Localization of Unbounded Operators on Guichardet Spaces

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Received 3 March 2015; accepted 23 June 2015; published 30 June 2015

Abstract

As stochastic gradient and Skorohod integral operators, \((\nabla, \delta)\) is an adjoint pair of unbounded operators on Guichardet Spaces. In this paper, we define an adjoint pair of operator \((\ell, \ell')\), where \(\ell_s = \nabla, E_s[C]\) with \(E_s[C]\) being the conditional expectation operator. We show that \(\ell_s\) (resp. \(\ell'_s\)) is essentially a kind of localization of the stochastic gradient operators (resp. Skorohod integral operators \(\delta\)). We examine that \(\ell_s\) and \(\ell'_s\) satisfy a local CAR (canonical anti-commutation relation) and \((\ell^\prime_s)_{s>0}\) forms a mutually orthogonal operator sequence although each \(\ell_s\) is not a projection operator. We find that \(\ell_s\) is s-adapted operator if and only if \(\nabla\) is s-adapted operator. Finally we show application exponential vector formulation of QS calculus.

Keywords
Stochastic Gradient Operator, Skorohod Integral Operator, Localization, Ex-Ponential Vector, Guichardet Spaces

1. Introduction

The quantum stochastic calculus [4] [6] developed by Hudson and Parthasarathy is essentially a noncommutative extension of classical Ito stochastic calculus. In this theory, annihilation, creation, and number operator processes in boson Fock space play the role of “quantum noises”, [2] which are in continuous time. On the other hand, the quantum stochastic calculus has been extended by Hitsuda is by means of the Hitsuda-Skorohod integral of anticipative process [3] [9] and the related gradient operator of Malliavin calculus. In this noncausal formulation the action of each QS integral is defined explicitly on Fock space vectors, and the essential quantum Ito formula is seen in terms of the Skorohod isometry.

In 2002, Attal [1] unify and extend both of the above approaches on Guichardet spaces. In this note, explicitly definitions of QS integrals provided and introduced no unnatural domain limitations. Moreover, maximality of operator domains is demonstrated for these QS integrals on Guichardet spaces.

In this argument, we define an adjoint pair of operator \((\ell, \ell')\), where \(\ell_s = \nabla, E_s[C]\) with \(E_s[C]\) being
the conditional expectation (operator). The motivation for this study comes from the following observations. It is known that \( E_s[C] \) is a projection operator on Guichardet Spaces. Hence, restricted to the range of \( E_s[C] \), \( \ell_s \) coincides with the stochastic gradient operator \( \nabla_s \). We show that \( \ell_s \) (resp. \( \ell'_s \)) is essentially a kind of localization of the stochastic gradient operators (resp. Skorohod integral operators \( \delta \)). We examine that \( \ell_s \) and \( \ell'_s \) can be called a local stochastic gradient operators (resp. local Skorohod integral operators \( \delta \)). Then, it is necessary and important to study a pair of operator \( (\ell_s, \ell'_s) \).

This paper is organized as follows. In Section 2, we fix some necessary notations and recall main notions and facts about unbounded operators on Guichardet spaces. In Section 3, Section 4 and Section 5, we state our main results. We first examined that \( \ell_s \) and \( \ell'_s \) satisfy a local CAR (canonical anti-communication relation) and \( (\ell'_s)_{s \geq 0} \) forms a mutually orthogonal operator sequence although each’s is not a projection operator. We find that \( \ell_s \) is \( s \)-adapted operator if and only if \( \nabla_s \) is \( s \)-adapted operator. Finally we show application exponential vector formulation of QS calculus.

2. Unbounded Operators on Guichardet Spaces

In this section, we fix some necessary notations and recall main notions and facts about unbounded operators on Guichardet spaces. For detail formulation of unbounded operators, we refer reader to [1].

Let \( \mathbb{R}_+ \) be the set of all nonnegative real numbers and \( \Gamma \) the finite power set of \( \mathbb{R}_+ \), namely

\[
\Gamma = \{ \sigma \in \mathbb{R}_+, \#\sigma < \infty \}
\]

where \( \#\sigma \) denotes the cardinality of \( \sigma \) as a set, with \( \Gamma^{(n)} \) denoting the collection of \( n \) element subsets. Obviously, \( \Gamma = \bigcup_{n=0}^{\infty} \Gamma^{(n)} \). Particularly, let \( \emptyset \in \Gamma^{(0)} \) be an atom of measure 1. We denote by \( L^2(\Gamma) \) the usual space of square integral real-valued functions on \( \Gamma \).

