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Skorohod Integral at Vacuum State on Guichardet-Fock Spaces

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Abstract

In this paper, we define expectation of $f \in F$, *i.e.* $E(f) = f(\varnothing)$, according to Wiener-Ito-Segal isomorphic relation between Guichardet-Fock space *F* and Wienerspace *W*. Meanwhile, we derive a formula for the expectation of random Hermite polynomial in Skorohod integral on Guichardet-Fock spaces. In particular, we prove that the anticipative Girsanov identities under the condition $E(H_n(\delta(x), \| x \|^2)) = 0, n \geq 1$ on Guichardet-Fock spaces.

Keywords

Moment Identities, Girsanov Identities, Hermitpolynomial, Skorohod Integral, Guichardet-Fock Spaces

1. Introduction

The quantum stochastic calculus developed by Hudson and Parthasarathy [1] is essentially a noncommutative extension of classical Ito stochastic calculus. In thistheory, annihilation, creation, and number operator processes in boson Fock space play the role of "quantum noises" [2] [3], which are in continuous time. In 2002, Attal [4] discussed and extended quantum stochastic calculus by means of the Skorohod integral of anticipation processes and the related gradient operator on Guichardet-Fock spaces. Usually, Fock spaces as the models of the Particle Systems are widely used in quantumphysics. Meanwhile, vacuum states described by empty set on Guichardet-Fockspaces play very important role at quantum physics.

Recently Privault [5] [6] developed a Malliavin-type theory of stochastic calculus on Wiener spaces and showed its several interesting applications. In his article, Privault surveyed the moment identities for Skorohod integral and derived a formula for the expectation of random Hermit polynomials in Skorohod integral on Wiener spaces. It is well known that Guichardet-Fock space F and Wiener space W are Wiener-Ito-Segal isomorphic. Motivated by the above, we would like to study the expectation of random Hermit polynomials in Skorohod integral on Guichardet-Fock spaces. However, how to define the expectation on Guichardet-Fock spaces is the primary problem.

In this argument, we define expectation of $f \in F$ according to isomorphic relation, *i.e.* $E(f) = f(\varnothing)$.

Meanwhile, we prove a moment identity for the Skorohod integrals and derive a formula for the expectation

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of random Hermite polynomial in Skorohod integral on Guichardet-Fock spaces. Particularly, under the condition $E(H_n(\delta(x),\|x\|^2))=0, n\geq 1$, we prove the anticipative Girsanov identities on Guichardet-Fock spaces.

This paper is organized as follows. Section 2, we fix some necessarynotations and recall main notions and facts about Skorohod integral in Guichardet-Fock spaces. Section 3 and Section 4 state our main results.

2. Notations

In this section, we fix some necessary notations and recall mainnotions in Guichardet-Fock spaces. For detail formulation of Skorohod integrals, we refer reader to [4].

Let R_+ be the set of all nonnegative real numbers and Γ the finite power set of R_+, namely

$$\Gamma := \{\sigma \mid \sigma \subset R_+, \#\sigma < \infty\},$$

where $\#\sigma$ denotes the cardinality of σ as a set. Particularly, let $\emptyset \in \Gamma^{(0)}$ be an atom of measure 1. We denote by $L^2(\Gamma)$ the usual space of square integral real-valued functions on Γ.

Fixing a complex separable Hilbert space η, Guichardet-Fock space tensor product $\eta \otimes L^2(\Gamma)$, which we identify with the space of square-integrable functions $L^2(\Gamma;\eta)$, is denoted by *F*.

For a Hilbert space-valued map $x:\Gamma\times R_+ \to \eta$, let

$$\delta(x):\sigma \mapsto \sum_{s\in\sigma} x_s(\sigma \setminus s)$$

denotes the Skorohod integral operator. For a vector space-valued map $f:\Gamma \to V$, let ∇f and Df be the maps $\Gamma\times R_+ \to V$ given by

$$\nabla f(\omega,s) = f(\omega\bigcup s),\ Df(\omega,s) = \mathbf{1}_{\{\omega<s\}} f(\omega\bigcup s)$$

respectively denote the stochastic gradient operator of *f* and the adapted gradient operator of *f*. Moreover, we write $\mathrm{Dom}\nabla$ for the domain of the stochastic gradient as anunbounded Hilbert apace operator:

$$\mathrm{Dom}\nabla := \{ f \in F : \nabla f \in L^2(\Gamma\times R_+;\eta)\}.$$

Definition 2.1 The value of $f \in F$ at empty set is called the expectation of *f* on Guichardet-Fock space and is denoted by $E(f)$.$i.e. E(f) = f(\emptyset)$

Definition 2.2 For the map $x:\Gamma\times R_+ \to \eta$, the value of Skorohod integral $\delta(x)$ at empty set is called the expectation of $\delta(x)$ on Guichardet-Fock space and is denoted by $E(\delta(x))$ *i.e.* $E(\delta(x)) = \delta(x)(\emptyset)$.

