Existence of Traveling Waves in Lattice Dynamical Systems

Xiaojun Li, Yong Jiang, Ziming Du
School of Science, Hohai University, Nanjing, China
Email: lixjun05@hhu.edu.cn, lixiaoj@yahoo.com

Received 28 December 2015; accepted 12 July 2016; published 15 July 2016

Abstract
Existence of traveling wave solutions for some lattice differential equations is investigated. We prove that there exists $c_0 > 0$ such that for each $c \geq c_0$, the systems under consideration admit monotonic nondecreasing traveling waves.

Keywords
Traveling Wave, Lattice Dynamical Systems, Schauder’s Fixed Point Theorem

1. Introduction
Consider the following lattice differential equation
\[
\begin{align*}
\dot{u}_i &= \nu (u_{i+1} - 2u_i + u_{i-1}) - f(u_i, (Bu)_i) + \alpha v_i, \quad i \in \mathbb{Z}, \\
\dot{v}_i &= -\sigma v_i + \beta u_i, \quad i \in \mathbb{Z},
\end{align*}
\]
where $\nu, \sigma$ are positive constants, $\alpha\beta > 0$, $f$ is a $C^2$-function, and $(Bu)_i = u_{i+1} - u_i$.

Lattice dynamical systems occur in a wide variety of applications, and a lot of studies have been done, e.g., see [1]-[4]. A pair of solutions $\{u_i\}_{i=-\infty}^{\infty}$, $\{v_i\}_{i=-\infty}^{\infty}$ of (1.1) is called a traveling wave solution with wave speed $c > 0$ if there exist functions $U, V: \mathbb{R} \to \mathbb{R}$ such that $u_i = U(i+ct)$, $v_i = V(i+ct)$ with $(U(-\infty), V(-\infty)) = (U_-, V_-)$ and $(U(+\infty), V(+\infty)) = (U_+, V_+)$. Let $\xi = i + ct$, note that (1.1) has a pair of traveling wave solutions if and only if $U, V$ satisfy the functional differential equation
\[
\begin{align*}
\dot{cU}(\xi) &= \nu(U(\xi + 1) - 2U(\xi) + U(\xi - 1)) - f(U(\xi), BU(\xi)) + \alpha V(\xi), \\
\dot{cV}(\xi) &= -\sigma V(\xi) + \beta U(\xi).
\end{align*}
\]
Without loss of generality, we can impose (1.1) with asymptotic boundary conditions
\[
\lim_{\xi \to -\infty} U(\xi) = 0, \quad \lim_{\xi \to +\infty} U(\xi) = k_1, \quad \lim_{\xi \to -\infty} V(\xi) = 0, \quad \lim_{\xi \to +\infty} V(\xi) = k_2.
\]

By the property of equation, we can assume that $\alpha, \beta > 0$. In the following, we give some assumptions on
nonlinear function $f:

\begin{align*}
(A_1) & \quad -f(k_1,0) + \alpha k_2 = 0, \quad f(0,0) = 0, \quad -\sigma k + \beta k = 0 . \\
(A_2) & \quad \text{There exists a positive-value continuous function } Q: R \to R \text{ such that} \\
& \quad \max_{u \in (0,1]} \left| f_u \right| + \max_{u \in (0,1]} \left| f_{uu} \right| \leq Q(r), \quad Q(2k) < v .
\end{align*}

\begin{align*}
(A_3) & \quad v < \frac{\partial f}{\partial x_2}(0,0) < 0, \quad \frac{\partial f}{\partial x_1}(0,0) < 3v + \alpha \kappa - \frac{\beta}{\kappa}, \quad \kappa = \frac{k_1}{k_2} . \\
(A_4) & \quad \frac{\partial^2 f}{\partial x_i \partial x_j}(x_i, x_j) > 0 \quad \text{for any } (x_i, x_j) \in [0, k_1] \times [0, k_2] , \quad i, j = 1, 2 ,
\end{align*}

where $\omega = (e^{2\kappa} - 1)k_1$, $\Lambda_*$ is given in Lemma 2.1.