Fixing a complex separable Hilbert space \( \eta \), Guichardet space tensor product \( L^2(\Gamma; \eta) \), which we identify with the space of square-integrable functions \( L^2(\Gamma; \eta) \), and is denoted by \( F \). Guichardet space enjoys a continuous tensor product structure: for each \( s \geq 0 \) the map

\[
f \otimes g(\omega) = f(\omega_t)g(\omega_s)
\]

where \( \omega_t = \omega \cap [0,t) \), \( \omega_s = \omega \cap (s, \infty) \).

For a Hilbert space-valued map \( x : \Gamma \times \mathbb{R}_+ \rightarrow \eta \), let \( \delta(x) \) be the map \( \Gamma \rightarrow \eta \) given by

\[
\delta(x)(\sigma) = \sum_{\sigma \subset \sigma'} x_{\sigma'}(\sigma \setminus s)
\]

when \( \delta(x) \in F \), we call \( x \) is Skorohod integrable, \( \delta(x) \) is Skorohod integral operator on \( F \) and

\[
\text{Dom}\delta = (x \in L^2(\Gamma \times \mathbb{R}_+, \eta) : \delta(x) \in F)
\]

For a map \( f : \Gamma \rightarrow \eta \), let \( \nabla f \) and \( Df \) be the maps \( \Gamma \times \mathbb{R}_+ \rightarrow \eta \) given by

\[
\nabla f(\omega, s) = f(\omega \cup s), \quad Df(\omega, s) = 1_{|s|<\infty} f(\omega \cup s)
\]

when \( f \in F \), we call \( \nabla f \) and \( Df \) the stochastic gradient of \( f \) and the adapted gradient of \( f \), respectively. Moreover,

\[
\text{Dom}\nabla = \{ f \in F : \nabla f \in L^2(\Gamma \times \mathbb{R}_+, \eta) \}
\]

\[
\text{Dom}D = \{ f \in F : Df \in L^2(\Gamma \times \mathbb{R}_+, \eta) \}
\]

where \( \Gamma_s = \{ \omega \in \Gamma : \omega \subset [0,s) \} \). Obviously, if \( f \in F_s \), \( \nabla f = Df \), where \( F_s = L^2(\Gamma_s, \eta) \).

Let \( \Gamma_{ad} = \{ (\omega, s) \in \Gamma \times \mathbb{R}_+ : \omega < s \} \), the adapted projection on \( L^2(\Gamma \times \mathbb{R}_+, \eta) \) is the orthogonal projection onto the closed subspace \( L^2(\Gamma_{ad} \times \mathbb{R}_+, \eta) \):

\[
\text{P}_{ad} : (\omega, s) \mapsto 1_{|s|<\infty} x(s)
\]

Remark 2.1 As Hilbert space operators \( \delta \), \( \nabla \) and \( D \) are unbounded operators. \( (\delta, \text{Dom}\delta) \) and \( (\nabla, \text{Dom}\nabla) \) are closed, densely defined operators. Especially, \( \delta \) is adjoint operator of \( \nabla \) and
\[ \text{Dom}\delta \supset \text{Dom}\sqrt{N \otimes I}; \text{Dom}\mathbb{V} = \text{Dom}\sqrt{N} \]

where \( N \) is the number operator, \( Nf(\sigma) = \# \sigma f(\sigma) \) with maximal domain and \( I \) is identical operator.

**Lemma 2.1** \[1\] Let \( f \in F \) and \( x : \Gamma \times \mathbb{R}_+ \to \eta \) be Skorohod integrable, if the map

\[ (\omega, s) \mapsto \{ x_s(\omega), f(\omega \cup s) \} \]

is integrable, then

\[ \langle \delta(x), f \rangle = \iint \{ x_s(\omega), \nabla f(\omega) \} d\omega ds . \]  \hspace{1cm} (1)

**Lemma 2.2** \[1\] Let \( x : \Gamma \times \mathbb{R}_+ \to \eta \) be measurable. If \( P_t x \in F \) for almost every \( t \geq 0 \), then

\[ D_t \delta(x) = \delta_t(\nabla_x x) + P_t x , \]  \hspace{1cm} (2)

where (1) may call the canonical-commutation relations.

