Schur Convexity and the Dual Simpson’s Formula

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Abstract
In this paper, we show that some functions related to the dual Simpson’s formula and Bullen-Simpson’s formula are Schur-convex provided that f is four-convex. These results should be compared to that of Simpson’s formula in Applied Math. Lett. (24) (2011), 1565-1568.

Keywords
Schur Convexity, 4-Convex Function, Dual Simpson’s Formula, Bullen-Simpson’s Formula

1. Introduction
Schur convexity is an important notion in the theory of convex functions, which were introduced by Schur in 1923 ([1] [2]), its definition is stated in what follows. Let \( R^n_+ \) be denoted as,
\[
R^n_+ = \left\{ x = (x_1, x_2, \cdots, x_n) \in R^n; x_1 \geq x_2 \geq \cdots \geq x_n \right\},
\]
and \( (R^n_+)^* \) be defined by,
\[
(R^n_+)^* = \left\{ y \in R^n; \sum_{j=1}^n y_j \geq 0 \text{ for all } j = 1, 2, \cdots, n-1 \text{ and } \sum_{j=1}^n y_j = 0 \right\}.
\]

Then we recall (see, e.g., [3]-[5]) that a function \( f : R^n \to R \) is Schur convex if
\[
\forall x, y \in R^n_+; y - x \in (R^n_+)^* \implies f (x) \leq f (y).
\]

Every Schur-convex function \( f : D \subseteq R^n \to R \) is a symmetric function, and if \( I \) is an open interval and \( f : I^n \to R \) is symmetric and of class \( C^1 \), then \( f \) is Schur-convex if and only if
\[
(x_i - x_j) \left( \frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j} \right) \geq 0, \text{ on } I^n
\]
for all \( i, j \in \{1, 2, \cdots, n \} \).

Let \( f : I \subseteq R \to R \) be a convex function defined on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). The following inequality

holds. This double inequality is called Hermite-Hadamard inequality for convex functions. Hermite-Hadamard inequality is improved though Schur convexity, c.f., [6]-[10]. Among these paper, it is proven that if \( I \in \mathbb{R} \) is an interval and \( f : I \to \mathbb{R} \) is continuous, then \( f \) is convex if and only if the mapping

\[
S_1(a,b) = \frac{1}{b-a} \int_a^b f(x) dx, \quad \text{if } b \neq a
\]

(Here and what follows, we use the mapping convention \( S_1(a,a) = \lim_{b \to a} S_1(a,b) \) for \( b = a \) case, which is no longer stated.) is Schur convex, and in this case, \( S_1(a,b) \) is convex. If \( I \in \mathbb{R} \) is an interval and \( f : I \to \mathbb{R} \) is continuous, then \( f \) is convex if and only if one of the following mappings

\[
S_2(a,b) = \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right), \quad \text{if } b \neq a,
\]

\[
S_3(a,b) = \frac{1}{2} \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{b-a} \int_a^b f(x) dx, \quad \text{if } b \neq a
\]

is Schur convex. Some exciting results on Schur’s majorization inequality can be found in [11]-[13].

Let \( f : [a, b] \to \mathbb{R} \) be a four times continuously differentiable mapping on \([a, b] \). Then the following quadrature rule is well-known:

\[
\frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{6} f(a) + f(b) + 4 f\left(\frac{a+b}{2}\right) - \frac{1}{2880} f^{(4)}(\xi)(b-a)^4, \quad \xi \in (a,b), \quad (1.3)
\]

which is called Simpson’s formula, c.f. [14] and [15]. For \( I \in \mathbb{R} \) is an interval and \( f : I \to \mathbb{R} \) is called four-convex, if \( f^{(4)}(t) \geq 0 \) for all \( t \in [a,b] \). In [15], the authors proved that if \( f^{(4)} : I \to \mathbb{R} \) is continuous, then \( f \) is four-convex is equivalent to the mappings defined by

\[
S_4(a,b) = \frac{1}{6} f(a) + f(b) + 4 f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx, \quad \text{if } b \neq a
\]

is Schur-convex, this is an improvement of the Simpson’s formula.

