Non-Negative Integer Solutions of Two Diophantine Equations $2^x + 9^y = z^2$ and $5^x + 9^y = z^2$

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Received 7 October 2015; accepted 24 April 2016; published 27 April 2016

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Abstract
In this paper, we study two Diophantine equations of the type $2^x + 9^y = z^2$, where $p$ is a prime number. We find that the equation $2^x + 9^y = z^2$ has exactly two solutions $(x, y, z)$ in non-negative integer i.e., \{(3,0,3),(4,1,5)\} but $5^x + 9^y = z^2$ has no non-negative integer solution.

Keywords
Exponential Diophantine Equation, Integer Solutions

1. Introduction
Recently, there have been a lot of studies about the Diophantine equation of the type $a^x + b^y = c^z$. In 2012, B. Sroysang [1] proved that $(1,0,2)$ is a unique solution $(x, y, z)$ for the Diophantine equation $3^x + 5^y = z^2$ where $x, y$ and $z$ are non-negative integers. In 2013, B. Sroysang [2] showed that the Diophantine equation $3^x + 17^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1,0,2)$. In the same year, B. Sroysang [3] found all the solutions to the Diophantine equation $2^x + 3^y = z^2$ where $x, y$ and $z$ are non-negative integers. The solutions $(x, y, z)$ are $(0,1,2), (3,0,3)$ and $(4,2,5)$. In 2013, Rabago [4] showed that the solutions $(x, y, z)$ of the two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$ where $x, y$ and $z$ are non-negative integers are $\{(1,0,2),(4,1,10)\}$ and $\{(1,0,2),(2,1,10)\}$, respectively. Different examples of Diophantine equations have been studied (see for instance [5]-[11]).

In this study, we consider the Diophantine equation of the type $p^x + 9^y = z^2$ where $p$ is prime. Particularly, we show that $2^x + 9^y = z^2$ has exactly two solutions in non-negative integer and $5^x + 9^y = z^2$ has no
non-negative integer solution.

2. Main Results

**Theorem 2.1.** (Catalan’s Conjecture [12]) The Diophantine equation \( a^x - b^y = 1 \), where \( a, b, x \) and \( y \) are integers with \( a, b, x, y > 1 \), has a unique solution \( (a, b, x, y) = (3, 2, 2, 3) \).

**Theorem 2.2.** The Diophantine equation \( 2^x + 1 = z^2 \) has a unique non-negative integer solution \( (x, z) = (3, 3) \).

**Proof:** Let \( x \) and \( z \) be non-negative integers such that \( 2^x + 1 = z^2 \). For \( x = 0 \), \( z^2 = 2 \) which is impossible. Suppose \( x \geq 1 \). Then, \( 2^x = z^2 - 1 = (z + 1)(z - 1) \) and \( (z + 1) = 2^i \) and \( (z - 1) = 2^j \), where \( \eta < \xi, \xi + \eta = x \).

Thus, \( 2^i - 2^j = 2 \) or, \( 2^j(2^i - 1) = 2 \). Now we have two possibilities.

Case-1: If \( 2^i = 2 \), then \( 2^{i-1} - 1 = 1 \). These give us \( \eta = 1 \) and \( \xi = 2 \). Then \( x = 3 \) and \( z = 3 \). Thus \( (x, z) = (3, 3) \) is a solution of \( 2^x + 1 = z^2 \).

Case-2: If \( 2^i = 1 \), then \( 2^{i-1} - 1 = 2 \). These give us \( \eta = 0 \) and \( \xi = 3 \) which is impossible.

Hence, \( (x, z) = (3, 3) \) is a unique non-negative integer solution for the equation \( 2^x + 1 = z^2 \).

**Theorem 2.3.** The Diophantine equation \( p^x + 1 = z^2 \), where \( p \) is an odd prime number, has exactly one non-negative integer solution \( (x, z, p) = (1, 2, 3) \).

**Proof:** Let \( x \) and \( z \) be non-negative integers such that \( p^x + 1 = z^2 \), where \( p \) be an odd prime. If \( x = 0 \), then \( z^2 = 2 \). It is impossible. If \( z = 0 \), then \( p^x = 1 \), which is also impossible. Now for \( x, z > 0 \),

\[ p^x + 1 = z^2 \]

or \( p^x = (z - 1)(z + 1) \).

