A Note on Discontinuous Functions with Continuous Second Iterate

Pingping Zhang¹, Xiaobing Gong²

¹Department of Mathematics, Binzhou University, Shandong, China
²Department of Mathematics, Neijiang Normal University, Sichuan, China

Email: zhangpingpingmath@163.com, xbgong@163.com

Received 6 May 2015; accepted 26 June 2015; published 30 June 2015

Abstract

This paper investigates four classes of functions with a single discontinuous point. We give the sufficient and necessary conditions under which the second order iterates are continuous functions. Furthermore, the sufficient conditions for the continuity of the even order iterates with finitely many discontinuous points are obtained.

Keywords

Iteration, Discontinuous Point, Continuous Function

1. Introduction

For a nonempty set $X$ and $n \in \mathbb{N}$, the $n$-th iterate of a self-mapping $f : X \to X$ is defined by $f^n(x) = f(f^{n-1}(x))$ and $f^0(x) = x$ for all $x \in X$ inductively. As a nonlinear operator, iteration usually amplifies the complexity of functions [1]-[7], computing the $n$-th iterate of functions is complicated, even for simple functions (see [8]-[12]). On the other hand, iteration can turn complex functions into simple ones. Recently, the following problem was first formulated by X. Liu, L. Liu and W. Zhang: what are discontinuous functions whose iterates of a certain order are continuous? This question, together with three classes of discontinuous functions defined on compact interval, was answered in the affirmative in [13]. That is, suppose that $f : [0,1] \to [0,1]$ with a single discontinuous point (removable discontinuous point, jumping discontinuous or oscillating discontinuous), the authors respectively gave the sufficient and necessary conditions under which the second order iterates are continuous functions.

The purpose of this paper is to study the discontinuous functions defined on open interval. For four classes of discontinuous functions with unique discontinuous point, we obtain the sufficient and necessary conditions for
functions being continuous ones under second iterate, which are easily verified respectively. As corollaries, the sufficient conditions for the continuity of the even order iterates with finitely many discontinuous points are obtained. Our results are illustrated by examples in Section 3.

2. Main Results

In this section the main results for the continuity of \( f^2 \) are stated. Throughout the paper we let \( I := (0,1) \).

**Theorem 1.** Suppose that \( f : I \to I \) has unique removable discontinuous point \( x_r \). Let \( c_r := f(x_r) \) and \( y_r := \lim_{x \to x_r} f(x) \).

Then \( f^2 \) is continuous on \( I \) if and only if the following conditions are fulfilled:

\[
(A_1) \quad f(c_r) = \lim_{x \to x_r} f(x),
\]

\[
(A_2) \quad x_r \not\in f(I \setminus x_r).
\]

**Proof.** \((\Rightarrow)\) Assume that \( f^2 \) is continuous on \( I \), the removable discontinuous point \( x_r \) of \( f \) is continuous point of \( f^2 \) under iteration. Whether \( y_r \) defined by (1) is continuous point of \( f \) or not, we have

\[
\lim_{x \to x_r} f^2(x) = \lim_{x \to x_r} f(x). \tag{2}
\]

On the other hand, using the definition of \( c_r \) and the continuity of \( f^2 \),

\[
f(c_r) = f^2(x_r) = \lim_{x \to x_r} f^2(x). \tag{3}
\]

Thus (2) and (3) lead to (A1). For an indirect proof of (A2), assume that \( x_r = f(\tilde{x}) \) for \( \tilde{x} \in I \setminus x_r \). Then

\[
\lim_{x \to x_r} f^2(x) = \lim_{x \to x_r} f(x) \neq f(x_r) = f^2(\tilde{x}),
\]

which contradicts the continuity of \( f^2 \) on \( I \) and gives a proof to (A2).