Lemma 2.1 Let *x* be a map $\Gamma\times R_+ \to \eta$, if *x* is square integrable and the function $(\omega,s,t) \to \langle x_s(\omega\bigcup t), x_t(\omega\bigcup s)\rangle$ is integrable, then $x \in \mathrm{Dom}\delta$ and

$$\|\delta(x)\|^2 = \int \|x\|^2 ds + \int\int\int \langle x_s(\omega\bigcup t), x_t(\omega\bigcup s)\rangle d\omega dt ds, \tag{2.1}$$

we denote

$$\begin{aligned}\mathrm{trace}(Dx)^2 &= \langle \nabla x, \nabla^* x\rangle \\ &= \int_0^\infty \int_0^\infty \langle \nabla_t x_s, \nabla_s x_t\rangle dt ds \\ &= \int_0^\infty \int_0^\infty \langle x_s(\omega\cup t), x_t(\omega\cup s)\rangle dt ds.\end{aligned}$$

Lemma 2.2 Let $f \in F$ and let $x:\Gamma\times R_+ \to \eta$ be Skorohod integrable, if the map

$$(\omega,s) \mapsto \langle x_s(\omega), f(\omega\cup s)\rangle$$

is integrable, then

$$\langle \delta(x), f\rangle = \int\int \langle x_\omega, \nabla_s f(\omega)\rangle d\omega ds. \tag{2.2}$$

Lemma 2.3 Let $x:\Gamma\times R_+ \to \eta$ be measurable. For $a.a.t$, we have

$$D_t\delta(x) = \delta_0^t(D_t x) + P_t x_t, \tag{2.3}$$

where $P_t x_t = \mathbf{1}_{\Gamma_t} x_t$, $\Gamma_t := \{\omega \in \Gamma : \omega \subset [0,t[\}$.

Theorem 2.1 For any $n \geq 1$ and $x \in F$, we have

$$E(\delta(x)^{n+1}) = \sum_{k=1}^{n} \frac{n!}{(n-k)!} E[\delta(x)^{n-k} (\langle (\nabla x)^{k-1} x, x \rangle + \mathrm{trace}(\nabla x)^{k+1} + \sum_{i=2}^{k} \frac{1}{i} \langle (\nabla x)^{k-i} x, \nabla \mathrm{trace}(\nabla x)^{i} \rangle)], \tag{2.4}$$

where

$$\mathrm{trace}(\nabla x)^{k+1} = \int_0^\infty \cdots \int_0^\infty \langle \nabla^*_{t_{k-1}} x_{t_k}, \nabla_{t_{k-2}} x_{t_{k-1}} \cdots \nabla_{t_0} x_{t_1} \nabla_{t_k} x_{t_0} \rangle dt_0 \cdots dt_k.$$

Lemma 2.4 Let $n \geq 1$ and $x \in F$. Then for all $1 \leq k \leq n$ we have

$$E(\delta(x)^{n-k} \langle (\nabla x)^{k-1} x, \nabla \delta(x) \rangle)(\varnothing) - (n-k)(\delta(x)^{n-k-1} \langle (\nabla x)^{k} x, \nabla \delta(x) \rangle)$$
$$= E[\delta(x)^{n-k} (\langle (\nabla x)^{k-1} x, x \rangle + \mathrm{trace}(\nabla x)^{k+1} + \sum_{i=2}^{k} \frac{1}{i} \langle (\nabla x)^{k-i} x, \nabla \mathrm{trace}(\nabla x)^{i} \rangle)].$$

3. Random Hermit Polynomials

In Theorem 3.1 below, we compute the expectation of the random Hermit polynomial $E(H_n(\delta(x), \|x\|^2))$ with respect to the Skorohod integral $\delta(x), n \geq 1$. This result will be applied in Section 4 to anticipate Girsanov identities on Guichardet-Fock spaces.