Let \( z + 1 = p^y \) and \( z - 1 = p^z \), where \( \psi < \xi, \psi + \xi = x \). Then,

\[ p^y - p^z = 2 \]

or \( \psi \left( p^y - p^z - 1 \right) = 2 \).

Thus, \( p^y = 1 \Rightarrow p^y = p^0 = \psi = 0 \) and \( p^y - 1 = 2 \Rightarrow p^z = 3 \), which is possible only for \( p = 3 \) and \( \xi = 1 \). So \( x = \psi + \xi = 0 + 1 = 1, z = p^y - 1 = 3^y - 1 = 2 \).

Therefore, \( (x, z, p) = (1, 2, 3) \) is the solution of \( p^x + 1 = z^2 \). This proves the theorem.

**Corollary 2.4.** The Diophantine equation \( 5^x + 1 = z^2 \) has no non-negative integer solutions.

**Theorem 2.5.** The Diophantine equation \( 1 + 9^x = z^2 \) has no unique non-negative integer solution.

**Proof:** Suppose \( x \) and \( z \) be non-negative integers such that \( 1 + 9^x = z^2 \). For \( x = 0 \), we have \( z^2 = 2 \). It is impossible. Let \( x \geq 1 \). Then \( 1 + 9^x = z^2 \) gives us \( 3^x = (z - 1)(z + 1) \). Let \( z + 1 = 3^{f_1} \) and \( z - 1 = 3^{f_2} \), where \( \Pi_2 < \Pi_1 \), \( \Pi_1 + \Pi_2 = 2x \). Therefore,

\[ 3^{f_1} - 3^{f_2} = 2 \]

or \( 3^{f_1} (3^{f_1 - f_2} - 1) = 2 \).

Thus, \( 3^{f_1} = 1 \) or \( \Pi_2 = 0 \) and \( 3^{f_1 - f_2} - 1 = 2 \) or \( \Pi_1 = 1 \). So \( 2x + 1 \Rightarrow x = \frac{1}{2} \), which is not acceptable since \( x \) is a non-negative integer. This completes the proof.

**Theorem 2.6.** The Diophantine equation \( 2^x + 9^y = z^2 \) has exactly two solutions \( (x, y, z) \) in non-negative integer i.e., \( (3, 0, 3), (1, 4, 5) \).

**Proof:** Suppose \( x, y \) and \( z \) are non-negative integers for which \( 2^x + 9^y = z^2 \). If \( x = 0 \), we have \( 1 + 9^y = z^2 \) which has no solution by theorem 2.5. For \( y = 0 \), by theorem 2.2 we have \( x = 3 \) and \( y = 3 \). Hence \( (x, y, z) = (3, 0, 3) \) is a solution to \( 2^x + 9^y = z^2 \). If \( z = 0 \), then \( 2^x + 9^y = 0 \) which is not possible for any non-negative integers \( x \) and \( y \).

Now we consider the following remaining cases.

Case-1: \( x = 1 \). If \( x = 1 \), then \( 2 + 9^y = z^2 \) or \( 2 = (z + 3^y)(z - 3^y) \). We have two possibilities. If \( z + 3^y = 1 \) and \( z - 3^y = 2 \), then \( 2x = 3 \) or \( z = \frac{3}{2} \) but which is not acceptable. On the other hand, if \( z + 3^y = 2 \) and \( z - 3^y = 1 \) same thing is occurred.

Case-2: \( y = 1 \). If \( y = 1 \), then \( 2^x + 9 = z^2 \) or \( 2^x = (z + 3)(z - 3) \). Let \( z + 3 = 2^i \) and \( z = 3 = 2^i \), where \( \eta < \xi, \xi + \eta = x \). Then \( 2^i - 2^j = 2.3 \) or \( 2^j (2^i - 1) = 2.3 \). Thus, \( 2^j = 2 \) and \( 2^i - 1 = 3 \), then this implies...
that $\eta = 1$ and $\xi - 1 = 2$ or $\xi = 3$. So $x = 4$ and $z = 5$. Here we obtain the solution $(x, y, z) = (4, 1, 5)$.

Case-3: $z = 1$. If $z = 1$, then $2^t + 9^r = 1$ which is not possible for any for any non-negative integers $x$ and $y$.

