New Model for $L^2$ Norm Flow

Jiaojiao Li, Meixia Dou

Department of Mathematics, Henan Normal University, Xinxiang, China
Email: lijiaojiao8219@163.com, 243410009@qq.com

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Abstract

We introduce a new $L^2$ norm preserving heat flow in matrix geometry. We show that this flow exists globally and preserves the positivity property of Hermitian matrices.

Keywords

Global Flow, Norm Conservation, Positivity

1. Introduction

In this paper we introduce a new evolution equation in the matrix geometry such that the $L^2$ norm is preserved. In [1], the author introduced the Ricci flow which exists globally when the initial matrix is a positive definite. The Ricci flow [2] [3] preserves the trace of the initial matrix and the flow converges the scalar matrix with the same trace as the initial matrix. In [4], we have introduced the heat equation, which also preserves the trace of the initial matrix. In [5]-[8], the authors introduce the norm preserving flows which are global flows and converge to eigenfunctions. We know that the fidelity of quantum state is an important subject in quantum computation and quantum information [9] [10], the $L^2$ norm flow we studied is very closely related to the fidelity. This is the motivation of the study of norm preserving flow in matrix geometry.

To introduce our new $L^2$ norm flow in matrix geometry, we need to use some language from the book [11] and the papers [1] [4] [12]. Let $x,y$ be two Hermitian matrices on $C^n$. Define $u = e^{2\pi i x}$, $v = e^{2\pi i y}$. We use $M_n$ to denote the algebra of all $n \times n$ complex matrices which generated by $u$ and $v$ with the bracket $[u,v] = uv - vu$. Then $CI$, which is the scalar multiples of the identity matrix $I$, is the commutant of the operation $[u,v]$. Sometimes we simply use $I$ to denote the $n \times n$ identity matrix.

We define two derivations $\delta_1$ and $\delta_2$ on the algebra $M_n$ by the commutators

$$\delta_1 := [\cdot, x], \delta_2 := -[\cdot, x]$$

and define the Laplacian operator on $M_n$ by

$$\hat{\Delta} = \delta_1^2 + \delta_2^2$$

where we have used the Einstein sum convention. We use the Hilbert-Schmidt norm $|\cdot|$ defined by the inner product

$$<a,b> := \tau(a^*b)$$

on the algebra $M_n$ and let $\sigma(a) = \langle a, a \rangle = \tau(a)$. Here $a^*$ is the Hermitian adjoint of the matrix $a$ and $\tau$ denotes the usual trace function on $M_n$. We now state basic properties of $\delta_1$, $\delta_2$ and $\hat{\Delta}$ (see also [1]) as follows.

Given a positive definite Hermitian matrix $A \in M_n$. For any $c \in M_n$, we define the Dirichlet energy

$$ D(c) = \sum \langle \delta_1 c, \delta_2 c \rangle + \langle A c, c \rangle $$

and the $L^2$ mass

$$ M(c) = \langle c, c \rangle. $$

Let, for $c \neq 0$,

$$ \lambda(c) = \frac{D(c)}{M(c)}. $$

Then the eigenvalues of the operator $\hat{\Delta} + A = \hat{L}$ correspond to the critical values of the Dirichlet energy $D(c)$ on the sphere

$$ \Sigma = \{ c \in M_n; M(c) = 1 \}.$$

We consider the evolution flow

$$ c_t = -\hat{L} c + \lambda(c) c = -\hat{\Delta} c - A c + \lambda(c) c $$

with its initial matrix $c_{|t=0} = c_0 \in M_n$. Assume $c = c(t)$ is the solution to the flow above. Then

$$ \frac{1}{2} \frac{d}{dt} M(c) = \langle c, \dot{c} \rangle = -\langle \hat{L} c, c \rangle + D(c). $$

Since $\langle c, \hat{L} c \rangle = D(c)$, we know that $\frac{d}{dt} M(c) = 0$. Then

$$ M(c) = M(c_0). $$

The aim of this paper is to show that there is a global flow to (1.1) with the initial data $c_0 \in M_n$ and the flow preserves the positivity of the initial matrix.