On the other hand, the dual Simpson’s formula ([14]) is stated as follows: if \( f^{(4)} \) is continuous, there exist \( \eta \in (a,b) \) such that

\[
\frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{3} \left[ 2 f\left(\frac{3a+b}{4}\right) + 2 f\left(\frac{a+3b}{4}\right) - f\left(\frac{a+b}{2}\right) \right] + \frac{1}{23040} f^{(4)}(\eta)(b-a)^4, \quad \eta \in (a,b). \quad (1.4)
\]

In [16], Bullen proved that, if \( f \) is four-convex, then the dual Simpson’s quadrature formula is more accurate than Simpson’s formula. That is, it holds that

\[
\frac{1}{b-a} \int_a^b f(x) dx \leq \frac{1}{12} \left[ f(a) + 4 f\left(\frac{3a+b}{4}\right) + 2 f\left(\frac{a+b}{2}\right) + 4 f\left(\frac{a+3b}{4}\right) + f(b) \right],
\]

provided that \( f \) is four-convex.

Now we can state our main results. In view of the dual Simpson’s formula and the above Bullen-Simpson formula, we construct two mappings as follows: for \( b \neq a \), we set

\[
S_5(a,b) = \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{3} \left[ 2 f\left(\frac{3a+b}{4}\right) + 2 f\left(\frac{a+3b}{4}\right) - f\left(\frac{a+b}{2}\right) \right],
\]

\[
S_6(a,b) = \frac{1}{12} \left[ f(a) + 4 f\left(\frac{3a+b}{4}\right) + 2 f\left(\frac{a+b}{2}\right) + 4 f\left(\frac{a+3b}{4}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx.
\]
We shall show that if \( \varphi^{(4)} : I \to \mathbb{R} \) is continuous, then \( f \) is four-convex if and only if the mapping \( S_{s}(a,b) \) or \( S_{s}(a,b) \) is Schur-convex. Obviously our results improve the dual-Simpson’s formula and the Bullen-Simpson’s formula, and hence complement the main result in [15].

2. Main Results

We now present our main theorem.

**Theorem 2.1.** Let \( I \subseteq \mathbb{R}, f \in C^{4}(I) \) be a mapping on \( I \), then the following statements are equivalent:

(a) The function \( S_{s}(a,b) \) is Schur-convex on \( I^{2} \).

(b) The function \( S_{s}(a,b) \) is Schur-convex on \( I^{2} \).

(c) The function \( S_{s}(a,b) \) is Schur-convex on \( I^{2} \).

(d) For any \( a, b \in I \) with \( a < b \), we have the Simpson inequality holds, i.e.

\[
\frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq \frac{1}{6} [f(a) + f(b) + 4f\left(\frac{a+b}{2}\right)].
\]

(e) For any \( a, b \in I \) with \( a < b \), we have the dual Simpson inequality holds, i.e.

\[
\frac{1}{3} \left[ 2f\left(\frac{3a+b}{4}\right) + 2f\left(\frac{a+3b}{4}\right) - f\left(\frac{a+b}{2}\right) \right] \leq \frac{1}{b-a} \int_{a}^{b} f(x) \, dx.
\]

(f) For any \( a, b \in I \) with \( a < b \), we have the Bullen-Simpson inequality holds, i.e.

\[
\frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq \frac{1}{12} \left[ f(a) + 4f\left(\frac{3a+b}{4}\right) + 2f\left(\frac{a+b}{2}\right) + 4f\left(\frac{a+3b}{4}\right) + f(b) \right].
\]

(g) The function \( f \) is four-convex on \( I \).

**Proof:**

The equivalence of (a) (d) (g) was already proven in [15]. Suppose that item (g) holds, then by the definition of the function \( S_{s}(a,b) \), we have

\[
(b-a)\left( \frac{\partial S_{s}}{\partial b} - \frac{\partial S_{s}}{\partial a} \right) = f(a) + f(b) - \frac{2}{b-a} \int_{a}^{b} f(x) \, dx - \frac{1}{3} \left[ f\left(\frac{a+3b}{4}\right) - f\left(\frac{3a+b}{4}\right) \right](b-a).
\]

\[
\geq f(a) + f(b) - \frac{1}{3} \left[ f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right] - \frac{1}{3} \left[ f\left(\frac{a+3b}{4}\right) - f\left(\frac{3a+b}{4}\right) \right](b-a),
\]

