Finite-Time Stabilization of Dynamical System with Adaptive Feedback Control

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Abstract
In this paper, an adaptive feedback controller is proposed to achieve the finite-time stability of dynamical system. In the proposed scheme, the feedback gain of the adaptive feedback controller is automatically tuned according to the adaptation law in order to stabilize unstable fixed points of the system. Based on the Lyapunov function method and the finite-time stability theory, we get a sufficient condition for the finite-time stability. Finally, simulation results show the effectiveness and feasibility of the proposed finite-time controller.

Keywords
Chaos, Finite-Time Stabilization, Adaptive Feedback Controller

1. Introduction
In the last two decades, chaos has been a hot topic due to its varied application in many fields, such as information processing, secure communication, power converters, biological system, engineering science, etc. Since Ott, Grebogi, and Yorke (OGY) [1] proposed the first approach of chaos control, and there are many variations reports based on the OGY method [2] [3] [4] [5], creating an entire new research domain in chaos. An important problem in chaos stabilization is how to design a controller to stabilize the chaotic system. In fact, there is a wide variety of control methods to approach stabilization or synchronization of chaotic systems, including sliding model control [6] [7], adaptive control [8] [9] [10] [11] [12], optimal control [13] [14] and feedback control [15] [16] [17] and other control methods [18] [19] [20].

All of the methods mentioned above have been proposed to guarantee the asymptotic stability of chaotic system, but these methods cannot guarantee the
stability of chaotic systems in a finite time. However, in many cases, we hope the chaotic system achieves stability in a finite time. Finite-time control is a useful technique for achieving finite-time stability. Moreover, the finite-time control technique has been demonstrated better rejection and robustness [21].

Recently, due to its useful applications in many areas, a lot of research work was done about the chaotic system stability based on the finite-time control technique. In Ref. [22], Hong and Wang proposed a continuous finite-time control design method to solve the finite-time stabilization problem for a class of nonlinear control systems and a class of finite-time stabilizing controller for linear systems appears in [23]. In Ref. [24], the researchers proposed a family of continuous time-invariant finite-time stabilizing controllers for double integrator. The most important problem in the study of finite-time chaos stabilization is how to design a physically available and simple controller to guarantee the stabilization of chaotic system in a finite time. However, in most of previous studies, the finite-time controller which they choose is too special and complicated to be physically practical. In addition, some of the finite-time controllers have a linear feedback part, but we know that the feedback constant is difficult to find.

Motivated by the above analysis, we propose a physically available and simple adaptive feedback controller to achieve the finite-time stability of chaotic system. Comparing to previous approaches, we employ a time-varying feedback gain in the controller which automatically converges to suitable constants, which make the controller simpler and lead to the speed of asymptotic stability of the system faster. Otherwise, the estimation of the convergence time is also given. The finite-time control technique has demonstrated better disturbance rejection and robustness against uncertainties. Based on the finite-time stability theory and the Lyapunov function method, sufficient condition for finite-time stabilization is obtained. Finally, some numerical examples are examined to illustrate the effectiveness of the analytical result.

The rest of this paper is organized as follows. In Section 2, we give some preliminary knowledge. In Section 3, the main result is derived based on Lyapunov function method and finite-time stability theory. In Section 4, numerical simulations are given to show the effectiveness of the theoretical result. Finally, a conclusion is drawn in Section 5.

2. Preliminary Knowledge

Consider a dynamical system described by:

$$\dot{x}(t) = Ax(t) + f(x(t)),$$  \hspace{1cm} (1)

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ is the state vector of the dynamical system, $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable nonlinear vector function. To stabilize the chaotic orbits in (1) to a fixed point, we consider the adaptive feedback control method. The controlled system (1) can be rewritten as:

$$\dot{x}(t) = Ax(t) + f(x(t)) + u(t),$$  \hspace{1cm} (2)
where \( u(t) \in \mathbb{R}^n \) is the input.