Theorem 3.1 For any $n \geq 0$ and $x : \Gamma \times R_+ \to \eta$, we have

$$E(H_{n+1}(\delta(x), \|x\|^2)) = \sum_{l=0}^{n-1} \frac{n!}{l!} E[\delta(x)^l \sum_{0 \leq 2k \leq n-1-l} \frac{(-1)^k}{k!} \frac{\|x\|^{2k}}{2^k} ? \langle \nabla x \ \nabla \ \nabla x ((\ )^{n-2k-l-1} x \rangle$$

Especially, for x and

$$\langle \nabla x, \nabla((\nabla x)^k x) \rangle = 0, 0 \leq k \leq n-2, \tag{3.1}$$

then we have

$$E(H_{n+1}(\delta(x), \|x\|^2)) = 0, \quad n \geq 1. \tag{3.2}$$

Proof We divide two steps to prove the stability result.

Step 1. We first prove that for any $n \geq 1$,

$$E(H_{n+1}(\delta(x), \|x\|^2)) = \sum_{0 \leq 2k \leq n-1} (-1)^k \frac{n!}{k! 2^k (n-2k-1)!} \cdot E(\delta(x)^{n-2k-1} \langle x, x \rangle \langle x, \delta(\nabla x) \rangle) +$$
$$\sum_{1 \leq 2k \leq n} (-1)^k \frac{n!}{k! 2^k (n-2k)!} \cdot E(\delta(x)^{n-2k} \langle x, \nabla \langle x, x \rangle^k \rangle),$$

For $f \in F$ and $l, k \geq 1$, we have

$$\delta(x)^{l+1} = \frac{l+2k+1}{2k} f \delta(x)^{l+1} - \frac{l+1}{2k} f \delta(x)^{l+1}$$
$$= \frac{l+2k+1}{2k} f \delta(x)^{l+1} - \frac{l+1}{2k} \langle x, \nabla(\delta(x)^l f) \rangle$$
$$= \frac{l+2k+1}{2k} f \delta(x)^{l+1} - \frac{l(l+1)}{2k} f \delta(x)^{l-1} \langle x, \nabla \delta(x) \rangle - \frac{l+1}{2k} \delta(x)^l \langle x, \nabla f \rangle$$
$$= \frac{l+2k+1}{2k} f \delta(x)^{l+1} - \frac{l(l+1)}{2k} f \delta(x)^{l-1} \langle x, x \rangle - \frac{l(l+1)}{2k} f \delta(x)^{l-1} \langle x, \delta(\nabla x) \rangle - \frac{l+1}{2k} \delta(x)^l \langle x, \nabla f \rangle,$$

replace 1 above with $n-2k$, we have

$$\delta(x)^{n-2k+1} + \frac{(n-2k)(n-2k+1)}{2k} f \delta(x)^{n-2k-1} \langle x, x \rangle$$
$$= \frac{n+1}{2k} f \delta(x)^{n-2k+1} - \frac{(n-2k)(n-2k+1)}{2k} f \delta(x)^{n-2k-1} \langle x, \delta(\nabla x) \rangle - \frac{n-2k+1}{2k} \delta(x)^{n-2k} \langle x, \nabla f \rangle,$$

