Moment Identities for Skorohod Integrals on Guichardet-Fock Spaces

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Abstract

In this paper, we define expectation of \( f \in F \), i.e. \( E(f) = f(\emptyset) \), according to Wiener-Ito-Segal isomorphic relation between Guichardet-Fock space \( F \) and Wiener space \( W \). Meanwhile, we prove a moment identity for the Skorohod integrals about vacuum state.

Keywords

Moment Identities, Skorohod Integral, Guichardet-Fock Spaces

1. Introduction

The quantum stochastic calculus [1] [2] developed by Hudson and Parthasarathy is essentially a noncommutative extension of classical Ito stochastic calculus. In this theory, annihilation, creation, and number operator processes in boson Fock space play the role of “quantum noises”, [3] which are in continuous time. In 2002, Attal [4] discussed and extended quantum stochastic calculus by means of the Skorohod integral of anticipation processes and the related gradient operator on Guichardet-Fock spaces. Usually, Fock spaces as the models of the Particle Systems are widely used in quantum physics. Meanwhile, vacuum states described by empty set on Guichardet-Fock spaces play a very important role in quantum physics.

Recently Privault [5] [6] developed a Malliavin-type theory of stochastic calculus on Wiener spaces and showed its several interesting applications. In his article, Privault surveyed the moment identities for Skorohod integral on Wiener spaces. It is well known that Guichardet-Fock space \( F \) and Wiener space \( W \) are Wiener-Ito-Segal isomorphic. Motivated by the above, we would like to study the moment identities for Skorohod integrals on Guichardet-Fock spaces.

This paper is organized as follows. Section 2, we fix some necessary notations and recall main notions and facts about Skorohod integrals in Guichardet-Fock spaces. Section 3 states our main results.

2. Notations

In this section, we fix some necessary notations and recall main notions in Guichardet-Fock spaces. For detail formulation of Skorohod integrals, we refer reader to [4].

Let \( \mathbb{R}_+ \) be the set of all nonnegative real numbers and \( \Gamma \) the finite power set of \( \mathbb{R}_+ \), namely...
where \(\#\) denotes the cardinality of \(\sigma\) as a set. Particularly, let \(\emptyset \in \Gamma^{(0)}\) be an atom of measure 1. We denote by \(L^2(\Gamma)\) the usual space of square-integrable functions on \(\Gamma\).

Fixing a complex separable Hilbert space \(\eta\), Guichardet-Fock space tensor product \(\eta \otimes L^2(\Gamma)\), which we identify with the space of square-integrable functions \(L^2(\Gamma; \eta)\), and is denoted by \(F\).

For a Hilbert space-valued map \(x : \Gamma \times R \rightarrow \eta\), let
\[
\delta(x) : \sigma \mapsto \sum_{s \in \sigma} x_s (\sigma \setminus s)
\]
denotes the Skorohod integral operator. For a vector space-valued map \(f : \Gamma \rightarrow V\), let \(\nabla f\) and \(Df\) be the maps \(\Gamma \times R \rightarrow V\) given by
\[
\nabla f(\omega, s) = f(\omega \bigcup s), \quad Df(\omega, s) = I_{(s \in \sigma)} f(\omega \cup s)
\]
respectively denote the stochastic gradient operator of \(f\) and the adapted gradient operator of \(f\). Moreover, we write \(\text{Dom} V\) for the domain of the stochastic gradient as an unbounded Hilbert space operator:
\[
\text{Dom} V := \{ f \in F : \nabla f \in L^2(\Gamma \times R; \eta) \}.
\]

**Definition 2.1** For the map \(x : \Gamma \times R \rightarrow \eta\), the value of Skorohod integral \(\delta(x)\) at empty set is called the expectation of \(\delta(x)\) on Guichardet-Fock space and is denoted by \(E(\delta(x))\).

**Lemma 2.1** Let \(x \in L^2(\eta \otimes \Gamma)\), if \(x\) is square integrable and the function \((\omega, s, t) \mapsto \langle x_t(\omega \cup s), x_s(\omega \cup s)\rangle\) is integrable, then \(x \in \text{Dom} \nabla\) and
\[
\| \delta(x) \|^2 = \int \int \| x \|^2 ds + \int \int \langle x_t(\omega \cup s), x_s(\omega \cup s)\rangle d\omega ds,
\]
where \(\| \cdot \|^2 = \langle \nabla x, \nabla x \rangle\).

**Lemma 2.2** Let \(f \in F\) and let \(x : \Gamma \times R \rightarrow \eta\) be Skorohod integrable, if the map \((\omega, s, t) \mapsto \langle x_t(\omega \cup s), f(\omega \cup s)\rangle\) is integrable, then
\[
\langle \delta(x), f \rangle = \int \int \langle x_t(\omega), \nabla f(\omega \cup s)\rangle d\omega ds.
\]

**Lemma 2.3** Let \(x : \Gamma \times R \rightarrow \eta\) be measurable. For \(a \in \Gamma\), we have
\[
D_a \delta(x) = \delta_a(D(x)) + P_a x_a
\]
where \(P_a x_a = 1_{\{a\}} x_a\) , \(\Gamma_a = \{ \omega \in \Gamma : \omega \subset [0, t]\} \).