### 2. Existence of the Global Flow

Firstly, we consider the local existence of the flow (1.1). We prefer to follow the standard notation and we let $L = \Delta - A$, where $\Delta$ is a positive definite Hermitian matrix. Let $a = a(t) \in M_n$ be such that

$$ a_t = L a + \lambda(t) a, $$

with the initial matrix $a_{|t=0} = a_0$. Here $a_0 \in M_n$ such that $|a|^2(0) = |a_0|^2 = 1$. Then for $a = a(t)$, we let

$$ \lambda(t) = \frac{-\langle L a, a \rangle}{\langle a, a \rangle}. $$

Formally, if the flow (2.1) exists, then we compute that

$$ \frac{d}{dt} |a|^2 = 2 \langle a, a_t \rangle = 2 \langle a, L a \rangle + 2 \lambda(t) < a, a > = 0. $$

Then $|a|^2(t) = |a|^2(0) = 1, \forall t > 0$.

In this section, our aim is to show that there is a global solution to Equation (2.1) for any initial matrix $a_0 \in M_n$ with $|a_0|^2 = 1$.

Assume at first that $\lambda(t) \geq 0$ is any given continuous function and $a = a(t)$ is the corresponding solution of (2.1). Define $b = e^{-\lambda(t)} a$. Then $b(0) = a(0)$ and we get
\[ b_i = e^{-\lambda(t)i}b + e^{-\lambda(t)i}a_i = -\lambda(t)e^{-\lambda(t)i}a + e^{-\lambda(t)i}(La + \lambda(t)a) \]  
\[ = e^{-\lambda(t)i}La = L(e^{-\lambda(t)i}a) = Lb. \]  

The Equation (2.3) can be solved by standard iteration method and we present it in below. Assume \((\phi_i)\) and \((\lambda_i)\) are eigen-matrices and eigenvalues of \(L\) as we introduced in [4], such that 
\[ -L\phi_i = \lambda_i\phi_i, <\phi_i, \phi_j> = \delta_{ij}. \]

Note that \(\lambda_i \geq 0\).

Assume that \(b = b(t)\) is the solution to (2.3). Set 
\[ b = \sum <b, \phi_i> \phi_i = \sum u_i \phi_i, u_i \in R, u_i = u_i(t). \]

Then by (2.3), we obtain 
\[ (u_i), \phi_i = L(u_i, \phi_i) = -u_i \lambda_i \phi_i. \]

Then \((u_i) = -u_i \lambda_i\), and \(u_i = u_i(0)e^{-\lambda_i t}\).

Hence 
\[ b = \sum u_i(0)e^{-\lambda_i t}\phi_i, \]
and 
\[ a = \sum u_i(0)e^{-\lambda_i t+\lambda(t)i}\phi_i \]

solves (2.1) with the given \(\lambda(t) \geq 0\).

Next we define a iteration relation to solve (2.1) for the unknown \(\lambda(t)\) given by (2.2).

Define \(a_i\) such that it solves the equation \(a_i = La + \lambda_i a\) with \(\lambda_0 = -\frac{<La_0, a_0>}{<a_0, a_0>}\).

Let \(k \geq 1\) be any integer. Define \(a_{k+1}\) such that 
\[ (a_{k+1}) = La_{k+1} + \lambda_{k+1}(t)a_{k+1}, a_{k+1}(0) = a_0, \]
with 
\[ \lambda_{k+1}(t) = -\frac{<La_k, a_k>}{<a_k, a_k>}. \]

Then using the Formula (2.4), we get a sequence \((a_k)\).

We claim that \((a_k) \subset M_k\) is a bounded sequence and \((\lambda_k(t))\) is also a bounded sequence.

It is clear that \((a_k) \subset M_k\). If this claim is true, we may assume 
\[ a_k \rightarrow a, \lambda_k(t) \rightarrow \lambda(t). \]

Then by (2.5) and (2.6), we obtain 
\[ a_i = La + \lambda(t)a \]
and 
\[ \lambda(t) = -\frac{<La, a>}{<a, a>}, \]

which is the same as (2.1). That is to say, \(a = a(t)\) obtained above is the desired solution to (2.1).