(by Simpson’s formula (1.4) and four-convexity of \( f \)) hence,

\[
\frac{\partial S_{s}}{\partial b} - \frac{\partial S_{s}}{\partial a} = \frac{2}{3} \left[ \frac{1}{b-a} \int_{a}^{b} f'(x) \, dx \right] - \frac{1}{2} f'\left(\frac{a+3b}{4}\right) - \frac{1}{2} f'\left(\frac{3a+b}{4}\right).
\]

\[
= \frac{1}{3} \left[ \frac{2}{b-a} \int_{a}^{b} f'\left(\frac{x+b-a}{2}\right) - f'(x) \, dx - f'\left(\frac{a+3b}{4}\right) - f'\left(\frac{3a+b}{4}\right) \right],
\]

\[
= \frac{1}{3} \left[ \frac{2}{b-a} \int_{a}^{b} h(x) \, dx - h\left(\frac{3a+b}{4}\right) \right],
\]

Here we denote \( h(x) = f'\left(\frac{x+b-a}{2}\right) - f'(x) \), for \( x \in \left[ a, \frac{a+b}{2} \right] \). Since \( f \) is four-convex, \( h(x) \) is convex.
Thus Hermite-Hadamard (1.2) holds for \( h(x) \) in \( \left[ a, \frac{a+b}{2} \right] \), this gives that 
\[
(b-a)\left( \frac{\partial S_2}{\partial b} - \frac{\partial S_2}{\partial a} \right) \geq 0,
\]
so by the criteria (1.1) \( S_2 \) is Schur-convex, item (b) is a consequence of item (g).

Now suppose that item (b) holds. Since 
\[
\left( \frac{a+b}{2}, \frac{a+b}{2} \right) \in (a, b),
\]
Schur-convexity of \( S_2 \) gives that
\[
0 = S_2 \left( \frac{a+b}{2}, \frac{a+b}{2} \right) \leq S_2(a, b), \quad \text{i.e., item (e) is valid if item (b) holds.}
\]

Next we prove item (e) implies item (g). By item (e) and the dual Simpson’s formula (1.6), we get
\[
0 \leq S_4(a, b) = \frac{1}{23040} f^{(4)}(\eta)(b-a)^4, \quad \eta \in (a, b).
\]

Since \( f \in C^4(I) \), and \( a, b \) are arbitrary, it follows that \( f \) is four-convex. Now the equivalence of (b) (e) (g) is proven. We follow the same pattern to show the equivalence of (c) (f) (g). If item (c) holds, then
\[
0 = S_4 \left( \frac{a+b}{2}, \frac{a+b}{2} \right) \leq S_4(a, b), \quad \text{i.e., item (f) is valid. Suppose that item (f) is valid. By the definitions and formulas (1.3) and (1.4), we get}
\]
\[
0 \leq 2S_4(a, b) = S_4(a, b) - S_4(a, b) = \frac{1}{2880} \left( f^{(4)}(\xi) - \frac{1}{8} f^{(4)}(\eta) \right)(b-a)^4, \quad \xi, \eta \in (a, b).
\]

Since \( f \in C^4(I) \), and \( a, b \) are arbitrary, item (g) follows again. It is only left to show that item (g) implies item (c). We give a lemma first.

**Lemma 2.1.** Let \( I \subseteq R, f \in C^4(I) \) be four-convex on \( I \), then the following inequalities hold for any \( a, b \in I \) with \( b \geq a \):

\[
\frac{1}{b-a} \int_a^b f(x) \, dx \geq f(a) + \frac{1}{6} f'(a)(b-a) + \frac{1}{3} f'' \left( \frac{a+b}{2} \right) (b-a).
\]

\[
\frac{1}{b-a} \int_a^b f(x) \, dx \geq f(b) - \frac{1}{6} f'(b)(b-a) - \frac{1}{3} f'' \left( \frac{a+b}{2} \right) (b-a).
\]

**Proof:**
We only prove the first inequality. Denote that
\[
T(b) := \int_a^b f(x) \, dx - \left[ f(a)(b-a) + \frac{1}{6} f'(a)(b-a)^2 + \frac{1}{3} f'' \left( \frac{a+b}{2} \right) (b-a)^2 \right],
\]
and that \( g(x) = f^*(x) \), then
\[
T(a) = 0; \quad T'(a) = 0; \quad T''(a) = 0.
\]

\[
T'(b) = f(b) - f(a) - \left( \frac{1}{3} f'(a) + \frac{2}{3} f'' \left( \frac{a+b}{2} \right) \right)(b-a) - \frac{1}{6} f'' \left( \frac{a+b}{2} \right) (b-a)^2.
\]

\[
T''(b) = f'(b) - \left[ \frac{1}{3} f'(a) + \frac{2}{3} f'' \left( \frac{a+b}{2} \right) \right] \left[ \frac{2}{3} f'' \left( \frac{a+b}{2} \right) (b-a) - \frac{1}{12} f'' \left( \frac{a+b}{2} \right) (b-a)^2 \right]
\]
\[
= \frac{2}{3} \int_a^b g(x) \, dx + \frac{1}{3} \int_a^b g(x) \, dx \left[ \frac{2}{3} g \left( \frac{a+b}{2} \right) (b-a) - \frac{1}{12} g \left( \frac{a+b}{2} \right) (b-a)^2 \right]
\]
\[
= T_1(b) + T_2(b).
\]