Throughout this paper we require the differentiable nonlinear vector function \( f(x(t)) \) satisfies the following assumption:

**Assumption 1.** For function \( f(x) \) there exists a positive constant \( l \) such that

\[
\left[ x(t) - y(t) \right]^T \left[ f(x(t)) - f(y(t)) \right] \leq \left[ x(t) - y(t) \right]^T l \left[ x(t) - y(t) \right], \forall x, y \in \mathbb{R}^n. \tag{3}
\]

**Remark 1.** Condition (3) is usually called global Lipschitz condition, and \( l \) is called Lipschitz constant. It should be pointed out that Condition (3) is very general, most well-known dynamical systems, such as Chua’s circuit, Rössler-like system, Genesio system, and hyperchaotic Lü system, satisfy Assumption 1.

In order to get our main result in the next section, we state here the definition of finite time stability and two lemmas.

**Definition 1.** System (2) can be finite time stabilized if there exists a constant \( T > 0 \), such that

\[ \lim_{t \to +\infty} \left\| x(t) - s(t) \right\| = 0, \]

and \( x(t) = s(t) \) if \( t > T \), where \( T \) is called the setting time and \( s(t) \) is a solution of an isolated node, satisfying \( \dot{s}(t) = A s(t) + f(s(t)) \).

**Lemma 1.** [25] Assume that a continuous, positive-definite function \( V(t) \) satisfies the following differential inequality:

\[ \dot{V}(t) \leq -\lambda \beta V(t), \]

where \( \lambda > 0 \), \( 0 < \beta < 1 \) are all constants. Then, for any given \( t_0 \), \( V(t_0) \) satisfies the following inequality:

\[ V^{1-\beta}(t) \leq V^{1-\beta}(t_0) - \lambda(1-\beta)(t-t_0), \quad t_0 \leq t \leq t_1, \]

and

\[ V(t) = 0, \quad \forall t \geq t_1, \]

with \( t_1 \) given by

\[ t_1 = t_0 + \frac{V^{1-\beta}(t_0)}{\lambda(1-\beta)}. \]

**Lemma 2.** [26] Let \( a_1, a_2, \cdots, a_n > 0 \) and \( 0 < r < p \). Then

\[ \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^{n} a_i^r \right)^{\frac{1}{r}}. \]

**3. Main Result**

In order to study the finite-time stability of dynamical system (2), we define the synchronization error \( e(t) = x(t) - s(t) \), then the stability of system (2) can be translated into the analysis of the finite-time stability of error system (4). The error system is described by:

\[ \dot{e}(t) = Ae(t) + f(x(t)) - f(s(t)) + u(t). \tag{4} \]
In this paper, we designed the controller as follows:

$$u(t) = -\left[ k(t)e(t) + \eta \text{sign}(e(t)) \right],$$

(5)

where $\eta$ is a constant and $\eta \geq 1$,

$$\text{sign}(e(t)) = \left[ \text{sign}(e_1(t)), \text{sign}(e_2(t)), \ldots, \text{sign}(e_n(t)) \right]^T.$$ 

The feedback gain $k(t)$ is adapted according to the following update law:

$$k(t) = -\gamma e^2(t),$$

(6)

where $\gamma$ is an arbitrary positive number. In this paper, for a better presentation, we set $\gamma = 1$, for other cases the extension is straightforward.

**Theorem 1.** Suppose that the Assumption (1) holds and there exists a sufficiently large positive constant $L$ such that $L > l + \lambda_{\text{max}}(A^T)$, where $A^T = \frac{A + A^T}{2}$. Then system (2) can be stabilized in a finite-time under the following adaptive feedback controller (5).

