A Strong Law of Large Numbers for Set-Valued Random Variables in $G_\alpha$ Space

Guan Li

College of Applied Sciences, Beijing University of Technology, Beijing, China
Email: guanli@bjut.edu.cn

Received 30 March 2015; accepted 23 June 2015; published 30 June 2015

Abstract

In this paper, we shall represent a strong law of large numbers (SLLN) for weighted sums of set-valued random variables in the sense of the Hausdorff metric $d_H$, based on the result of single-valued random variable obtained by Taylor [1].

Keywords

Set-Valued Random Variable, the Laws of Large Numbers, Hausdorff Metric

1. Introduction

We all know that the limit theories are important in probability and statistics. For single-valued case, many beautiful results for limit theory have been obtained. In [1], there are many results of laws of large numbers at different kinds of conditions and different kinds of spaces. With the development of set-valued random theory, the theory of set-valued random variables and their applications have become one of new and active branches in probability theory. And the theory of set-valued random variables has been developed quite extensively (cf. [2]-[7] etc.). In [1], Artstein and Vitale used an embedding theorem to prove a strong law of large numbers for independent and identically distributed set-valued random variables whose basic space is $\mathbb{R}^d$, and Hiai extended it to separable Banach space $\mathcal{X}$ in [8]. Taylor and Inoue proved SLLN’s for only independent case in Banach space in [7]. Many other authors such as Giné, Hahn and Zinn [9], Puri and Ralescu [10] discussed SLLN’s under different settings for set-valued random variables where the underlying space is a separable Banach space.

In this paper, what we concerned is the SLLN of set-valued independent random variables in $G_\alpha$ space. Here the geometric conditions are imposed on the Banach spaces to obtain SLLN for set-valued random variables. The results are both the extension of the single-valued’s case and the extension of the set-valued’s case.

This paper is organized as follows. In Section 2, we shall briefly introduce some definitions and basic results of set-valued random variables. In Section 3, we shall prove a strong law of large numbers for set-valued independent random variables in $G_\alpha$ space.

2. Preliminaries on Set-Valued Random Variables

Throughout this paper, we assume that $(\Omega, \mathcal{A}, \mu)$ is a nonatomic complete probability space, $(\mathcal{X}, \| \cdot \|)$ is a real separable Banach space, $\mathbb{N}$ is the set of nature numbers, $\mathcal{K}(\mathcal{X})$ is the family of all nonempty closed subsets...
of $\mathcal{X}$, and $\mathbf{K}_b(\mathcal{X})$ is the family of all nonempty bounded closed convex subsets of $\mathcal{X}$.

Let $A$ and $B$ be two nonempty subsets of $\mathcal{X}$ and let $\lambda \in \mathbb{R}$, the set of all real numbers. We define addition and scalar multiplication as

$$A + B = \{a + b : a \in A, b \in B\},$$

$$\lambda A = \{\lambda a : a \in A\}.$$

The Hausdorff metric on $\mathbf{K}(\mathcal{X})$ is defined by

$$d_H(A, B) = \max\{\inf_{a \in A} \|a - b\|, \inf_{b \in B} \|a - b\|\},$$

for $A, B \in \mathbf{K}(\mathcal{X})$. For an $A$ in $\mathbf{K}(\mathcal{X})$, let $\|A\| = d_H(\{0\}, A)$. The metric space $(\mathbf{K}_b(\mathcal{X}), d_H)$ is complete, and $\mathbf{K}_b(\mathcal{X})$ is a closed subset of $(\mathbf{K}_b(\mathcal{X}), d_H)$ (cf. [6], Theorems 1.1.2 and 1.1.3). For more general hyperspaces, more topological properties of hyperspaces, readers may refer to a good book [11].

For each $A \in \mathbf{K}_b(\mathcal{X})$, define the support function by

$$s(x^*, A) = \sup_{a \in A} <x^*, a >, \quad x^* \in \mathcal{X}^*,$$

where $\mathcal{X}^*$ is the dual space of $\mathcal{X}$.

Let $S^*$ denote the unit sphere of $\mathcal{X}^*$, $C(S^*)$ the all continuous functions of $S^*$, and the norm is defined as $\|v\|_c = \sup_{x^* \in S^*} <x^*, v>$. The following is the equivalent definition of Hausdorff metric.

For each $A, B \in \mathbf{K}_b(\mathcal{X})$,

$$d_H(A, B) = \sup\{|s(x^*, A) - s(x^*, B)| : x^* \in S^*\}.$$

A set-valued mapping $F : \Omega \to \mathbf{K}(\mathcal{X})$ is called a set-valued random variable (or a random set, or a multifunction) if, for each open subset $O$ of $\mathcal{X}$, $F^{-1}(O) = \{\omega \in \Omega : F(\omega) \cap O \neq \emptyset\} \in \mathcal{A}.$

For each set-valued random variable $F$, the expectation of $F$, denoted by $E[F]$, is defined as

$$E[F] = \{\int_{\mathcal{X}} f d\mu : f \in S_F\},$$

where $\int_{\mathcal{X}} f d\mu$ is the usual Bochner integral in $L^1[\Omega, \mathcal{X}]$, the family of integrable $\mathcal{X}$-valued random variables, and $S_F = \{f \in L^1[\Omega; \mathcal{X}] : f(\omega) \in F(\omega), a.e.(\mu)\}$. This integral was first introduced by Aumann [3], called Aumann integral in literature.