\((\Leftarrow)\) It follows from (A1)

\[
\lim_{x \to x_r} f^2(x) = \lim_{x \to x_r} f(x) = f(c_r) = f^2(x_r),
\]

implying that \( f^2 \) is continuous at \( x_r \). The condition (A2), i.e., \( x_r \not\in f(I \setminus x_r) \), shows that all points \( x \in I \setminus x_r \) are continuous points of \( f^2 \). Therefore \( f^2 \) is continuous on \( I \). This completes the proof. \( \square \)

**Corollary 1.** Suppose that \( f : I \to I \) has finitely many removable discontinuous points \( x_1, x_2, \ldots, x_m \). If the following conditions

\[
(A_1) \quad f(c_{i}) = \lim_{x \to x_{i}} f(x),
\]

\[
(A_2) \quad x_i \not\in f(I \setminus \{x_1, x_2, \ldots, x_m\}),
\]

are fulfilled for all \( i = 1, 2, \ldots, m \), where \( c_i := f(x_i) \), \( y_i := \lim_{x \to x_i} f(x) \). Then \( f^{2n} \) is continuous on \( I \) for arbitrary integer \( n \geq 1 \).

**Proof.** By using the sufficiency of Theorem 1, the assumption \((\overline{A}_i)\) implies that \( f^2 \) is continuous on those points \( x_1, x_2, \ldots, x_m \) and \((\overline{A}_2)\) guarantees that all points \( x \in I \setminus \{x_1, x_2, \ldots, x_m\} \) are continuous points of \( f^2 \). Thus \( f^2 \) is continuous on \( I \). Since the composition of continuous functions is continuous, we get the continuity of \( f^{2n} \) for all integers \( n \geq 1 \) inductively. This completes the proof. \( \square \)

**Theorem 2.** Suppose that \( f : I \to I \) has unique jumping discontinuous point \( x_j \). Let \( c_j := f(x_j) \) and \( y_{j,0} := \lim_{x \to x_j, 0} f(x) \) and \( y_{j,0} := \lim_{x \to x_j, 0} f(x) \).

Then \( f^2 \) is continuous on \( I \) if and only if the following conditions are fulfilled:

\[
(B_1) \quad f(c_j) = \lim_{x \to x_j, 0} f(x) = \lim_{x \to x_j, 0} f(x),
\]

\[
(B_2) \quad x_j \not\in f(I \setminus x_j) \).
Proof. \((\Rightarrow)\) In view of the definitions of \(e_j, y_{j,0}, y_{j,0}^{+}\) and the continuity of \(f^2\), we get
\[
f(c_j) = f^2(x_j) = \lim_{x \to y_j} f^2(x) = \lim_{x \to y_j^+} f(x), \tag{4}
\]
and
\[
f(c_j) = f^2(x_j) = \lim_{x \to y_j} f^2(x) = \lim_{x \to y_j^-} f(x). \tag{5}
\]
Clearly, (4) and (5) yield (B₁). Suppose the contrary to (ii), there is \(x_j = f(\bar{x})\) for some \(\bar{x} \in I \setminus x_j\). The limit
\[
\lim_{x \to x_j} f^2(x) = \lim_{x \to x_j} f(x)
\]
is nonexistence since \(x_j\) is a jumping discontinuous point of \(f\), which contradicts the fact that \(f^2\) is continuous at the point \(\bar{x}\). This contradiction proves (B₂).

\((\Leftarrow)\) The condition (B₁) implies
\[
\lim_{x \to x_j} f^2(x) = \lim_{x \to x_j} f(x) = f(c_j) = f^2(x_j), \tag{6}
\]
and
\[
\lim_{x \to x_j} f^2(x) = \lim_{x \to x_j} f(x) = f(c_j) = f^2(x_j). \tag{7}
\]
Thus, (6) and (7) lead to
\[
\lim_{x \to x_j} f^2(x) = f^2(x_j),
\]
which implies that the jumping discontinuous point \(x_j\) of \(f\) change into the continuous point of \(f^2\). Using the similar argument as the sufficiency for (B₂) in Theorem 1, we can prove that all points \(x \in I \setminus x_j\) are continuous points of \(f^2\). Thus \(f^2\) is continuous on \(I\). That is, we prove the sufficiency. This completes the proof. \(\square\)

Corollary 2. Suppose that \(f : I \to I\) has finitely many jumping discontinuous points \(x_1, x_2, \ldots, x_m\). If the following conditions
\begin{enumerate}
\item[(B₁)] \(f(c_j) = \lim_{x \to y_j} f(x) = \lim_{x \to y_j^+} f(x),\)
\item[(B₂)] \(x_j \not\in f\left(I \setminus \{x_1, x_2, \ldots, x_m\}\right),\)
\end{enumerate}
are fulfilled for all \(i = 1, 2, \ldots, m\), where \(c_j := f(x_j), y_{j,0} := \lim_{x \to y_{j,0}} f(x), y_{j,0}^+ := \lim_{x \to y_{j,0}^+} f(x)\). Then \(f^{2n}\) is continuous on \(I\) for arbitrary integer \(n \geq 1\).