Case-4: $x, y, z > 1$. Now

\[2^t + 9^r = z^2 \quad \text{or} \quad 2^t = (z + 3^r)(z - 3^r).\]

Let $z + 3^r = 2^\eta$, and $z - 3^r = 2^\xi$, where $\Pi_1 < \Pi_1, \Pi_1 + \Pi_2 = x$. So $2^\eta - 2^\xi = 2^t$ or $2^\eta (2^\eta - 2^\xi - 1) = 2^t$.

Thus, $2^\eta = 2$ and $2^\eta (2^\eta - 2^\xi - 1) = 3^r$ then these imply that $\Pi_2 = 1$ and $2^\eta - 1 = 3^r$.

So we get

\[2^\eta - 1 = 3^r = 1 \quad (1)\]

The Diophantine Equation (1) is a Diophantine equation by Catalan’s type $a^t - b^t = 1$ because for $y > 1$, the value of $\Pi_1$ must be greater than 1. So by the Catalan’s conjecture Equation (1) has no solution. This proves the theorem.

**Theorem 2.7.** The Diophantine equation $5^t + 9^r = z^2$ has no non-negative integer solution.

**Proof.** Suppose $x, y$ and $z$ are non-negative integers for which $5^t + 9^r = z^2$. If $x = 0$, we have $1 + 9^r = z^2$ which has no solution by Theorem 2.5. For $y = 0$ we use corollary 2.4. If $z = 0$, then $5^t + 9^r = 0$ which is not possible for any for any non-negative integers $x$ and $y$.

Now we consider the following remaining cases.

Case-1: $x = 1$. If $x = 1$, then $5^t + 9^r = z^2$ or $5 = (z + 3^r)(z - 3^r)$. We have two possibilities. If $z + 3^r = 5$ and $z - 3^r = 1$, it follows that $2z = 6$ or $z = 3$ and $3^r = 2$, a contradiction. On the other hand, $z + 3^r = 1$ and $z - 3^r = 5$, it follows that $2z = 6$ or $z = 3$ and $3^r = 2$ which is impossible.

Case-2: $y = 1$. If $y = 1$, then $5^t + 9^r = z^2$ or $5^t = (z + 3^r)(z - 3^r)$. Let $z + 3^r = 5^t$ and $z - 3^r = 5^r$, where $\eta < \xi, \xi + \eta = x$. Then $5^t - 5^r = 2.3$ or $5^t (5^t - 5^r - 1) = 2.3$. Thus, $5^t = 1$ and $5^t - 5^r - 1 = 6$, this implies that $\eta = 0$ and $5^t = 7$, a contradiction.

Case-3: $z = 1$. If $z = 1$, then $5^t + 9^r = 1$ which is not possible for any for any non-negative integers $x$ and $y$.

Case-4: $x, y, z > 1$. Now

\[5^t + 9^r = z^2 \quad \text{or} \quad 5^t = (z + 3^r)(z - 3^r).\]

Let $z + 3^r = 5^\eta$, and $z - 3^r = 5^\xi$, where $\Pi_1 < \Pi_1, \Pi_1 + \Pi_2 = x$. So $5^\eta - 5^\xi = 2.3$ or $5^\eta (5^\eta - 5^\xi - 1) = 2.3$. Thus, $5^\eta = 1$ and $5^\eta - 5^\xi - 1 = 2.3$ then these imply that $\Pi_2 = 0$ and $5^\eta - 1 = 2.3$.

Since $5 \equiv 1 (\text{mod} \ 4)$, it follows that $5^\eta \equiv 1 (\text{mod} \ 4)$ i.e., $5^\eta - 1 \equiv 0 (\text{mod} \ 4)$. But we see that $2.3 \equiv 0 (\text{mod} \ 4)$. This is impossible.

**3. Conclusion**

In the paper, we have discussed two Diophantine equation of the type $p^t + q^r = z^2$, where $p$ is a prime number. We have found that $(3, 0, 3)$ and $(4, 1, 5)$ are the exact solutions to $2^t + 9^r = z^2$ in non-negative integers. On the contrary, we have also found that the Diophantine equation $5^t + 9^r = z^2$ has no non-negative integer solution.

**References**


