which will be done below.

**Lemma 3.1.** There exist non-negative constants \( \alpha_1 \) and \( \alpha_2 \) satisfying

\[ \mathcal{E}'(t) \leq -\alpha_1 [u_i] - \alpha_2 [u_i(t-\tau)]^2 - \frac{1}{2} g(t) a(u,u) + \frac{1}{2} g\Box \partial^2 u. \]
**Proof.** Applying (14) to the last term in the right hand side of (13), we have

\[
E'(t) = -a_0 \|u_t\|^2 - a_1 (u_t (t-\tau), u_t) - \frac{1}{2} g(t) a(u, u) + \frac{1}{2} \Delta^2 u
\]

+ \frac{p}{2} \|u_t\|^2 - \frac{p}{2} \|u_t (t-\tau)\|^2.

By Young’s inequality,

\[-a_1 (u_t (t-\tau), u_t) \leq \frac{|a_1|}{2} \|u_t\|^2 + \frac{|a_1|}{2} \|u_t (t-\tau)\|^2.
\]

Thus, we have

\[
E'(t) \leq -\left( a_0 - \frac{p}{2} \frac{|a_1|}{2} \right) \|u_t\|^2 - \left( \frac{p}{2} \frac{|a_1|}{2} \right) \|u_t (t-\tau)\|^2
\]

\[-\frac{1}{2} g(t) a(u, u) + \frac{1}{2} \Delta^2 u.
\]

Putting \( a_1 = a_0 - \frac{p}{2} \frac{|a_1|}{2} \), \( a_2 = \frac{p}{2} \frac{|a_1|}{2} \) and considering (16), we complete the proof.

Now, let us define the perturbed modified energy by

\[
L(t) = NE(t) + e\Psi(t) + \Theta(t) + \Phi(t),
\]

where

\[
\Psi(t) = \frac{1}{\rho + 1} \left( u_t^n u_t, u \right) + \alpha (\nabla u, \nabla u),
\]

\[
\Upsilon(t) = \int_{t-\tau}^{t} e^{-s-\tau} \|u_t(s)\|^2 ds,
\]

\[
\Phi(t) = -\frac{1}{\rho + 1} \int_0^t g(s) (u_t(s) - u_t, u_t^n u_t) ds
\]

\[-\alpha \int_0^t g(s) (\nabla u_t(s) - \nabla u_t(s), u_t) ds.
\]

Then, it is easily shown that \( L(t) \) is equivalent with \( E(t) \) for all \( t \geq 0 \).

**Lemma 3.2.** There exist positive constants \( C_3, C_4 \) and \( t_0 > 0 \) satisfying

\[
\frac{d}{dt} L(t) \leq -C_3 E(t) + C_4 \Delta^2 u \text{ for } t \geq t_0.
\]

**Proof.** Poincare’s inequality gives

\[
\Upsilon'(t) = -\int_{t-\tau}^{t} e^{-s-\tau} \|u_t(s)\|^2 ds - \|u_t(t-\tau)\|^2
\]

\[-e^{-\tau} \int_{t-\tau}^{t} \|u_t(s)\|^2 ds + \lambda_t^2 \|\nabla u_t(t)\|^2 - e^{-\tau} \|u_t(t-\tau)\|^2,
\]

where \( \lambda_t \) is the embedding constant from \( H^1(\Omega) \) to \( L^2(\Omega) \). Using the problem (1) and (14), we have

\[
\Psi'(t) = \frac{1}{\rho + 1} \|u_t^n\|^2 + \alpha \|\nabla u_t\|^2 + \alpha \left( \frac{\partial u_t}{\partial \nu} \right)_{t_1}^2 - a(u, u)
\]