**Proof.** Take the Lyapunov function

$$V(t) = \frac{1}{2}e^T(t)e(t) + \frac{1}{2}\left[ k(t) + L \right]^2,$$

(7)

where $L$ is a constant bigger that $l$. Differentiating the function $V$ along the solution of the system (2),

$$\dot{V}(t) = e^T(t)e(t) + \frac{1}{2}\left[ \dot{k}(t) + L \right]\dot{k}(t).$$

(8)

Substituting $\dot{e}(t)$ and $\dot{k}(t)$ (given by (5) and (6)) into the right-hand of Equation (8), we have

$$\dot{V}(t) = e^T(t)\left[ Ae(t) + f(x(t)) - f(y(t)) + u(t) \right] - \left[ k(t) + L \right]e^2(t).$$

(9)

Note that

$$e^T(t)Ae(t) \leq \lambda_{\text{max}}(A^T)e^T(t)e(t).$$

(10)

Under Assumption 1, from (5) and (10), we obtain

$$\dot{V}(t) \leq \lambda_{\text{max}}(A^T)e^T(t)e(t) + le^T(t)e(t) - Le^T(t)e(t) - \eta e^T(t)\text{sign}(e(t)).$$

(11)

Moreover, we notice that $e^T(t)\text{sign}(e(t)) = \|e(t)\|$, we can get

$$\dot{V}(t) \leq -\left[ L - l - \lambda_{\text{max}}(A^T) \right]e^T(t)e(t) - \eta \|e(t)\|.$$ 

If $L > l + \lambda_{\text{max}}(A^T)$, we have

$$\dot{V} \leq -\eta \|e(t)\|.$$ 

(12)

From Lemma 2, the following inequality is established

$$\sum_{i=1}^{n} e^2_i(t) \geq \left[ \sum_{i=1}^{n} e^2_i(t) \right]^{\frac{1}{2}}.$$ 

Thus, we obtain from (12) that

$$\dot{V}(t) \leq -\eta \left[ \sum_{i=1}^{n} e^2_i(t) \right]^{\frac{1}{2}} \triangleq -\eta (2V^2)^{\frac{1}{2}},$$

(13)
where \( V_1 = \frac{1}{2} e^T(t)e(t) \). It is to see that \( V_1 \leq V \).

Furthermore, from (13), \( V \) is non-increasing. Therefore, there exists a upper bound \( V^* \), such that

\[
V_1 \leq V \leq V^*.
\]

Let \( \theta(t) = \frac{V_1}{V^*} \leq 1 \), then

\[
\theta(t)\dot{V} \leq \theta(t)V^* = V_1.
\]

Finally, we have

\[
\dot{V} \leq -\eta\sqrt{2\theta(t)V^*} = -\eta\sqrt{2V^*},
\]

(14)

where \( \dot{\theta} = \min \theta(t) \).

According to Lemma 1, we have

\[
V(t) = 0, \ \forall t \geq T
\]

which further results in

\[
e(t) = 0, \ \forall t \geq T
\]

and the settling time

\[
T = t_0 + \frac{\sqrt{2V^*}(t_0)}{\eta\sqrt{\theta}},
\]

(15)

where \( V(t_0) = \begin{bmatrix} e^T(t_0)e(t_0) + (k(t_0)L)^2 \end{bmatrix} / 2 \). This means that, for any arbitrary initial value \( e(t_0) \), system (4) can be stabilized by the above controller within the time \( T \). This obviously implies the dynamical system (2) can be stabilized by controller (5). This completes the proof. \( \square \)

**Remark 2.** In Theorem 1, for the case \( e(t) = 0 \), we assume that \( k(t) = -L \), which can guarantee the positive definiteness of the Lyapunov function in (7). In fact, without this hypothesis, the feedback gain \( k(t) \) will also converge to other suitable constants when \( e(t) = 0 \). Thus, in practical engineering process, the feedback gain \( k(t) \) can only adapted according to the updated law in (6). Consequently, the design of controller in (5) is independent of the Lipchitz constant of the chaotic system.

**4. Simulation Results**

In this section, two examples are used to illustrate the feasibility and effectiveness of the above theoretical result.