Hence, taking $f = \langle x, x \rangle^k$, we get

$$
\begin{aligned}
&E(\delta(x)^{n+1}) = E(\langle x, \nabla\delta(x)^n\rangle) = E(n\delta(x)^{n-1}\langle x, \nabla\delta(x)\rangle) \\
&= E(n\delta(x)^{n-1}\langle x, x\rangle) + E(n\delta(x)^{n-1}\langle x, \delta(\nabla x)\rangle) \\
&= E(n\delta(x)^{n-1}\langle x, \delta(\nabla x)\rangle) - \sum_{1\le 2k\le n+1}(-1)^k \\
&\times \frac{n!}{(k-1)!2^{k-1}(n+1-2k)!}E(\delta(x)^{n-2k+1}\langle x, x\rangle^k) \\
&+ \frac{(n-2k+1)(n-2k)}{2k}E(\delta(x)^{n-2k-1}\langle x, x\rangle^{k+1}) \\
&= E(n\delta(x)^{n-1}\langle x, \delta(\nabla x)\rangle) - \sum_{1\le 2k\le n+1}(-1)^k \\
&\times \frac{n!}{(k-1)!2^{k-1}(n+1-2k)!}E(\frac{n+1}{2k}\delta(x)^{n-2k+1}\langle x, x\rangle^k) \\
&- \frac{(n-2k)(n-2k+1)}{2k}E(\delta(x)^{n-2k-1}\langle x, x\rangle^k\langle x, \delta(\nabla x)\rangle) \\
&- \frac{n-2k+1}{2k}E(\delta(x)^{n-2k}\langle x, \nabla\langle x, x\rangle^k\rangle) \\
&= -\sum_{1\le 2k\le n+1}(-1)^k \frac{(n+1)!}{k!2^k(n+1-2k)!}E(\delta(x)^{n-2k+1}\langle x, x\rangle^k) \\
&+ \sum_{0\le 2k\le n-1}(-1)^k \frac{n!}{k!2^k(n-1-2k)!}E(\delta(x)^{n-2k-1}\langle x, x\rangle^k\langle x, \delta(\nabla x)\rangle) \\
&+ \sum_{1\le 2k\le n}(-1)^k \frac{n!}{k!2^k(n-2k)!}E(\delta(x)^{n-2k}\langle x, \nabla\langle x, x\rangle^k\rangle).
\end{aligned}
$$

Step 2. For $f \in F$, and $0 \le i \le l$, we have

$$
\begin{aligned}
&E(f\delta(x)^l\langle(\nabla x)^i x, \delta(\nabla x)\rangle) - lE(f\delta(x)^{l-1}\langle(\nabla x)^{i+1} x, \delta(\nabla x)\rangle) \\
&= E(\langle\nabla x, \nabla(f\delta(x)^l(\nabla x)^i x)\rangle) - lE(f\delta(x)^{l-1}\langle(\nabla x)^{i+1} x, \delta(\nabla x)\rangle) \\
&= lE(f\delta(x)^{l-1}\langle\nabla x, (\nabla x)^i x\otimes\nabla\delta(x)\rangle) - lE(f\delta(x)^{l-1}\langle(\nabla x)^{i+1} x, \delta(\nabla x)\rangle) \\
&\quad + E(\delta(x)^l\langle\nabla x, \nabla(f(\nabla x)^i x)\rangle) \\
&= lE(f\delta(x)^{l-1}\langle\nabla x, (\nabla x)^i x\otimes x\rangle) + lE(f\delta(x)^{l-1}\langle\nabla x, (\nabla x)^i x\otimes\delta(\nabla x)\rangle) \\
&- lE(f\delta(x)^{l-1}\langle(\nabla x)^{i+1} x, \delta(\nabla x)\rangle) + E(\delta(x)^l\langle\nabla x, \nabla(f(\nabla x)^i x)\rangle) \\
&= lE(f\delta(x)^{l-1}\langle(\nabla x)^{i+1} x, x\rangle) + E(\delta(x)^l\langle(\nabla x)^{i+1} x, \nabla f\rangle) + E(f\delta(x)^l\langle\nabla x, \nabla((\nabla x)^i x)\rangle).
\end{aligned}
$$

Hence, replacing 1 above with $l-i$, we get

$$
\begin{aligned}
&E(f\delta(x)^l\langle x, \delta(\nabla x)\rangle) \\
&= l!E(f\langle(\nabla x)^l x, \delta(\nabla x)\rangle) + lE(f\delta(x)^{l-1}\langle x, \delta(\nabla x)\rangle) \\
&= l!E(f\langle(\nabla x)^l x, \delta(\nabla x)\rangle) + \sum_{i=0}^{l-1}\frac{l!}{(l-i)!}E(f\delta(x)^{l-i}\langle(\nabla x)^i x, \delta(\nabla x)\rangle) \\
&-(l-i)E(f\delta(x)^{l-i-1}\langle(\nabla x)^{i+1} x, \delta(\nabla x)\rangle) \\
&= l!E(\langle(\nabla x)^{l+1} x, \nabla f\rangle) + \sum_{i=0}^{l-1}\frac{l!}{(l-i-1)!}?E\ \ f\delta((x)^{n-2k-1}\langle(\nabla x)^{i+1} x, x\rangle) \\
&+ \sum_{i=1}^{l}\frac{l!}{l-i+1}E(\delta(x)^{l-i+1}\langle(\nabla x)^i x, \nabla f\rangle) + \sum_{i=0}^{l}\frac{l!}{(l-i)!}E(f\delta(x)^{l-i}\langle\nabla x, \nabla((\nabla x)^i x)\rangle),
\end{aligned}
$$