### 3. Main Results

In this section, we will give the limit theorems for independent set-valued random variables in $G_\alpha$ space. The following definition and lemma are from [1], which will be used later.

**Definition 3.1** A Banach space $\mathcal{X}$ is said to satisfy the condition $G_\alpha$ for some $0 < \alpha \leq 1$, if there exists a mapping $G : \mathcal{X} \to \mathcal{X}^*$ such that

(i) $\|G(x)\| = \|x\|^\alpha$;

(ii) $G(x)x = \|x\|^{\alpha}x$;

(iii) $\|G(x) - G(y)\| \leq A \|x - y\|^\alpha$ for all $x, y \in \mathcal{X}$ and some positive constant $A$.

Note that Hilbert spaces are $G_1$ with constant $A = 1$ and identity mapping $G$.

**Lemma 3.1** Let $\mathcal{X}$ be a separable Banach space which is $G_\alpha$ for some $0 < \alpha \leq 1$ and let $\{V_1, V_2, \ldots, V_n\}$ be single-valued independent random elements in $\mathcal{X}$ such that $E[V_k] = 0$ and $E[\|V_k\|^{\alpha}] < \infty$ for each $k = 1, 2, \ldots, n$. then

$$E[\|V_1 + \cdots + V_n\|^{\alpha}] \leq A^{\frac{n}{\alpha}}E[\sum_{k=1}^n \|V_k\|^{\alpha}]$$
Theorem 3.1 Let \( X \) be a separable Banach space which is \( G_\alpha \) for some \( 0 < \alpha \leq 1 \). Let \( \{ F_n : n \geq 1 \} \) be a sequence of independent set-valued random variables in \( K_\alpha(X) \), such that \( E[F_n] = \{0\} \) for each \( n \). If

\[
\sum_{j=1}^{\infty} E[\| F_j \|_\kappa]^{1+\alpha} < \infty
\]

where \( \phi(t) = t^{1+\alpha} \) for \( 0 \leq t \leq 1 \) and \( \phi(t) = t \) for \( t \geq 1 \), then \( \sum_{j=1}^{\infty} F_j \) converges with probability 1 in the sense of \( d_H \).

Proof. Define

\[
U_j = F_j I_{\{\|F\|_\kappa \leq 1\}} \quad \text{and} \quad W_j = F_j I_{\{\|F\|_\kappa > 1\}}.
\]

Note that \( F_j = U_j + W_j \) for each \( j \) and that both \( \{U_j : j \geq 1\} \) and \( \{W_j : j \geq 1\} \) are independent sequences of set-valued random variables. Next, for each \( m \) and \( n \)

\[
E[\| \sum_{j=n}^{\infty} W_j \|_\kappa] \leq \sum_{j=n}^{\infty} E[\| W_j \|_\kappa] \leq \sum_{j=n}^{\infty} E[\phi(\| F_j \|_\kappa)].
\]

That means \( \{E[\sum_{j=n}^{\infty} W_j \|_\kappa] : m \geq 1\} \) is a Cauchy sequence and hence

\[
E[\sum_{j=1}^{\infty} W_j \|_\kappa] \]

converges

as \( m \to \infty \). Since convergence in the mean implied convergence in probability, Ito and Nisio’s result in [12] for independent random elements (rf. Section 4.5) provides that

\[
\| \sum_{j=1}^{\infty} W_j \|_\kappa \]

converges in probability 1 as \( n \to \infty \).

Then for \( n, m \geq 1, m > n \), by triangular inequality we have

\[
d_H (\sum_{j=1}^{m} W_j, \sum_{j=1}^{n} W_j) = d_H (\sum_{j=m+1}^{n} W_j, \sum_{j=1}^{n} W_j)
\]

\[
\leq d_H (\{0\}, \sum_{j=m+1}^{n} W_j) = \| \sum_{j=m+1}^{n} W_j \|_\kappa \to 0, \ a.e. \ as \ n, m \to \infty.
\]

By the completeness of \( (K_\alpha(X), d_H) \), we can have \( \sum_{j=1}^{\infty} W_j \) converges almost everywhere in the sense of \( d_H \).