Proof. The discussion is similar as that of Corollary 1. By using the sufficiency of Theorem 2, the assumption \((\mathcal{B}_1)\) implies that \(f^2\) is continuous on those points \(x_1, x_2, \ldots, x_m\) and \((\mathcal{B}_2)\) implies that \(x \in I \setminus \{x_1, x_2, \ldots, x_m\}\) are all continuous points of \(f^2\). Thus \(f^2\) is continuous on \(I\). Consequently, we obtain the continuity of \(f^{2n}\) for all integers \(n \geq 1\) inductively. This completes the proof. \(\square\)

Theorem 3. Suppose that \(f : I \to I\) has unique oscillating discontinuous point \(x_o\). Then \(f^2\) is continuous on \(I\) if and only if the following conditions are fulfilled:
\begin{enumerate}
\item[(C₁)] \(f^2(x) = f^2(x_o)\) on a neighborhood \(U(x_o)\),
\item[(C₂)] \(x_o \not\in f\left(I \setminus x_o\right)\).
\end{enumerate}

Proof. \((\Rightarrow)\) We first show that the condition (C₁) holds. Suppose the contrary, for any \(\delta > 0\) there exists a corresponding point \(y_{\delta} \in U(x_o, \delta)\) satisfying \(f^2(y_{\delta}) \neq f^2(x_o)\). Put \(\varepsilon_0 = \left| f^2(y_{\delta}) - f^2(x_o) \right|\), then for \(\delta > 0\) there is \(y_{\delta} \in U(x_o, \delta)\) such that \(\left| f^2(y_{\delta}) - f^2(x_o) \right| = \varepsilon_0\), implying \(f^2\) is discontinuous at \(x_o\), a contradiction. This gives a proof to (C₁). To prove (C₂), by reduction to absurdity, we assume that \(x_o \not\in f\left(I \setminus x_o\right)\), there is \(\hat{x} \in I \setminus x_o\) such that \(x_o = f(\hat{x})\). Note that \(x_o\) is oscillating discontinuous point of \(f\), the limit
\[
\lim_{x \to x_o} f^2(x) = \lim_{x \to x_o} f(x)
\]
is nothingness, which contradicts the continuity of \( f^2 \). Therefore, the claim (C2) is proved.

(\( \Leftarrow \)) From the assumption (C1) we see that
\[
\lim_{x \to x_0} f^2(x) = \lim_{x \to x_0} f^2(x) = f^2(x_0),
\]
implying the oscillating discontinuous point \( x_0 \) of \( f \) is a continuous point of \( f^2 \). On the other hand, one can use the similar argument as the sufficiency for the condition (C2) in Theorem 1 and prove that all points \( x \in I \setminus x_0 \) are continuous points of \( f^2 \). This completes the proof. \( \blacksquare \)

**Corollary 3.** Suppose that \( f : I \to I \) has finitely many oscillating discontinuous points \( x_1, x_2, \ldots, x_m \). If the following conditions
\[
(\overline{C}_1) \quad f^2(x) = f^2(x) \quad \text{on a neighborhood } U(x_i), \\
(\overline{C}_2) \quad x_i \not\in f(I \setminus \{x_1, x_2, \ldots, x_m\}),
\]
are fulfilled for all \( i = 1, 2, \ldots, m \). Then \( f^{2n} \) is continuous on \( I \) for arbitrary integer \( n \geq 1 \).