### 3. Local Skorohod Integral and Stochastic Gradient Operators

In the present section we state and prove our main results. We first make some preparations.

Let \( C \) be an operator on \( F \) with domain \( V \), we define an conditioned expectation operator \( E_s[C] \) on \( F \) by the a.e. prescription

\[ (E_s[C]f)(\omega) = (CP_D \delta_s, f)(\omega_s), \]

with domain

\[ \{ f \in D_s[V] : \sigma \mapsto 1_{s, \omega \cap \sigma} \} \text{ is square integrable } \Gamma \to F \}

where \( D_s[V] \equiv \{ f \in F : P_D \delta_s, f \in V(\subset F) \text{ for a.a. } \sigma > s \}, \omega_s = \omega \cap [0,s), \omega_s = \omega \cap (s,\infty) \).

Clearly, \( D_s[V] \) is a subspace of \( F \), and for any \( f \in D_s[V] \), we have \( p_s f \in D_s[V], d_s f \in D_s[V] \) for a.a. \( t > s \). Thus \( D_s[V] \) is an s-adapted subspace.

**Remark 3.1** If \( C \) is s-adapted(i.e. for all \( f \in \text{Dom} C \), \( P_t f = P_s f, P_D \delta t, f = CD_t f \) for a.a. \( t > s \)), then the subspaces \( F_s \cap \text{Dom} C \) and \( F_s \cap \text{Dom} E_s[C] \) coincide, and \( E_s[C] g = C g \) for \( g \) in this subspace, it follows that

\[ E_s[C] P_t f = CP_s f, E_s[C] D_s f = CD_t f \]

Whenever \( P_t f \) belongs to \( \text{Dom} C \). If \( C \) is densely defined, s-adapted and \( E_s[C] \) is closable, then \( E_s[C]^* = E_s[C] \).

**Remark 3.2** \( E_s[C] \) is s-adapted operator and \( E_s[C] \in F \).

**Definition 3.1** For \( s \in \mathbb{R}_+ \), we call \( \ell_s = \nabla_x E_s[C] \) the local stochastic gradient operator and its adjoint operator \( \ell_s^* = \delta E_s[C] \) is the local Skorohod integral operator. And operator domain of \( \ell_s \) is given by

\[ \text{Dom} \ell_s = \text{Dom} \nabla_s \cap \text{Dom} E_s[C] \]

where \( C \) is operator on \( F \).

We note that for \( f \in \text{Dom} \ell_s \),

\[ (\nabla_x E_s[C] f)(\omega) = \nabla_x (E_s[C] f)(\omega) = CP_s \delta_s f(\omega \cup s) \]

\[ (E_s[C] \nabla_x f)(\omega) = E_s[C] (\nabla_x f)(\omega) = CP_s \delta_s f(\omega \cup s) \]

hence, \( \nabla E_s[C] = E_s[C] \nabla_s \). Especially, when \( C = P_s, f(\omega) = P_s \delta_s f(\omega_s) \), we have \( \ell_s = \nabla_s \).

**Theorem 3.1** By lemma2.2, we can get the following relations

\[ \ell_s \ell_s^* = \ell_s^* \ell_s + E_s[C] \]  \hspace{1cm} (3)

which we may call the local CAR(canonical anti-commutation relations).
Proof we note that
\[ \ell_t \ell_s^* = \mathcal{V}_t \mathcal{E}_s[C] \delta E_s[C] = E_s[C] \mathcal{V}_t \mathcal{E}_s[C] = E_s[C] (\delta (\mathcal{V}_t \mathcal{E}_s[C]) + E_s[C]) = \ell_t^* \ell_s^* + E_s[C] \]

The next theorem shows that \( \ell_s \) is not a projection operator on \( F \).