Here,
\[
T_1(b) = \frac{1}{3} \int_a^b \left[ g \left( \frac{a+b}{2} \right) \right] (b-a).
\]
From the Hermite-Hadamard inequality for convex function \( g(x) \), we see that \( T_1(b) \geq 0 \). Besides, it follows from convexity of \( g(x) \) that for any \( x \leq y \):

\[
g(y) \geq g(x) + g'(x)(y-x).
\]

Take integration w.r.t \( y \), we get

\[
\int_y x g(y) \, dy \geq g(x)(y-x) + \frac{1}{2} g'(x)(y-x)^2,
\]

applying this inequality in \( \left( \frac{a+b}{2}, b \right) \), we see that \( T_2(b) \geq 0 \). It follows that \( T^*(b) \geq 0 \) for any \( b \geq a \), hence by (2.1) we know \( T(b) \geq 0 \) for any \( b \geq a \). The second inequality in the lemma is just the first inequality with \( b \leq a \), we omit its proof. The lemma is proven.

Now we continue the proof of our main theorem. By the definition of \( S_6(a,b) \), we have

\[
(b-a) \left( \frac{\partial^2 S_6}{\partial b \partial a} - \frac{\partial^2 S_6}{\partial a^2} \right) = \frac{2}{b-a} \int_a^b f(x) \, dx - \left[ f(a) + f(b) \right]
\]

\[
+ \frac{1}{12} \left[ f'(b) - f'(a) \right] (b-a) + \frac{1}{6} \left[ f' \left( \frac{a+b}{4} \right) \right] - \frac{1}{6} \left( \frac{a+b}{4} \right) (b-a)
\]

\[
= K_1(b) + K_2(b),
\]

here \( K_1(b), K_2(b) \) is denoted as

\[
K_1(b) = \frac{2}{b-a} \int_a^b f(x) \, dx - f(a) - \frac{1}{12} f'(a)(b-a) - \frac{1}{6} f' \left( \frac{a+b}{4} \right) (b-a)
\]

\[
K_2(b) = \frac{2}{b-a} \int_a^b f(x) \, dx - f(b) + \frac{1}{12} f'(b)(b-a) + \frac{1}{6} f' \left( \frac{a+b}{4} \right) (b-a)
\]

Suppose that item (g) holds, by applying the lemma to \( f \) in \( \left[ a, \frac{a+b}{2}, \frac{a+b}{2}, b \right] \), we get both \( K_1, K_2 \geq 0 \), thus \( (b-a) \left( \frac{\partial^2 S_6}{\partial b \partial a} - \frac{\partial^2 S_6}{\partial a^2} \right) \geq 0 \), so by the criteria (1.1) \( S_6(a,b) \) is Schur-convex, item (c) follows.

**Remark 2.1.** From Lemma 2.1, we add the two inequalities together to see that the following holds for four-convex functions \( f \):

\[
\int_a^b f(x) \, dx \geq \frac{1}{2} \left[ f(a) + f(b) \right] - \frac{1}{12} \left[ f'(b) - f'(a) \right] (b-a)
\]

(2.2)

It is well-known, c.f., [14] or [15].

Starting from this inequality (2.2), we deduce some properties for four-convex functions. As in the above, we define a pair of mappings \( S_6, S_8 \) by

\[
S_6(a,b) = \frac{1}{b-a} \int_a^b f(x) \, dx - \frac{1}{2} \left[ f(a) + f(b) \right] + \frac{1}{12} \left[ f'(b) - f'(a) \right] (b-a);
\]

\[
S_8(a,b) = \frac{1}{2} \left[ f(a) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx - \frac{1}{12} f' \left( \frac{a+b}{2} \right) (b-a)^2.
\]
Then we have

**Theorem 2.2.** Let \( I \subseteq \mathbb{R}, f \in C^4(I) \) be four-convex on \( I \), then the mappings \( S_8, S_9 \) are non-negative and Schur-convex on \( I^2 \).