\[+ \left( B_t u, \frac{\partial u}{\partial \nu} \right)_{t_1} - \left( B_t u, u \right)_{t_1} + a(g * u, u)
\]
\[
- \int_{t}^{t'} g(t-s) \left[ \left( B_0 u(s), \frac{\partial u(t)}{\partial \nu} \right)_{\Gamma_i} - \left( B_0 u(s), u(t) \right)_{\Gamma_i} \right] \, ds \\
- a_0(u, u) - a_1(u, (t - \tau), u) - \left( \Delta^2 v, v \right)
\]
\[
= \frac{1}{\rho + 1} \left\| u \right\|_{\nu^{p+2}}^{p+2} + \alpha \left\| \nabla u \right\|^2 - a(u, u) + a(g \ast u, u)
- a_0(u, u) - a_1(u, (t - \tau), u) - \left\| \Delta v \right\|^2
\]
\[
\leq \frac{1}{\rho + 1} \left\| u \right\|_{\nu^{p+2}}^{p+2} + \alpha \left\| \nabla u \right\|^2 - \left(1 - 2\int_{0}^{1} \bar{g}(s) \, ds\right) a(u, u)
+ \frac{5}{8c_1^2} g \circ \partial^2 u - a_0(u, u) - a_1(u, (t - \tau), u) - \left\| \Delta v \right\|^2.
\]

Young and Poincaré's inequalities produce
\[
-a_0(u, u) \leq \eta a(u, u) + \frac{\lambda^2 a_0^2 C_p}{4\eta} \left\| \nabla u \right\|^2,
\]
\[
-a_1(u, (t - \tau), u) \leq \eta a(u, u) + a^2 C_p \left\| u(t - \tau) \right\|^2.
\]

Substituting these into (21), we derive
\[
\Psi'(t) \leq \frac{1}{\rho + 1} \left\| u \right\|_{\nu^{p+2}}^{p+2} + \left( \alpha + \frac{\lambda^2 a_0^2 C_p}{4\eta} \right) \left| \nabla u \right| - (1 - 2\lambda_0 - 2\eta) a(u, u)
+ \frac{5}{8c_1^2} g \circ \partial^2 u + \frac{a^2 C_p}{4\eta} \left\| u(t - \tau) \right\|^2 - \left\| \Delta v \right\|^2
\]
\[
(22)
\]

Similarly, we get from (1) that
\[
\Phi'(t) = -\int_{t}^{t'} g(t-s) \int_{t}^{t} g(t-s) a(u(\tau), u(t) - u(s)) \, d\tau \, ds
+ \int_{0}^{t} g(t-s) \left( a(u(t), u(t) - u(s)) \, ds - a_0(t) \int_{0}^{t} g(t-s) \, ds \right)
- a_1(t) \int_{0}^{t} g(t-s) \, ds \left( a(t) - u(s) \right) \, ds
- \frac{1}{\rho + 1} \left( \int_{0}^{t} \left| u \right|^2 \, ds \right) \left( u(t) - u(s) \right) \, ds
- \frac{1}{\rho + 1} \int_{0}^{t} \left( \int_{0}^{t} g(s) \, ds \right) \, ds \left( \nabla u \right) \left( \nabla u \right) \, ds
- \alpha \left( \nabla u, \int_{0}^{t} g(t-s) \, ds \right) \left( \nabla u \right) \left( \nabla u \right) \, ds
- \alpha \left( \nabla u, \int_{0}^{t} g(t-s) \, ds \right) \left( \nabla u \right) \left( \nabla u \right) \, ds
\]
\[
(23)
\]