**Theorem 3.2** \( \ell_t \ell_s = 0 \), whenever \( t \geq s \) and \( t, s > 0 \).

**Proof** Let \( t, s > 0 \) with \( t \geq s \). The following algebraic relations are evident for \( t \geq s \):
\[ P_s P_s f = P_s f, \]
\[ D_s D_s f = D_s f = 0, \]
\[ D_s P_s f = P_s D_s f = 0. \]

We show that \( E_s[C] \mathcal{V}_t \mathcal{E}_s[C] = \mathcal{V}_t E_s[C] \mathcal{E}_s[C] \), thus
\[ \ell_t \ell_s^* = \mathcal{V}_t E_s[C] \mathcal{E}_s[C] \delta = (\mathcal{V}_t E_s[C]) \delta = 0. \]

Now, if \( t < s \), then by the result of the first step we have
\[ \ell_t \ell_s^* (\ell_t \ell_s^* )^* = 0. \]

This completes the proof.

**Theorem 3.3** \( \ell_t \ell_s = 0 \), whenever \( t \neq s \) and \( t, s > 0 \).

**Proof** Let \( t \neq s \) and \( t, s > 0 \). If \( t > s \), then we can show that \( \mathcal{V}_t E_s[C] = 0 \), from which it follows that
\[ \ell_t \ell_s^* = \mathcal{V}_t E_s[C] \mathcal{E}_s[C] \delta = (\mathcal{V}_t E_s[C]) \delta = 0. \]

On the other hand, if \( \ell_s \) is s-adapted, \( \mathcal{V}_s \) is also s-adapted.

### 4. Application to Exponential Vector Formulation of QS Calculus

Recall that in the exponential vector formulation of QS calculus, all processes are defined on a domain of the algebraic tensor product form \( V_0 \odot \mathcal{S}(S) \), where \( V_0 \) is a dense subspace of \( \eta \) and \( \mathcal{S}(S) = \text{Lin}(\varepsilon(\phi) : \phi \in S) \) which \( S \) is a subset of \( L(\mathbb{R}_+) \) and \( \varepsilon(\phi) \) denotes the exponential vector of the test function \( \phi \) which in Guichardet spaces given by \( \varepsilon(\phi) = \prod_{s > 0} \phi(s) \).

For all \( s \) and a.e. \( t \), we have
\[ P_s \varepsilon(\phi) = \varepsilon(\phi_{[0,s]}), \quad \text{and} \quad D_t \varepsilon(\phi) = \phi(t) \varepsilon(\phi_{[0,t]}), \quad \text{(4)} \]

since, the domain of the form \( V_0 \odot \mathcal{S}(S) \) are s-adapted. Note the a.e. identity
\[ D_t \varepsilon(\phi) = \epsilon(\phi_{[0,t]}), \quad \text{where} \quad t = \land t. \quad \text{(5)} \]

**Theorem 4.1** \( \ell_s \) be an operator on \( F \) with domain of the form \( V_0 \odot \mathcal{S}(S) \). Then \( \ell_s \) is s-adapted if and only if, for all \( \nu \in V_0 \) and \( \phi \in S \):
\[ \ell_s \varepsilon(\phi_{[0,s]}), \quad \ell \varepsilon(\phi) = \ell \varepsilon(\phi_{[0,t]}), \quad \text{(6)} \]

where \( F_s = \eta \odot L(\Gamma_s) \).

**Proof** By definition of \( \ell_s \), \( \ell_s \) be an operator on \( F \) with domain of the form \( V_0 \odot \mathcal{S}(S) \). We note that if \( \ell_s \) is s-adapted, then \( \ell_s \varepsilon(\phi_{[0,s]}), \quad \ell \varepsilon(\phi) = \ell \varepsilon(\phi_{[0,t]}), \quad \text{for} \quad \omega > s. \)
\[ PD_w \varphi = \nu(\varphi_{(0,a)}) \ast \nu(\varphi_{(\sigma,\infty)})(\omega), \]

and so, for a.a. \( \omega \),

\[
(\ell, \nu(\varphi))(\omega) = (\ell, \nu(\varphi_{(0,a)}))(\omega)(\varphi_{(\sigma,\infty)})(\omega),
\]

\[
= (\ell, \nu(\varphi_{(0,a)}))(\ast)(\nu(\varphi_{(\sigma,\infty)}))(\omega).
\]

**Acknowledgements**

The authors are extremely grateful to the referees for their valuable comments and suggestions on improvement of the first version of the present paper. The authors are supported by National Natural Science Foundation of China (Grant No. 11261027 and No. 11461061).

**References**