Select positive constants $\mu_1, \mu_2$ such that $\mu_1 > 2v + 2Q(2k_1), \mu_2 > \sigma$, and define operators

\begin{align*}
H_1, H_2 : C(\mathbb{R}^2, R) & \to C(\mathbb{R}^2, R) \\
H_i(U, V)(\xi) & = \mu_i U(\xi) + v(U(\xi + 1) - 2U(\xi) + U(\xi - 1)) - f(U(\xi), U(\xi + 1) - U(\xi) + \alpha V(\xi)) \\
& \quad - f(U(\xi), U(\xi) - U(\xi)) + \alpha V(\xi) .
\end{align*}

Then, (1.2) can be rewritten as

\begin{align*}
e^J(\xi) = -\mu U(\xi) + H_1(U, V)(\xi), \quad e^V(\xi) = -\mu V(\xi) + H_2(U, V)(\xi). 
\end{align*}

Define the operators

\begin{align*}
F_i : C(\mathbb{R}^2, R) & \to C(\mathbb{R}^2, R) \\
F_i(U, V)(\xi) & = \frac{1}{c} e^{-\frac{\alpha}{c} \xi} \int_{-\infty}^{\xi} e^{\frac{\alpha}{c} s} H_i(U, V)(s) ds, \quad i = 1, 2 .
\end{align*}

Note that

\begin{align*}
F_i & \text{ satisfy } \quad cF_i(U, V)(\xi) = -\mu_i F_i(U, V)(\xi) + H_i(U, V)(\xi), \quad i = 1, 2, \quad \text{and a fixed point of } F = (F_1, F_2) \text{ is a solution of (1.2).}
\end{align*}

Denote $\mathcal{L} = \text{ the Euclidean norm in } \mathbb{R}^2$. Define

\begin{align*}
B_\mu(R, R^2) & = \left\{ \Phi \in C(\mathbb{R}, \mathbb{R}^2) : \sup_{\xi \in \mathbb{R}} \| \Phi \| < \infty, \| \Phi \| = \sup_{\xi \in \mathbb{R}} \| \Phi(t) e^{-\mu t} \| , \mu < c \right\},
\end{align*}

where $0 < \mu < \min \left\{ \frac{\mu_1}{c}, \mu_2 \right\}$. Note that $B_\mu(R, R^2)$ is a Banach space.

**Definition 1.1.** If the continuous functions $(\overline{U}(\xi), \overline{V}(\xi)) : R \to R^2$ are differentiable almost everywhere and satisfy

\begin{align*}
\left\{ \begin{array}{l}
\frac{c \overline{U}(\xi)}{\overline{V}(\xi)} \geq v(\overline{U}(\xi + 1) - 2\overline{U}(\xi) + \overline{U}(\xi - 1)) - f(\overline{U}(\xi), B \overline{U}(\xi)) + \alpha \overline{V}(\xi), \\
\frac{c \overline{V}(\xi)}{\overline{V}(\xi)} \geq -\sigma \overline{V}(\xi) + \beta \overline{U}(\xi),
\end{array} \right.
\end{align*}

Then, $(\overline{U}(\xi), \overline{V}(\xi))$ is called an upper solution of (1.2).

Similarity, we can define a lower solution of (1.2). The main result of this paper is

**Theorem 1.1.** Assume that $(A_1) \sim (A_4)$ hold. Then there exists $c > 0$ such that for every $c \geq c_0$, (1.2) admits a traveling wave solution $(U(\xi), V(\xi))$ connecting $(0,0)$ and $(k_1, k_2)$. Moreover, each component of traveling wave solution is monotonically nondecreasing in $\xi \in R$, and for each $c \geq c_0$, $U(\xi), V(\xi)$ also satisfy

\begin{align*}
\lim_{\xi \to -\infty} U(\xi) e^{-(\lambda - c)\xi} = 1, \quad \lim_{\xi \to -\infty} V(\xi) e^{-(\lambda - c)\xi} \leq \kappa , \quad \lambda = \Lambda_*(c) \text{ is the smallest solution of the equation}
\end{align*}

\begin{align*}
c\lambda - \left[ (v - \frac{\partial f}{\partial x_2}(0,0)) e^\lambda + ve^{\lambda - \lambda} \right] + 2v + \frac{\partial f}{\partial x_1}(0,0) - \frac{\partial f}{\partial x_2}(0,0) - \alpha \kappa = 0 .
\end{align*}
2. Upper-Lower Solutions of (1.2)

