A Maximum Principle Result for a General Fourth Order Semilinear Elliptic Equation

A. Mareno

Penn State University, Middletown, USA
Email: aum24@psu.edu

Received 2 June 2016; accepted 27 August 2016; published 30 August 2016

Abstract

We obtain maximum principles for solutions of some general fourth order elliptic equations by modifying an auxiliary function introduced by L.E. Payne. We give a brief application of these maximum principles by deducing apriori bounds on a certain quantity of interest.

Keywords

Nonlinear, Fourth Order, Partial Differential Equation, Semilinear

1. Introduction

In [1], Payne obtains maximum principle results for the semilinear fourth order elliptic equation

\[ \Delta^2 u = f(u) \]  

(1)

by proving that certain functionals defined on the solution of (1) are subharmonic. In this work, functionals containing the terms \( \nabla^2 u - u_j \Delta u_j \) are utilized and apriori bounds on the integral of the square of the second gradient and on the square of the gradient of the solution are deduced. Since then, many authors [2]-[11] and references therein have used this technique to obtain maximum principle results for other fourth order elliptic differential equations whose principal part is the biharmonic operator.

Other works deal with the more general fourth order elliptic operator \( L^2 u \), where \( Lu := a_{ij} u_{ij} \) and \( a_{ij} = a_{ji} \). In [12], Dunninger mentions that functionals containing the term \( (Lu)^2 \) can be used to obtain maximum principle results for such linear equations as

\[ L^2 u + aLu + bu = 0. \]

A similar approach is taken in [13] for a class of nonlinear fourth order equations.

In this paper, we modify the results in [1] and a matrix result from [14] to deduce maximum principles defined on the solutions to semilinear fourth order elliptic equations of the form:

\[ L^2 u = f(u). \]  

(2)

Then we briefly indicate how these maximum principles can be used to obtain apriori bounds on a certain quantity of interest.

2. Results

Throughout this paper, the summation convention on repeated indices is used; commas denote partial differentiation. Let \( a_j(x) \) be a symmetric matrix. Moreover let \( Lu := a_j u_j \), be a uniformly elliptic operator, i.e., the symmetric matrix \( a_j(x) \) is positive definite and satisfies the uniform ellipticity condition:

\[ a_j(x) v_j \geq \| v \|^2, \quad x \in \Omega, \quad v \in \mathbb{R}^n, \]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) and \( n \geq 2 \).

Let \( u \) be a \( C^2 \) solution to the equation

\[ L^2 u = f(u) \quad \text{in} \quad \Omega. \]  

(3)

where \( f \) is say, a \( C^1 \) function. Now we define the functional

\[ P = c_1 (Lu)^2 - (a_m Lu)_m u_m + c_2 \| \nabla u \|^2 + 2(1-c_1) \int_\Omega f(s) \, ds + \beta(x). \]

We show that \( L(P) \) is subharmonic and note that the constants \( c_1 \) and \( c_2 \) and any constraints on \( f \) are yet to be determined.

By a straight-forward calculation, we have

\[ P_j = 2c_1 LuLu_j - (a_m Lu)_m u_m - (a_m Lu)_m u_m + 2c_2 u_m u_m + 2(1-c_1) f(u)u_j + \beta. \]

Now we write

\[ L(P) = a_j P_j \]

\[ = 2c_1 a_j LuLu_j + 2c_2 a_j Lu_j u_j - a_j (a_m Lu)_m u_m + 2c_1 a_j u_m u_m + 2c_2 u_m u_m + 2(1-c_1) f(u)u_j + \beta. \]

(4)

By expanding out the derivative terms in parentheses, we see that \( L(P) \) is

\[ = 2c_1 f(u)u_j + 2c_2 a_j Lu_j u_j - a_j u_m \left[ a_m + \frac{1}{2} a_m Lu_j + a_m Lu_j + a_m Lu_j + a_m Lu_j + a_m Lu_j \right] + 2c_2 a_j u_m u_m + 2(1-c_1) f(u)u_j + L(\beta). \]

(5)