**Example 1:** In the first example, we take Chua’s circuit as the first example, which is governed by the following three-dimensional differential equations [27]:

\[
\dot{x} = \begin{bmatrix} -p - pb & p & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{bmatrix} x + \begin{bmatrix} \psi(x_1) \\ 0 \\ 0 \end{bmatrix} = Ax + f(x),
\]

(16)
where \( x = (x_1, x_2, x_3) \) is the state vector, 

\[
\psi(x) = 0.5p(b-a)(|x_1| + |x_2| - |x_3|) 
\]

In all of the simulations, we always choose the system parameters of the Chua’s circuit as \( p = 10, q = 14.87, a = -1.27, b = -0.68 \) which causes the Chua’s circuit to exhibit a double-scroll chaotic attractor. It is easy to compute that 

\[
\lambda_{\max} \left( \frac{A + A^T}{2} \right) = 7.8572. 
\]

Take \( \epsilon = 2.95 \), it is easy to verify that

\[
|\phi(s) - \phi(s')| \leq |s_1 - s_2|. 
\]

Applying (16) we have

\[
(x - y)^T (f(x) - f(y)) = p(b-a)(x_1 - y_1)(\phi(x_1) - \phi(y_1)) 
\leq p(b-a)(x - y)^T (x - y). 
\]

Therefore the Assumption 1 is satisfied with \( l = p(b-a) = 5.9 \). Here we take \( L = 14 \), it is easy to see the condition \( L > l + \lambda_{\max} \left( \frac{A + A^T}{2} \right) \) is satisfied. Take \( \eta = 1 \), we simulate the evolution of the system (16) according to the controller defined in (5). According to Theorem 1, system (2) can be stabilized in a finite time. 

**Figure 1** shows the temporal evolution of the Chua’s circuit, where the initial values of the system are taken as \( (1, -3, 2) \), respectively. By computing (15) with matlab, we get \( T = 1.7864 \). It can be observed that the finite-time stable is achieved successfully by the proposed control scheme (5). The simulation matches the theoretical result perfectly. The temporal evolutions of variable strengths \( k_i (i = 1, 2, 3) \) are also simulated in this paper, the initial values are set as zero. From **Figure 2** we can see that feedback strengths \( k_i \) reach some certain constants when the system is stabilized.

**Example 2:** To show the generality of the presented method, the second example is the hyperchaotic Rössler system [28]:

\[
\dot{x} = \begin{pmatrix}
0 & -1 & 1 & 0 \\
1 & \alpha & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & -\delta & \zeta \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
x_3x_1 + \theta \\
0 \\
\end{pmatrix} \approx Ax + f(x),
\]

where \( x = (x_1, x_2, x_3, x_4) \) is the state vector.

The system has a hyperchaotic attractor with two positive Lyapunov exponents. Similarly, for the hyperchaotic Rössler system, we choose the initial conditions of the system (2) are random but the feedback strengths \( k_i \) are set as zero. The system parameters set as \( \alpha = 0.25, \delta = 0.5, \zeta = 0.05 \) and \( \theta = 3 \). **Figure 3** shows the temporal evolution of hyperchaotic Rössler system, and **Figure 4** shows the temporal evolutions of the corresponding feedback strengths \( k_i \). From **Figure 3** one can find that the system (17) can be achieved stabilization within a finite time under the controller of (5).

The above numerical examples show that the stabilization of chaotic or hyperchaotic system can be quickly achieved by the present controller in the form of (5). In addition, we also simulated the time-varying feedback gains \( k_i \), we can find that the feedback gains are automatically converge to suitable constants. Moreover, by comparing the converged feedback strengths and the
Figure 1. Trajectories of the Chua’s circuit system with adaptive controller (5).

Figure 2. Feedback strength \( k_i \ (i = 1, 2, 3) \) of adaptive controller (5) for dynamical system (2).

Figure 3. Trajectories of the hyperchaotic Rössler system with adaptive controller (5).
corresponding feedback signals, we find that the coupling is indeed small in the above two examples.

5. Conclusion

In this paper, we have investigated the stabilization of chaotic system based on the finite-time stability theory of differential equations, and proposed a simple, systematic and rigorous adaptive feedback control method to stabilize finite-dimensional chaotic systems within a finite time. In comparison with previous methods, the proposed scheme is simple to implement in practice. Numerical simulations are provided to illustrate the effectiveness of the method. The present study does not consider the effect of time delays, however, research is being pursued in this direction.

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