Set \( \Delta(c, \lambda) = c \lambda - [(v - \frac{\partial f}{\partial x_1}(0, 0)) e^{\lambda} + v e^{-\lambda}] + 2v + \frac{\partial f}{\partial x_1}(0, 0) - \frac{\partial f}{\partial x_2}(0, 0) - \alpha \kappa \).

**Lemma 2.1.** Assume that \((A_1)\) holds. Then there exists a unique \( c_0 > 0 \) such that \( (i) \) if \( c > c_0 \), then there exist two positive numbers \( \Lambda_1(c) \) and \( \Lambda_2(c) \) with \( \Lambda_1(c) < \Lambda_2(c) \) such that \( \Delta(c, \Lambda_1(c)) = \Delta(c, \Lambda_2(c)) = 0 \), \( \Delta(c_0) > 0 \) in \( (\Lambda_1(c), \Lambda_2(c)) \), and \( \Delta(c) < 0 \) in \( R \setminus [\Lambda_1(c), \Lambda_2(c)] \); \( (ii) \) if \( c < c_0 \), then \( \Delta(c, \lambda) < 0 \) for all \( \lambda \geq 0 \); \( (iii) \) if \( c = c_0 \), then \( \Lambda_1(c) = \Lambda_2(c) = \Lambda_0 \), and \( \Delta(c, \Lambda_0) = 0 \).

**Proof.** Using assumption \((A_1)\), we can get the result directly. \( \square \)

**Lemma 2.2.** Assume that \((A_1), (A_2)\) and \((A_3)\) hold. Let \( c_0 \), \( \Lambda_1(c) \), and \( \Lambda_2(c) \) be defined as in Lemma 2.1, and \( c > c_0 \) be any number. Then for every \( \theta \in (1, \min \{\Lambda_2(c), \Lambda_1(c)\} \) and \( 0 < \kappa < \alpha \), there exists \( Q(c, \theta) \geq 1 \) such that for any \( q \geq Q(c, \theta) \),

\[
\phi^+(\xi) := \min \left\{ k_1 e^{\Lambda_1(c) \xi} + q e^{\theta \Lambda_1(c) \xi} \right\}, \psi^+(\xi) := \min \left\{ k_2 \kappa \left( e^{\Lambda_1(c) \xi} + q e^{\theta \Lambda_1(c) \xi} \right) \right\}, \xi \in R,
\]

and

\[
\phi^-(\xi) := \max \left\{ 0, e^{\Lambda_1(c) \xi} - q e^{\theta \Lambda_1(c) \xi} \right\}, \psi^-(\xi) := \max \left\{ 0, h \left( e^{\Lambda_1(c) \xi} - q e^{\theta \Lambda_1(c) \xi} \right) \right\}, \xi \in R,
\]

are a pair of upper solutions and a pair of lower solutions of (1.2), respectively.

**Proof.** Let

\[
N_1^1 \left[ \phi, \psi \right](\xi) := c \phi(\xi) - v(\phi(\xi + 1) + 2 \phi(\xi) + \phi(\xi - 1)) + f(\phi(\xi), \phi(\xi + 1) - \phi(\xi)) - \alpha \psi(\xi),
\]

\[
N_2^1 \left[ \phi, \psi \right](\xi) := c \psi(\xi) + \sigma \psi(\xi) - \beta \psi(\xi).
\]