**Proof:**

We observe that

\[
\begin{align*}
(b-a) \left( \frac{\partial S_8}{\partial b} - \frac{\partial S_8}{\partial a} \right) &= \frac{2}{b-a} \int f(x) dx - \left[ f(a) + f(b) \right] \\
&+ \frac{1}{2} \left[ f'(b) - f'(a) \right] (b-a) - \frac{1}{3} f'' \left( \frac{a+b}{2} \right) (b-a)^2 \\
&\geq \frac{1}{2} \left[ f'(b) - f'(a) \right] (b-a) - \frac{1}{3} f'' \left( \frac{a+b}{2} \right) (b-a)^2 \\
&\geq 0
\end{align*}
\]

(2.3)

(2.4)

Here inequality (2.3) is due to inequality (2.2), and inequality (2.4) is a consequence of the Hermite-Hadamard inequality for convex function \( f'' \), thus by the criteria (1.1) \( S_8 \) are Schur-convex on \( I^2 \). Hence we get \( S_8(a,b) \geq S_8 \left( \frac{a+b}{2}, \frac{a+b}{2} \right) = 0 \).

Since \( S_8 \) is non-negative, we observe that

\[
\begin{align*}
(b-a) \left( \frac{\partial S_7}{\partial b} - \frac{\partial S_7}{\partial a} \right) &= - \frac{2}{b-a} \int f(x) dx + \left[ f(a) + f(b) \right] \\
&- \frac{1}{3} f'' \left( \frac{a+b}{2} \right) (b-a)^2 \\
&\geq 0
\end{align*}
\]

(2.5)

It is shown in [7] for a convex function \( g \) that the function

\[
S_7(a,b) = \frac{1}{4} \left[ g(a) + g(b) \right] + \frac{1}{2} \int g \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int f(x) dx \quad \text{(if \( b \neq a \))}
\]

is Schur-convex, specially we have \( S_7(a,b) \geq 0 \). We set \( g = f'' \), then it is convex, we see that RHS of inequality (2.5) is non-negative, so by the criteria (1.1), \( S_7 \) is Schur-convex.

Furthermore, we give a Schur-convexity theorem for the following mapping:

\[
S_{10}(a,b) = f \left( \frac{a+b}{2} \right) - \frac{1}{2} \left[ f(a) + f(b) \right] + \frac{1}{12} \left[ f''(b) - f''(b) \right] (b-a) + \frac{1}{24} f'' \left( \frac{a+b}{2} \right) (b-a)^2.
\]

**Theorem 2.3.** Let \( I \subseteq \mathbb{R}, f \in C^4(I) \) be four-convex on \( I \), then the mappings \( S_{10} \) are non-negative and Schur-convex on \( I^2 \).

**Proof:** We observe that

\[
\begin{align*}
(b-a) \left( \frac{\partial S_{10}}{\partial b} - \frac{\partial S_{10}}{\partial a} \right) &= - \frac{1}{3} f'' \left( \frac{a+b}{2} \right) (b-a)^2 \\
&\geq 0
\end{align*}
\]

Since \( S_7(a,b) \geq 0 \) for convex function \( g = f'' \), as in the above, we can conclude that \( S_{10}(a,b) \) are non-negative and Schur-convex.

**Remark 2.2.** For smooth four-convex functions, we see that both \( S_8 \) and \( S_{10} \) are non-negative and Schur-convex functions, then the sum of \( S_8 \) and \( S_{10} \) is also non-negative and Schur-convex function, especially it holds that
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\[
f\left(\frac{a+b}{2}\right) + \frac{1}{24} f''\left(\frac{a+b}{2}\right) (b-a)^2 \geq \frac{1}{b-a} \int_e^b f(x) \, dx
\]

**Remark 2.3.** For positive real numbers \(x, y\), we denote the arithmetic mean, geometric mean, and logarithmic mean of \(x, y\) by \(A, G, L\). Applying non-negativity of \(S_r\) and \(S_s\) to function \(f(t) = e^t, \ t \in [\ln x, \ln y]\) then we have

\[
\frac{1}{12} G \cdot \left(\frac{\ln y}{x}\right)^2 \leq A - L \leq \frac{1}{12} L \cdot \left(\frac{\ln y}{x}\right)^2.
\]

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**References**