The terms in lines 2 and 3 above containing two or more derivatives of \( Lu \) can be rewritten using (3) in the form \( A^{ij} f(u) = Lu_j \), where \( A^{ij} \) denotes the matrix which is the inverse of the positive definite matrix \( a_{ij} \). Furthermore, we use the identity \( a_{i}u_{m} = Lu_{m} - a_{m}u_{m} \) to rewrite the last two terms in line 4. Hence, we obtain

\[ L(P) = 2f(u)u_j + 2c_2 a_j Lu_j - a_j u_m \left[ A^{ij} f(u) + A^{ij} f(u)u_j \right] - 2a_j a_m u_{m} A^{ij} f(u) \]

(6)
Using the identity above for \( a_{ij} u_{mij} \) and the additional identity, \( A_{ij}^{m} u_{ij} = -a_{ij} A_{ij}^{m} \), which can be obtained by computing \( \left( A_{ij}^{m} \right)_{ij} \), for the terms at the ends of lines 6 and 3 respectively, we obtain

\[
L(P) = L(\beta) + (1 - 2c_i) a_{ij} f'(u) u_{ij} + 2a_{mn, ru} f(u) + 2c_i u_{ij} L u_{ij}
\]

\[
+ 2c_i a_{ij} u_{ijk} u_{jk} - 2a_{mn, ru} a_{ij} u_{ij} - a_{mn, ru} L ru_{ij} + a_{mn, ru} L u_{ij} + a_{mn, ru} u_{ij} L u_{ij} - 2a_{ij} a_{mn, ru} u_{ij}
\]

\[
- 2a_{ij} a_{mn, ru} L u_{ij} + 2a_{ij} a_{mn, ru} u_{ij} L u_{ij} + (2c_i - 1) a_{ij} L u_{ij} L u_{ij}
\]

\[
- a_{ij} a_{mn, ru} u_{ij} L u_{ij} - 2a_{ij} a_{mn, ru} u_{ij} L u_{ij}.
\]

(7)

To show that \( L(P) \) is nonnegative, we establish a series of inequalities based on the following one from [14]: Let \( \left( s_{ki} \right) \) be any \( n \times n \) matrix. From the inequality

\[
a_{ij} u_{ik} u_{ki} + s_{ki} u_{ki} \geq \frac{1}{4} A_{ij}^{m} s_{kj} s_{kj}.
\]

(9)

Repeated use of (9) on terms in lines 2, 3, 4, 5 in (7) yields the following:

\[
a_{ij} u_{ik} u_{jk} + a_{mn, ru} u_{ij} L u_{ij} \geq \frac{1}{4} A_{ij}^{m} \left( a_{mn, ru} u_{ij} a_{ij} a_{ij} L u_{ij} \right)
\]

(10)

\[
a_{ij} u_{ik} u_{jk} - 2a_{mn, ru} u_{ij} L u_{ij} \geq -A_{ij}^{m} \left( a_{mn, ru} u_{ij} a_{mn, ru} a_{ij} L u_{ij} \right)
\]

(11)

\[
a_{ij} u_{ik} u_{jk} - 2a_{mn, ru} u_{ij} L u_{ij} \geq -A_{ij}^{m} \left( L u_{ij} a_{mn, ru} u_{ij} a_{ij} \right)
\]

(12)

\[
a_{ij} u_{ik} u_{jk} + a_{mn, ru} u_{ij} L u_{ij} \geq -\frac{1}{4} A_{ij}^{m} \left( a_{mn, ru} u_{ij} L u_{ij} a_{ij} \right)
\]

(13)

\[
a_{ij} u_{ik} u_{jk} - a_{ij} a_{mn, ru} u_{ij} u_{ij} \geq -\frac{1}{4} A_{ij}^{m} \left( a_{ij} a_{mn, ru} a_{mn, ru} a_{ij} a_{ij} u_{ij} u_{ij} \right)
\]

(14)

\[
a_{ij} L u_{ij} - 2a_{mn, ru} u_{ij} L u_{ij} \geq -A_{ij}^{m} \left( a_{mn, ru} u_{ij} a_{mn, ru} u_{ij} \right)
\]

(15)

\[
a_{ij} L u_{ij} - a_{ij} a_{mn, ru} u_{ij} L u_{ij} \geq -\frac{1}{4} A_{ij}^{m} \left( a_{ij} a_{mn, ru} a_{mn, ru} u_{ij} \right)
\]