Since \( \kappa = \frac{k_2}{k_1} \), there exists \( \xi_1 \) such that \( \phi^+(\xi_1) = k_1 \), \( \psi^+(\xi_1) = k_2 \). If \( \xi \geq \xi_1 \), then \( \phi^+(\xi) = k_1 \), \( \psi^+(\xi) = k_2 \). By \((A_1)\), we get that

\[
N_1^1 \left[ \phi^+, \psi^+ \right](\xi) \geq f(k_1, 0) - \alpha k_2 = 0, \ N_2^1 \left[ \phi^+, \psi^+ \right](\xi) \geq \sigma k_2 - \beta k_1 = 0.
\]

If \( \xi < \xi_1 \), then \( \phi^+(\xi) = e^{\Lambda_1(c) \xi} + q e^{\theta \Lambda_1(c) \xi}, \psi^+(\xi) = \kappa \left( e^{\Lambda_1(c) \xi} + q e^{\theta \Lambda_1(c) \xi} \right) \). By \((A_1), (A_2) - (A_3)\), and using Lemma 2.1, we get that

\[
N_1^1 \left[ \phi^+, \psi^+ \right](\xi) \geq c \Lambda_1(c) e^{\Lambda_1(c) \xi} + q \Theta \Lambda_1(c) e^{\theta \Lambda_1(c) \xi} - v \left[ e^{\Lambda_1(c) \xi} + q e^{\theta \Lambda_1(c) \xi} \right] - 2 e^{\Lambda_1(c) \xi} - q e^{\theta \Lambda_1(c) \xi} + e^{\theta \Lambda_1(c) \xi} + q e^{\theta \Lambda_1(c) \xi} + e^{\theta \Lambda_1(c) \xi} - e^{\theta \Lambda_1(c) \xi} - q e^{\theta \Lambda_1(c) \xi} - q e^{\theta \Lambda_1(c) \xi} - q e^{\theta \Lambda_1(c) \xi} - q e^{\theta \Lambda_1(c) \xi} - q e^{\theta \Lambda_1(c) \xi} - q e^{\theta \Lambda_1(c) \xi} - q e^{\theta \Lambda_1(c) \xi} \geq 0.
\]

Lemma 2.1 and \((A_1)\) yields

\[
c \kappa \Lambda_1(c) \theta + 2 \beta > c \kappa \Lambda_1(c) + \kappa \sigma - \beta > 0.
\]

Thus,

\[
N_2^1 \left[ \phi^+, \psi^+ \right](\xi) = c \kappa \left( \Lambda_1(c) e^{\Lambda_1(c) \xi} + q \Theta \Lambda_1(c) e^{\theta \Lambda_1(c) \xi} \right) + \kappa \sigma \left( e^{\Lambda_1(c) \xi} + q e^{\theta \Lambda_1(c) \xi} \right) - \beta \left( e^{\Lambda_1(c) \xi} + q e^{\theta \Lambda_1(c) \xi} \right) = (c \kappa \Lambda_1(c) + \kappa \sigma - \beta) e^{\Lambda_1(c) \xi} + q \left( c \kappa \Lambda_1(c) \theta + \kappa \sigma - \beta \right) e^{\theta \Lambda_1(c) \xi} > 0.
\]

Therefore, \((\phi^+, \psi^+)\) is an upper solution of (1.2). Similarly, we can prove that \((\phi^-, \psi^-)\) is a lower
solution. □

3. Existence of Traveling Wave

Let $K = (k_1, k_2)$, $C_{[0,k]}(R, R^2) = \{(U, V) \in C(R, R^2) : 0 \leq U(s) \leq k_1, 0 \leq V(s) \leq k_2, s \in R\}$. We have the following result.

**Lemma 3.1** Assume that $(A_1)$ and $(A_2)$ hold. Then

(i) $F_1(U, V, \phi) \geq F_1(U,\phi) \geq F_2(U, V, \phi)$ for $\phi \in R$ if $(U, V)(\xi) \in C_{[0,k]}(R, R^2)$ satisfy $U_1(\xi) \geq U_2(\xi)$, $V_1(\xi) \geq V_2(\xi)$ for $\xi \in R$;

(ii) $F_1(U, V, \phi) \geq F_2(U, V, \phi)$ are nondecreasing in $\xi \in R$ if $(U, V)(\xi) \in C_{[0,k]}(R, R^2)$ is nondecreasing in $\xi \in R$.