**Proof.** The discussion is similar as that of Corollary 1. By using the sufficiency of Theorem 3, the second iterate \( f^2 \) is continuous on \( x_1, x_2, \ldots, x_m \) by \( (\overline{C}_1) \) and is continuous on all points \( x \in I \setminus \{x_1, x_2, \ldots, x_m\} \) from \( (\overline{C}_2) \), thus \( f^2 \) is continuous on \( I \). Consequently, we have the continuity of \( f^{2n} \) for all integers \( n \geq 1 \) inductively. This completes the proof. \( \blacksquare \)

**Theorem 4.** Suppose that \( f : \mathbb{R} \to \mathbb{R} \) has unique infinite discontinuous point \( x_i \). Then \( f^2 \) is continuous on \( \mathbb{R} \) if and only if the following conditions are fulfilled:

\[
(D_1) \quad f^2(x_i) = \lim_{x \to x_i} f^2(x) = \lim_{x \to x_i} f(x), \\
(D_2) \quad x_i \not\in f(\mathbb{R} \setminus x_i).
\]

**Proof.** (\( \Rightarrow \)) Note that \( f^2 \) is continuous on \( \mathbb{R} \), then the infinite discontinuous point \( x_i \) of \( f \) is a continuous point of \( f^2 \), i.e.,
\[
f^2(x_i) = \lim_{x \to x_i} f^2(x) = \lim_{x \to x_i} f(x),
\]
which shows the limit \( \lim_{x \to x_i} f(x) \) exists and is equivalent to \( f^2(x_i) \). This implies the result \( (D_1) \). To prove \( (D_2) \), suppose the contrary, there exists a point \( \bar{x} \in \mathbb{R} \setminus x_i \) such that \( x_i = f(\bar{x}) \). Since \( x_i \) is infinite discontinuous point of \( f \), the limit
\[
\lim_{x \to x_i} f^2(x) = \lim_{x \to x_i} f(x)
\]
is infinite, which contradicts the continuity of \( f^2 \). Thus, the necessary proof of \( (D_2) \) is completed.

(\( \Leftarrow \)) From the assumption \( (D_1) \) and the fact \( \lim_{x \to x_i} f^2(x) = \lim_{x \to x_i} f(x) \), one can see that
\[
f^2(x_i) = \lim_{x \to x_i} f^2(x) = A,
\]
implying the infinite discontinuous point \( x_i \) of \( f \) is a continuous point of \( f^2 \). If \( (D_2) \) holds, then all real numbers \( x \neq x_i \) are continuous points of \( f^2 \). This completes the proof. \( \blacksquare \)

**Corollary 4.** Suppose that \( f : (a, +\infty) \to (a, +\infty) \) (or \( f : (a, +\infty) \to (a, \infty) \)) has unique infinite discontinuous point \( x_i \), where \( a, b \in \mathbb{R} \). Then \( f^2 \) is continuous on \( (a, +\infty) \) (or \( (a, \infty) \)) if and only if the following conditions are fulfilled:

\[
(D_1) \quad f^2(x_i) = \lim_{x \to x_i} f^2(x) = \lim_{x \to x_i} f(x) = \text{constant}, \\
(D_2) \quad x_i \not\in f((a, +\infty) \setminus x_i) \text{ (or } x_i \not\in f((a, \infty) \setminus x_i)).
\]

Proceding similarly as Theorem 4 one can show this corollary.

**Corollary 5.** Suppose that \( f : \mathbb{R} \to \mathbb{R} \) has finitely many infinite discontinuous points \( x_1, x_2, \ldots, x_m \). If the following conditions
\[
(D_1) \quad f^2(x_i) = \lim_{x \to x_i} f^2(x) = \text{constant}, \\
(D_2) \quad x_i \not\in f(\mathbb{R}),
\]
are fulfilled for all \( i = 1,2,\cdots,m \). Then \( f^{2n} \) is continuous on \( \mathbb{R} \) for arbitrary integer \( n \geq 1 \).

**Proof.** We obtain the result by using the similar argument as Corollary 1. In view of the sufficiency of Theorem 4, the second iterate \( f^2 \) is continuous on those points \( x_1,x_2,\cdots,x_m \) from \( (\mathcal{D}_1) \) and is continuous on all points \( x \in I \setminus \{x_1,x_2,\cdots,x_m\} \) from \( (\mathcal{D}_2) \), thus \( f^2 \) is continuous on \( I \). Then we have the continuity of \( f^{2n} \) for all integers \( n \geq 1 \) inductively. This completes the proof. \( \square \)

3. Examples

In this section we demonstrate our theorems with examples.