In what follows we will estimate the terms in right hand side of (23). By similar arguments given in [8], we have
\[
|I_1| \leq \left( \delta \bar{a}_0^2 + \left( \frac{L_0}{8c_1^2} + \frac{5L_0}{8c_1^2} \right) g \circ \partial^2 u, \right.
\]
\[
|I_2| \leq \delta a \left\| \nabla u \right\|^2 - \frac{\alpha g \left( \frac{C_p}{4\delta} \right) \left( g \circ \partial^2 u \right),}{\delta}
\]
\[
|I_3| \leq \delta a \left( \nabla u \right) \left( \nabla u \right) + \frac{5}{8c_1^2} g \circ \partial^2 u,
\]
\[
|I_4| \leq \delta a \left( \nabla u \right) \left( \nabla u \right) + \frac{5}{8c_1^2} g \circ \partial^2 u,
\]
\[
|I_5| \leq \delta a \left( \nabla u \right) \left( \nabla u \right) + \frac{5}{8c_1^2} g \circ \partial^2 u.
\]
and
\[ |I_3| \leq \delta \left\| u \right\|^2 + \frac{1}{4\delta} \left( \int_0^t g(t-s)(u(t)-u(s))\,ds \right) \]
\[ \leq \delta c \left( \left\| u \right\|^2 \right) + \frac{C_p l_0}{4\delta} g \vartheta^2 u \]
\[ \leq \delta c a(u,u) + \frac{C_p l_0}{4\delta} g \vartheta^2 u. \]

Using Young inequality and the fact that imbedding \( H^1(\Omega) \hookrightarrow L^{2(\rho+1)} \) is continuous, we infer
\[ |I_4| \leq \frac{1}{\rho+1} \left( \delta \left\| u \right\|^2 + \frac{1}{4\delta} \left( \int_0^t g(t-s)(u(t)-u(s))\,ds \right) \right) \]
\[ \leq \frac{1}{\rho+1} \left( \delta \left\| u \right\|^2 + \frac{1}{4\delta} \left( \int_0^t g(t-s)(u(t)-u(s))\,ds \right) \right) \]
\[ \leq \frac{\delta \left\| u \right\|^2}{\rho+1} \left( \frac{2\varepsilon(0)}{\alpha} \right)^{\rho} \left\| \nabla u \right\|^2 \]
\[ + \frac{g(0)C_p}{4\delta} g \vartheta^2 u, \]
where \( \lambda \) is the embedding constant from \( V \) to \( L^{2(\rho+1)}(\Omega) \).

Young’s inequality and (10) give
\[ |I_5| \leq \delta \left\| \nabla u \right\|^2 + \frac{a^2 l_0 C_p \lambda \delta}{4\delta} g \vartheta^2 u \]
and
\[ |I_4| \leq \delta \left\| u_t \right\|^2 + \frac{a^2 l_0 C_p \lambda \delta}{4\delta} g \vartheta^2 u. \]

Combining these estimates with (23), we get
\[ \Phi'(t) \leq \delta \left( l_0 + l_0^2 + c \right) a(u,u) + \left\| u \right\| + \frac{\delta \left\| u \right\|^2}{\rho+1} \left( \frac{2\varepsilon(0)}{\alpha} \right)^{\rho} \left\| \nabla u \right\|^2 \]
\[ + \delta \left\| u_t \right\|^2 + \frac{1}{\rho+1} \left( \int_0^t g(s)\,ds \right) \left\| u \right\|^2 + cg \vartheta^2 u - \frac{\alpha g(0)C_p + g(0)C_p}{4\delta} g \vartheta^2 u. \]

Since \( g \) is positive, for any \( t_0 > 0 \) we have \( \int_0^t g(s)\,ds \geq \int_0^{t_0} g(s)\,ds \geq g_0 \) for all \( t \geq t_0 \). Thus, combining (17), (22) and (24), we arrive
\[ \frac{d}{dt} L(t) = NE'(t) + c \Psi'(t) + \Phi'(t) \]
\[ \leq \left\{ \alpha g_0 - \delta \left( \alpha + 1 + \frac{\lambda \delta}{\rho+1} \right) \right\} \left\| \nabla u \right\|^2 \]
\[ - \frac{1}{\rho+1} (g_0 - \epsilon) \left\| u \right\|^2 + \left( \epsilon (1 - 2\lambda \delta + \epsilon) \right) a(u,u) \]
\[
-e \| \nabla \|^2 - \left( \partial e^{-z} - \frac{\epsilon a_1^2 C_p}{4\eta} - \delta \right) \| u_0 (t-s) \|^2 \leq e^{-z} \int_s^t \| u_0 (s) \|^2 \, ds
\]
\[
+ \left[ \frac{N}{2} - \frac{\alpha g(0) C_p + g(0) C_p}{4\eta} \right] e^{-\delta^2 s} \| u \|^2 + (e+c) g \| \nabla^2 u \|.
\] (25)