thus letting $f = \langle x, x\rangle^k$ *and* $l = n-2k-1$ above, and use (2.3) in step 1, we get

$$
\begin{aligned}
&E(H_{n+1}(\delta(x),\|x\|^2))\\
&=\sum_{0\le 2k\le n-1}(-1)^k\frac{n!}{k!2^k(n-2k-1)!}E(\delta(x)^{n-2k-1}\langle x,x\rangle^k\langle x,\delta(\nabla x)\rangle)\\
&+\sum_{1\le 2k\le n}(-1)^k\frac{n!}{k!2^k(n-2k)!}E(\delta(x)^{n-2k}\langle x,\nabla\langle x,x\rangle^k\rangle)\\
&=\sum_{0\le 2k\le n}(-1)^k\frac{n!}{k!2^k}E(\langle(\nabla x)^{n-2k}x,\nabla\langle x,x\rangle^k\rangle)\\
&+\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!2^k}\sum_{i=0}^{n-2k-2}\frac{n!}{(n-2(k+1)-i)!}\cdot E(\langle x,x\rangle^k\delta(x)^{n-2(k+1)-i}\langle(\nabla x)^{i+1}x,x\rangle)\\
&+\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!2^k}\sum_{i=1}^{n-2k-1}\frac{n!}{(n-2k-i)!}\cdot E(\delta(x)^{n-2k-i}\langle(\nabla x)^i x,\nabla\langle x,x\rangle^k\rangle)\\
&+\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!2^k}\sum_{i=0}^{n-2k-1}\frac{n!}{(n-2k-1-i)!}\cdot E(\langle x,x\rangle^k\delta(x)^{n-2k-i-1}\langle\nabla x,\nabla((\nabla x)^i x)\rangle)\\
&+\sum_{0\le 2k\le n-1}(-1)^k\frac{n!}{k!2^k(n-2k)!}E(\delta(x)^{n-2k}\langle x,\nabla\langle x,x\rangle^k\rangle)\\
&=\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!2^k}\sum_{i=0}^{n-2k-1}\frac{n!}{(n-2k-1-i)!}\cdot E(\langle x,x\rangle^k\delta(x)^{n-2k-i-1}\langle\nabla x,\nabla((\nabla x)^i x)\rangle).
\end{aligned}
$$

4. Girsanov Identities

Corollary 4.1 Assume that $x:\Gamma\times R_+\to\eta$ with $E(e^{|\delta(x)|+\frac{1}{2}\|x\|^2})<\infty$ and that ∇x holds (3.1). Then, we have

$$E(\exp(\delta(x)-\frac{1}{2}\|x\|^2))=1.$$

Proof We have

$$|H_n(x,\sigma^2)|\le\sum_{0\le 2k\le n}\frac{(-1)^k}{k!2^k}\frac{n!}{(n-2k)!}|x|^{n-2k}(-\sigma^2)^k=H_n(|x|,-\sigma^2),$$

hence

$$\sum_{n=0}^{\infty}\frac{1}{n!}E|H_n(\delta(x),\|x\|^2)|\le\sum_{n=0}^{\infty}\frac{1}{n!}E(H_n(|\delta(x)|,-\|x\|^2))=E(e^{|\delta(x)|+\frac{1}{2}\|x\|^2})<\infty.$$

By Theorem 3.1 and Fubini theorem, we have

$$
\begin{aligned}
E(\exp(\delta(x)-\frac{1}{2}\|x\|^2))&=1+\sum_{n=0}^{\infty}\frac{1}{(n+1)!}E(H_n(|\delta(x)|,\|x\|^2))\\
&=1+\sum_{n=0}^{\infty}\frac{1}{(n+1)!}E(H_{n+1}(|\delta(x)|,\|x\|^2))=1.
\end{aligned}
$$

This shows that $\|x\|$ is deterministic and ∇x holds (3.1), we have

$$E(e^{\delta(x)})=E(e^{\frac{1}{2}\|x\|^2}),$$

i.e. $\delta(x)$ has a centered Gaussian distribution with variance $\|x\|^2$ on Guichardet-Fock spaces.

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