Since by equivalent definition of Hausdorff metric, we have

\[
E[\| \sum_{j=n}^{\infty} U_j \|_\kappa^{1+\alpha} ] = E[d_H (\sum_{j=n}^{\infty} U_j, \{0\})^{1+\alpha}]
\]

\[
= E[\sup_{x^* \in S} | s(x^*, \sum_{j=n}^{\infty} U_j) |]^{1+\alpha}.
\]

For any fixed \( n, m \), there exists a sequence \( x^*_k \in S^* \), such that

\[
\lim_{k \to \infty} | s(x^*_k, \sum_{j=n}^{m} U_j) | = \sup_{x^* \in S^*} | s(x^*, \sum_{j=n}^{m} U_j) |.
\]

Then by dominated convergence theorem, Minkowski inequality and Lemma 3.1, we have
\[ E[\sum_{j=1}^{\infty} U_j] = E[\lim_{k \to \infty} s(x_k, \sum_{j=1}^{m} U_j)] = \lim_{k \to \infty} E[\sum_{j=1}^{m} U_j] \]

\[ \leq \lim_{k \to \infty} E[|s(x_k, \sum_{j=1}^{m} U_j) - s(x_k, U_j)| + |E[s(x_k, \sum_{j=1}^{m} U_j)]|]^{1+\alpha} \]

\[ = \lim_{k \to \infty} E[|s(x_k, \sum_{j=1}^{m} U_j) - s(x_k, U_j)| + |E[s(x_k, \sum_{j=1}^{m} U_j)]|]^{1+\alpha} \]

\[ \leq 2^{1+\alpha} \left\{ A \lim_{k \to \infty} \sum_{j=1}^{m} E[|s(x_k, U_j)|] + \lim_{k \to \infty} \sum_{j=1}^{m} E[|s(x_k, W_j)|] \right\}^{1+\alpha} \]

for each \( n \) and \( m \). Thus, \( \{E[\sum_{j=1}^{\infty} U_j]: m \geq 1\} \) is a Cauchy sequence, and hence converges. Hence, by the similar way as above to prove \( \sum_{j=1}^{\infty} W_j \) converges with probability one in the sense of \( d_{\mu} \). We also can prove that

\[ \sum_{j=1}^{\infty} U_j \] converges

with probability one in the sense of \( d_{\mu} \). The result was proved. \( \square \)

From theorem 3.1, we can easily obtain the following corollary.

**Corollary 3.2** Let \( \mathcal{X} \) be a separable Banach space which is \( G_\alpha \) for some \( 0 < \alpha \leq 1 \). Let \( \{F_n\} \) be a sequence of independent set-valued random variables in \( \mathcal{K}_\alpha(\mathcal{X}) \) such that \( E[F_n] = \{0\} \) for each \( n \). If \( \phi_n : \mathbb{R}^+ \to \mathbb{R}^+ \), \( n = 1, 2, \cdots \), are continuous and such that \( \frac{\phi_n(t)}{t} \) and \( \frac{t^{1+\alpha}}{\phi_n(t)} \) are non-decreasing, then for each \( \alpha_n \subset \mathbb{R}^+ \) the convergence of

\[ \sum_{n=1}^{\infty} E[\frac{\phi_n(\|F_n\|)}{\phi_n(\alpha_n)}] \]

implies that

\[ \sum_{n=1}^{\infty} \frac{F_n}{\alpha_n} \]

converges with probability one in the sense of \( d_{\mu} \).

**Proof.** Let

\[ U_j = \frac{F_j}{\alpha_j} I_{[\|F_j\| < \alpha_j]} \quad \text{and} \quad W_j = \frac{F_j}{\alpha_j} I_{[\|F_j\| \geq \alpha_j]} \cdot \]

If \( \|F_n\| > \alpha_n \), by the non-decreasing property of \( \frac{\phi_n(t)}{t} \), we have

\[ \frac{\phi_n(\alpha_n)}{\alpha_n} \leq \frac{\phi_n(\|F_n\|)}{\|F_n\|} \cdot \]

That is
\[ \frac{\| F_n \|_K}{\alpha_n} \leq \frac{\phi_\alpha(\| F_n \|_K)}{\phi_\alpha(\alpha_n)}. \]  

(4.1)

If \( \| F_n \|_K \leq \alpha_n \), by the non-decreasing property of \( \frac{t^{1+a}}{\phi_\alpha(t)} \), we have

\[ \frac{\| F_n \|_K^{1+a}}{\phi_\alpha(\| F_n \|_K)} \leq \frac{\alpha_n^{1+a}}{\phi_\alpha(\alpha_n)}. \]

That is

\[ \frac{\| F_n \|_K^{1+a}}{\alpha_n^{1+a}} \leq \frac{\phi_\alpha(\| F_n \|_K)}{\phi_\alpha(\alpha_n)}. \]  

(4.2)

Then as the similar proof of theorem 3.1, we can prove both \( \sum_{j=1}^{\infty} U_j \) and \( \sum_{j=1}^{\infty} W_j \) converges with probability one, and the result was obtained. \( \square \)

**Acknowledgements**

The research was supported by NSFC(11301015, 11401016, 11171010), BJNS (1132008).

**References**