**Example 1.** Consider the mapping \( F_1 : (-3,3) \to (-3,3) \) defined by

\[
F_1(x) = \begin{cases} 
\frac{1}{2}x + 2, & x \in (-3,-2), \\
1, & x = -2, \\
x + 3, & x \in (-2,-1), \\
2, & x \in (-1,3).
\end{cases}
\]

Clearly, \( x_1 = -2 \) is the unique removable discontinuous point of \( F_1 \). By simple calculation, we have

\[
c_1 := F_1(-2) = -1, \quad y_1 := \lim_{x \to -2} F_1(x) = 1,
\]

\[
F_1(c_1) = \lim_{x \to y_1} F_1(x) = 2.
\]

Moreover, the set \( F_1((-3,3) \setminus \{-2\}) \subset \left[\frac{1}{2}, 2\right] \bigcup \{-1\} \) is not include the point \(-2\). By using the sufficiency of Theorem 1, we obtain the continuity of \( F_1^2 \) on \((-3,3)\).

**Example 2.** Consider the mapping \( F_2 : (-3,4) \to (-3,4) \) defined by

\[
F_2(x) = \begin{cases} 
\frac{1}{2}x + \frac{7}{2}, & x \in (-3,-1], \\
\frac{3}{5}x + \frac{21}{10}, & x \in \left(-1,\frac{3}{2}\right], \\
-6x + 12, & x \in \left[\frac{3}{2},2\right], \\
3x - 6, & x \in (2,4).
\end{cases}
\]

Clearly, \( x_2 = -1 \) is the unique jumping discontinuous point of \( F_2 \). By calculating we have

\[
c_2 := F_2(-1) = 3, \quad y_{1,-1} := \lim_{x \to -1} F_2(x) = 3, \quad y_{1,3} := \lim_{x \to 3} F_2(x) = \frac{3}{2},
\]

\[
F_2(c_2) = \lim_{x \to y_{1,-1}} F_2(x) = \lim_{x \to y_{1,3}} F_2(x) = 3,
\]

and \( F_2((-3,4) \setminus \{-1\}) = [0,6) \) is not include the points \(-1\). Then \( F_2^2 \) is continuous on \((-3,4)\) using the sufficiency of Theorem 2.

**Example 3.** Consider the mapping \( F_3 : (4,8) \to (4,8) \) defined by

\[
F_3(x) = \begin{cases} 
7 - \sin x, & x \in \left(\frac{4}{2}, \frac{9}{2}\right], \\
7 + \sin \frac{1}{x - 5}, & x \in \left(\frac{9}{2}, 6\right], \\
7 + \sin 1, & x \in (6,8).
\end{cases}
\]
Clearly, \( x_j = 5 \) is an oscillating discontinuous point of \( F_j \). By calculating, \( F_j ((4.8) \setminus \{5\}) \subset [6, 8] \) is not
include 5. Moreover, \( F_j^2 (x) = F_j^2 (5) = 7 + \sin 1 \) for \( x \in \left( \frac{9}{2}, 6 \right) \). Thus the function \( F_j \) is continuous on
\((4, 8)\) by the sufficiency of Theorem 3.

**Example 4.** Consider the mapping \( F_4 : (-\infty, +\infty) \rightarrow (-\infty, +\infty) \) defined by
\[
F_4 (x) = -\frac{1}{|x-2|}.
\]

Clearly, \( x_4 = 2 \) is an infinite discontinuous point of \( F_4 \). By calculating we have
\[
F_4^2 (2) = \lim_{x \to \infty} F_4 (x) = 0
\]
and \( F_4 \left( (-\infty, +\infty) \setminus \{2\} \right) \ni (-\infty, 0) \) is not include 2, then \( F_4 \) is continuous on \((-\infty, +\infty)\) by the sufficiency of
Theorem 4.

**Acknowledgements**
We thank the Editor and the referee for their comments. Project supported by Shandong Provincial Natural
Science Foundation of China (ZR2014AL003), Scientific Research Fund of Sichuan Provincial Education
Departments (12ZA086), Scientific Research Fund of Shandong Provincial Education Department (J12L59) and
Doctoral Fund of Binzhou University (2013Y04).

**References**


http://dx.doi.org/10.1007/978-1-84996-486-9

http://dx.doi.org/10.1007/978-1-4757-4017-2