**Proof.** If $(U_1, V_1)(\xi), (U_2, V_2)(\xi) \in C_{[0,k]}(R, R^2)$ such that $U_1(\xi) \geq U_2(\xi)$ and $V_1(\xi) \geq V_2(\xi)$ for $\xi \in R$, then by $(A_1)$ we have

$$
\begin{align*}
\left| f(U_1(\xi), BU_1(\xi)) - f(U_2(\xi), BU_2(\xi)) \right| \\
\leq \left| f(U_2(\xi), BU_2(\xi)) \right| + \left| f(U_1(\xi), BU_1(\xi)) \right| \\
\leq 2M_1(\xi) - U_1(\xi) + U_1(\xi) + U_2(\xi) - U_2(\xi),
\end{align*}
$$

where $M_1 = Q(2k_1)$. Note that

$$
\begin{align*}
H_1(U_1, V_1)(\xi) - H_2(U_2, V_2)(\xi) \\
= (\mu_1 - 2v_1)(U_1(\xi) - U_2(\xi)) + \theta(U_2(\xi) - U_2(\xi) + U_1(\xi) - U_1(\xi - 1)) \\
- \left[ f(U_1(\xi), BU_1(\xi)) - f(U_2(\xi), BU_2(\xi)) \right] + \alpha(V_1(\xi) - V_2(\xi)).
\end{align*}
$$

Thus, from (3.1)-(3.2), we have

$$
\begin{align*}
H_1(U_1, V_1)(\xi) - H_2(U_2, V_2)(\xi) \\
\geq (\mu_1 - 2v - 2M_1)(U_1(\xi) - U_2(\xi)) + (v - M_1)(U_1(\xi) - U_2(\xi + 1)) \\
+ v(U_1(\xi - 1) - U_2(\xi - 1)) + \alpha(V_1(\xi) - V_2(\xi)) \geq 0,
\end{align*}
$$

which implies that $H_1(U_1, V_1)(\xi) \geq H_2(U_2, V_2)(\xi)$. A similar argument can be done for $H_2(U, V)(\xi)$. Thus, we can get the desired results. □

**Lemma 3.2.** Assume that $(A_1)$ and $(A_2)$ hold. Then $F = (F_1, F_2) : B_\mu(R, R^2) \to B_\mu(R, R^2)$ is continuous with respect to the norm $\| \cdot \|_{\mu}$ with $0 < \mu < \min\left\{ \frac{H_1}{c}, \mu_2 \right\}$.

**Proof.** We first prove that $H_1, H_2 : B_\mu(R, R^2) \to B_\mu(R, R^2)$ are continuous. Denote $\Phi_1 = (U_1, V_1)$, $\Phi_2 = (U_2, V_2)$. For any $\varepsilon > 0$, choose $0 < \delta < \frac{\varepsilon}{N}$, where $N = \max\{\mu_1 - 2v + 2M_1 + 2v + M_1\}$. If $\Phi_1$ and $\Phi_2$ satisfy

$$
\| \Phi_1 - \Phi_2 \|_\mu = \sup_{\xi \in R} \left| \phi_1(\xi) - \phi_2(\xi) \right| e^{-\alpha|\xi|} \leq \delta,
$$

then by (3.1),

$$
\begin{align*}
\left| H_1(U_1, V_1)(\xi) - H_2(U_2, V_2)(\xi) \right| e^{-\alpha|\xi|} \\
= \left| \mu_1 - 2v \right| \left| U_1(\xi) - U_2(\xi) \right| + v \left| U_1(\xi + 1) - U_2(\xi + 1) \right| \\
+ \left| U_1(\xi - 1) - U_2(\xi - 1) \right| + \alpha \left| V_1(\xi) - V_2(\xi) \right| e^{-\alpha|\xi|} \\
\leq \left| \mu_1 - 2v + 2M_1 + (2v + M_1) \right| e^{-\alpha|\xi|} \| \phi_1(\xi) - \phi_2(\xi) \|_\mu < \varepsilon.
\end{align*}
$$
Similarly, \( H_2(U_i, V_i)(\xi) \) is continuous. By definition of \( F_i \), we have