**Proof** In view of the identity
\[
1_{(\sigma \subset \Gamma)} \delta(x)(\sigma \cup t) = \sum_{s \in \sigma} 1_{(\sigma \subset \Gamma)} 1_{[0,t]}(s) x_s((\sigma \cup s) \cup t) + 1_{(\sigma \subset \Gamma)} x_t(\sigma),
\]
we have
\[
D_a \delta(x)(\sigma) = \delta(1_{[0,t]}(a)D_t x)(\sigma) + P_a x_a(\sigma).
\]

### 3. Moment Identities for Skorohod Integrals

**Theorem 3.1** For any \(n \geq 1\) and \(x \in F\), we have
\[
E(\delta(x)^{n+1}) = \sum_{k=1}^{n+1} \frac{n!}{(n-k)!} E[\delta(x)^{n-k}(\nabla x)^{k-1} x x] + \text{trace}(\nabla x)^{k+1} + \sum_{k=1}^{n+1} \frac{1}{k} \langle x, \text{trace}(\nabla x) \rangle, \tag{3.1}
\]
where
\[ \text{trace}(\nabla x)^{k+1} = \int_0^\infty \cdots \int_0^\infty \langle \nabla x^1, x_1, \nabla x_2, x_2, \ldots \rangle \, dt_0 \cdots \, dt_k. \]

For \( n = 1 \) the above identity coincides with (2.1).

We will need the following lemma.

**Lemma 3.1** Let \( n \geq 1 \) and \( x \in F \). Then for all \( 1 \leq k \leq n \) we have
\[
E(\delta(x)^{n-k} \langle (\nabla x)^{k-1} x, \nabla, \delta(x) \rangle) - (n-k)(\delta(x)^{n-k} \langle (\nabla x)^{k-1} x, \nabla, \delta(x) \rangle)
= [E[\delta(x)^{n-k} \langle (\nabla x)^{k-1} x, + \text{trace}(\nabla x) \rangle] + \sum_{i=2}^{k} \langle (\nabla x)^{k-1} x, \nabla, \text{trace}(\nabla x) \rangle].
\]

**Proof** Using relation (2.2), (2.3), we obtain
\[
\delta(x)^{n-k} \langle (\nabla x)^{k-1} x, \nabla, \delta(x) \rangle = \delta(x)^{n-k} \langle (\nabla x)^{k-1} x, + (\delta(x)^{n-k} \langle (\nabla x)^{k-1} x, \nabla, \delta(x) \rangle)
= \delta(x)^{n-k} \langle (\nabla x)^{k-1} x, + (\delta(x)^{n-k} \langle (\nabla x)^{k-1} x, \nabla, \delta(x) \rangle)
= \delta(x)^{n-k} \langle (\nabla x)^{k-1} x, + (\delta(x)^{n-k} \langle (\nabla x)^{k-1} x, \nabla, \delta(x) \rangle)
= \delta(x)^{n-k} \langle (\nabla x)^{k-1} x, + (\delta(x)^{n-k} \langle (\nabla x)^{k-1} x, \nabla, \delta(x) \rangle)
= \delta(x)^{n-k} \langle (\nabla x)^{k-1} x, + (\delta(x)^{n-k} \langle (\nabla x)^{k-1} x, \nabla, \delta(x) \rangle)
\]

and
\[
\langle \nabla x, \nabla ((\nabla x)^{k-1} x) \rangle = \int_0^\infty \cdots \int_0^\infty \langle \nabla x^1, x_1, \nabla x_2, x_2, \ldots \rangle \, dt_0 \cdots \, dt_k
= \int_0^\infty \cdots \int_0^\infty \langle \nabla x^1, x_1, \nabla x_2, x_2, \ldots \rangle \, dt_0 \cdots \, dt_k
= \text{trace}(\nabla x)^{k+1} + \sum_{i=0}^{k-2} \int_0^\infty \cdots \int_0^\infty \langle \nabla x^1, x_1, \nabla x_2, x_2, \ldots \rangle \, dt_0 \cdots \, dt_k
= \text{trace}(\nabla x)^{k+1} + \sum_{i=0}^{k-2} \int_0^\infty \cdots \int_0^\infty \langle \nabla x^1, x_1, \nabla x_2, x_2, \ldots \rangle \, dt_0 \cdots \, dt_k
= \text{trace}(\nabla x)^{k+1} + \sum_{i=0}^{k-2} \int_0^\infty \cdots \int_0^\infty \langle \nabla x^1, x_1, \nabla x_2, x_2, \ldots \rangle \, dt_0 \cdots \, dt_k
= \text{trace}(\nabla x)^{k+1} + \sum_{i=0}^{k-2} \int_0^\infty \cdots \int_0^\infty \langle \nabla x^1, x_1, \nabla x_2, x_2, \ldots \rangle \, dt_0 \cdots \, dt_k
\]

Proof of Theorem 3.1, We decompose
\[
E(\delta(x)^{n-k}) = E(\langle x, \nabla, \delta(x) \rangle) = E(n(\delta(x)^{n-k} \langle x, \nabla, \delta(x) \rangle)
= \sum_{i=2}^{k} \langle (\nabla x)^{k-1} x, \nabla, \text{trace}(\nabla x) \rangle]
\]

then we apply lemma 3.1, which yields (3.1).

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**References**