(16)

\[
a_{ij} L u_{ij} \geq -2c_i a_{mn, ru} u_{ij} u_{ij} \geq -c_i^2 A_{ij}^{m} \left( a_{mn, ru} u_{ij} a_{ij} \right)
\]

(17)

\[
a_{ij} u_{ij} \geq -2a_{ij} a_{mn, ru} L u_{ij} \geq -A_{ij}^{m} \left( a_{mn, ru} L u_{ij} a_{ij} a_{ij} \right)
\]

(18)

Furthermore, by completing the square, we obtain useful inequalities for the last two terms in line 1 and the third term in line 2 of (7):

\[
2c_i u_{ij} L u_{ij} \geq -c_i u_{ij} L u_{ij} - c_i L u_{ij} L u_{ij}
\]

(19)

\[
-2a_{mn, ru} f(u) \geq -a_{mn, ru} a_{mn, ru} u_{ij} - f^2
\]

(20)

\[
-a_{mn, ru} L L u_{ij} \geq -a_{mn, ru} L u_{ij} - L u_{ij} L u_{ij}
\]

(21)

We add (10)-(21) and label the resulting inequality, for part of \( L(P) \), as
\[
\hat{L}(P) = 7a_{ij}u_j + 2a_{ijj}u_{jj} + 2c_{ij}u_j - 2a_{ij,jj}f(u) + a_{ij} \partial_i u_j
\]

\[
\geq -A^{pq}(a_{ij}u_j + a_{ij,jj}u_{jj}) - \frac{1}{4}A^{pq}(a_{ij}u_j + a_{ij,jj}u_{jj})
\]

Now,

\[
L(P) = \hat{L}(P) + L(\beta) + (1 - 2c_1)a_{ij}f(u) + (2c_2 - 7)a_{ij}u_j u_{jj} + (2c_1 - 3)a_{ij}u_j u_{jj}
\]

Since \(a_{ij}(x)\) is positive definite, for a sufficiently large value of \(c_2\), where \(c_2\) depends on the coefficients \(a_{ij}\) and their derivatives, and for a sufficiently large value of \(c_1\), say \(1\), where \(c_1\) depends on the constants \(c_2, \gamma, a_{ij}\), and various derivatives of \(a_{ij}\), \(L(P)\) can be made nonnegative as desired. Thus we have the following result.

**Theorem 1.** Suppose that \(u \in C^3(\Omega) \cap C^1(\bar{\Omega})\) is a solution of (2) and \(f \in C^1(R)\). If \(f^2 \leq \gamma\), where \(\gamma > 0\), \(f'(\alpha) \leq a, \alpha < 0\), \(\beta(x)\) is a nonnegative function such that \(L(\beta) \geq \gamma\) then there exists positive constants \(c_2, c_1\) sufficiently large \((c_1 > 1)\) such that \(P\) cannot attain its maximum value in \(\Omega\) unless it is a constant.

We note that the function \(f(u) = -\left(u + u^3\right)\) satisfies the conditions stated in Theorem 1 for a solution that is bounded above.

### 3. Bounds

Here we give a brief application of Theorem 1.

Suppose that

\[
u = \frac{\hat{u}}{\partial\Omega} = 0 \text{ on } \partial\Omega
\]

By Theorem 1,

\[
P \leq \max_{\partial\Omega} \left(c_1(Lu)^2 + \beta(x)\right).
\]

Using integration by parts on the first two terms of \(P\) yields the identity

\[
\int_{\Omega} a^{mm}_j u_m a^{rr}_n u_n - (a^{mm}_j Lu_j) u_m dx = 2\int_{\Omega} (Lu)^2 dx.
\]
Upon integrating both sides of the previous inequality we deduce

\[
2\int_{\Omega} (Lu)^2 \, dx + 2(1-c_i)\int_{\Omega} \left( \int_0^1 f(s) \, ds \right) \, dx \\
\leq \left[ \max_{\partial \Omega} \left( c_i (Lu)^2 + \beta(x) \right) \right] \text{area}(\Omega).
\]

(22)

(23)

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