\[
\left| F_i(U_1, V_1)(\xi) - F_i(U_2, V_2)(\xi) \right| = \frac{1}{c} e^{\frac{\mu_1}{c}} \left| \int_{-\infty}^{\xi} \left( H_1(U_1, V_1) - H_1(U_2, V_2) \right)(s) \, ds \right|
\]

\[
\leq \frac{1}{c} \left\| H_1(U_1, V_1)(\xi) - H_1(U_2, V_2)(\xi) \right\|_\mu e^{\frac{\mu_1}{c}} \int_{-\infty}^{\xi} e^{\frac{\mu_1}{c} \mu} \, ds.
\]  

(3.4)

If \( \xi < 0 \), it follows that

\[
\left| F_i(U_1, V_1)(\xi) - F_i(U_2, V_2)(\xi) \right| e^{\mu_1 \mu} \leq \frac{1}{\mu_1 - c \mu} \left\| H_1(U_1, V_1)(\xi) - H_1(U_2, V_2)(\xi) \right\|_\mu.
\]

(3.5)

If \( \xi \geq 0 \), it follows that

\[
\left| F_i(U_1, V_1)(\xi) - F_i(U_2, V_2)(\xi) \right| e^{\mu_1 \mu} \leq \frac{1}{\mu_1 - c \mu} \left\| H_1(U_1, V_1)(\xi) - H_1(U_2, V_2)(\xi) \right\|_\mu.
\]

(3.6)

Combining (3.5) and (3.6), we get that \( F_i \) is continuous with respect to the norm \( \| \|_\mu \). A similar argument can be done for \( F_2 \). \( \square \)

Define

\[
\Gamma = \Gamma \left( \left[ \phi, \psi \right] \, , \, \left[ \phi^*, \psi^* \right] \right)
\]

\[
\subset \left\{ (\phi, \psi) \in C \left( R, R^2 \right) \mid \begin{array}{l}
(i) \text{ } \phi(\xi), \psi(\xi) \text{ are nondecreasing in } R; \\
(ii) \text{ } \phi(\xi) \leq \phi^*(\xi) \text{ and } \psi^*(\xi) \leq \psi(\xi) \text{ for all } \xi \in R; \\
(iii) \text{ } \left| \phi(\xi) - \phi^*(\xi) \right| \leq \frac{2 \mu_1 k_1}{c} \left| \xi_1 - \xi_2 \right| \text{ and } \left| \psi(\xi) - \psi^*(\xi) \right| \leq \frac{2 \mu_1 k_2}{c} \left| \xi_1 - \xi_2 \right|
\end{array} \right\}
\]

It is easy to verify that \( \Gamma \) is nonempty, convex and compact in \( B_\mu \left( R, R^2 \right) \). As the proof of Claim 2 in the proof of Theorem A in [5], we have

Lemma 3.3. Assume that \( (A_1) \neq (A_2) \) hold. Then \( F(\Gamma) \subset \Gamma \).

Proof of Theorem 1.1. By the definition of \( \Gamma \), Lemma 3.2-3.3 and Schauder’s fixed point theorem, we get that there exists a fixed point \( (\phi^*(\xi), \psi^*(\xi)) \in \Gamma \). Note that \( (\phi^*(\xi), \psi^*(\xi)) \) is nondecreasing in \( \xi \in R \), assumption \( (A_1) \) and Lemma 2.2 imply that \( \lim_{\xi \to -\infty} (\phi^*(\xi), \psi^*(\xi)) = (0, 0) \), \( \lim_{\xi \to +\infty} (\phi^*(\xi), \psi^*(\xi)) = (k_1, k_2) \). Therefore, \( (\phi^*(\xi), \psi^*(\xi)) \) is a traveling wave solution of (1.1). \( \square \)

Acknowledgements

This work was supported by the NNSF of China Grant 11571092.

References