First, we fix \( \eta > 0 \) and \( \theta > 0 \) such that \( 1 - 2l_0 - 2\eta > 0 \) and \( \alpha g_0 - \theta \lambda^2 > 0 \), respectively. Next, we choose \( N > 0 \) so large that \( \frac{N}{2} - \frac{\alpha g(0) C_p + g(0) C_p}{4\eta} > 0 \), and \( \epsilon > 0 \) sufficiently small such that \( g_0 - \epsilon > 0 \),
\[
\alpha g_0 - \theta \lambda^2 - \epsilon \left( \alpha + \frac{\lambda^2 C_p a_0^2}{4\eta} \right) > 0 \quad \text{and} \quad \theta e^{-\epsilon} - \frac{\epsilon a_1^2 C_p}{4\eta} > 0.
\]
Finally, taking \( \delta > 0 \)
so small that \( \alpha g_0 - \theta \lambda^2 - \epsilon \left( \alpha + \frac{\lambda^2 C_p a_0^2}{4\eta} \right) - \delta \left( \alpha + 1 + \frac{2\xi(0)}{\rho} \right) > 0 \),
\[
\epsilon (1 - 2l_0 - 2\eta) - \delta \left( l_0 + l_0^2 + \tilde{c} \right) > 0 \quad \text{and} \quad \theta e^{-\epsilon} - \frac{\epsilon a_1^2 C_p}{4\eta} - \delta > 0,
\]
we complete the proof. \( \square \)

**Theorem 3.1.** There exist positive constants \( C_0, \omega \) and \( t_0 > 0 \) such that
\[
E(t) \leq C_0 e^{-\rho t} \quad \text{for} \quad t \geq t_0.
\]

**Proof.** Multiplying (18) by \( \zeta(t) \), using (11) and (17), we get
\[
\zeta(t) L'(t) \leq -C_7 \zeta(t) E(t) + C_7 \zeta(t) \| g \| \| \nabla^2 u \|
\leq -C_7 \zeta(t) E(t) - C_4 g \| \nabla^2 u \|
\leq -C_7 \zeta(t) E(t) + C_4 \| -2E'(t) \|.
\]
Since \( \zeta \) is nonincreasing, we have
\[
\left( \zeta(t) L(t) + 2C_4 E(t) \right) \leq -C_7 \zeta(t) E(t) \quad \text{for} \quad t \geq t_0.
\]
Thus, by letting \( L(t) = \zeta(t) \| u \| + 2C_4 E(t) \), we get
\[
L'(t) \leq -C_7 \zeta(t) E(t) \quad \text{for} \quad t \geq t_0.
\]
Since \( \zeta(t) \) is a nonincreasing positive function, we can easily observe that \( L(t) \) is equivalent to \( E(t) \). Subsequently, it follows that
\[
L'(t) \leq -C_7 \zeta(t) E(t) \quad \text{for} \quad t \geq t_0.
\]
Integrating this over \( (t_0, t) \), we conclude that
\[
L(t) \leq L(t_0) e^{-\zeta(t_0) \rho t} \quad \text{for} \quad t \geq t_0.
\]
Consequently, the equivalent relations of \( L, L \) and \( E \) yield the desired result. \( \square \)

**4. Conclusion**

In this paper we proved decay rates of energy for a viscoelastic von Karman
equation with constant time delay in the velocity by establishing proper Lyapunov functionals corresponding to the delay effect. In the future work, we will consider the equation with time-varying delay effect.

References