Firstly we prove the claim in a small interval \([0, T]\). Assume \(|a_k| \leq A = 1.5\) and \(|\lambda_k| \leq B = \log(4 / 3)\) on \([0, T], T = 1/2\). Then, by (2.5), 
\[ \frac{1}{2} |a_{k+1}|^2 = <a_{k+1}, a_{k+1}, > <a_{k+1}, La_{k+1}, > + \lambda_{k+1} |a_{k+1}|^2 . \]  
\[ (2.7) \]
By (2.6), we obtain \( \lambda_{k+1} = \frac{-<La_{k+1}, a_{k+1}>}{|a_{k+1}|^2} \). Then

\[
-<La_{k+1}, a_{k+1}> = \lambda_{k+1} |a_{k+1}|^2.
\]

By (2.7), we get

\[
\frac{1}{2} |a_{k+1}|^2 = -\lambda_{k+1} |a_{k+1}|^2 + \lambda_k |a_{k+1}|^2 = (\lambda_k - \lambda_{k+1}) |a_{k+1}|^2.
\]

Then \( |a_{k+1}|^2 = e^{2[\lambda_k - \lambda_{k+1} - \beta]} |a_0|^2 = e^{2\beta} \leq A \).

Note that \( \exists D > 0 \) such that \( |\delta \rho a|^2 \leq D |a|^2 \) for any \( a \in M_a \). We have

\[
\lambda_{k+1} |a_{k+1}|^2 = <La_{k+1}, a_{k+1}> = <\Delta a_{k+1}, a_{k+1}> + <\Lambda a_{k+1}, a_{k+1}> \\
\leq (D+|\Lambda|)|a_{k+1}|^2 = C |a_{k+1}|^2.
\]

Then \( \lambda_{k+1} \leq C \). Hence the claim is true in \([0,T]\).

Therefore, (2.1) has a solution in \([0,T]\). By iteration we can get a solution in \([T,2T]\) with \( u(T) \) as the initial data. We can iterate this step on and on and we get a global solution to (2.1) with initial data \( a_0 \).

In conclusion we have the below.

**Theorem 2.1** For any given initial matrix \( a_0 \in M_a \) with \( |a_0|^2 = 1 \), the Equation (2.1) has a global solution with \( a_0 \) as its initial data and \( |a(t)|^2 = 1 \) for all \( t > 0 \).

### 3. Positive Property Preserved by the Flow

In this section we show that positivity of the initial matrices is preserved along the flow. That is to say, we show that if the initial matrix is positive definite, then along the flow (2.1), the evolution matrix is also positive definite.

**Theorem 3.1** Assume \( a_0 > 0 \), that is \( a_0 \) is a Hermitian positive definite. Then \( a(t) > 0, \forall t > 0 \) along the flow equation

\[
a_t = La + \lambda(t)a
\]

with \( a(0) = a_0 \), where \( \lambda(t) \) is given by (2.2).

**Proof.** By an argument as in [4], we know \( a = a(t) \) is Hermitian matrix. Then we know that \( a = a(t) > 0 \) for small \( t > 0 \) by continuity. Compute

\[
\frac{d}{dt} \log |det a| = <a^{-1}, a_> <a^{-1}, La> + N\lambda(t),
\]

where \( N = \sigma(I) = <1, I> \).

Since

\[
<a^{-1}, La> = <a^{-1}, \Delta a> = <a^{-1}, \Lambda a> = <a^{-1}, \Delta a> = -\tau(\Lambda),
\]

and

\[
<a^{-1}, La> = -<\delta \rho (a^{-1}), \delta \rho a> = <a^{-1} \delta \rho a, a^{-1} \delta \rho a> = |a^{-1} \delta \rho a|^2.
\]

We know that

\[
\frac{d}{dt} \log \epsilon^{i\tau(\Lambda)} det a = \frac{d}{dt} [\log det a + \tau(\Lambda)] = |a^{-1} \delta \rho a|^2 + \tau(\Lambda) - \tau(\Lambda) + N\lambda(t) \geq N\lambda(t) \geq 0.
\]

Then we have

\[
\epsilon^{i\tau(\Lambda)} det a \geq det a_0 > 0.
\]

Hence \( det a > 0 \) and \( a = a(t) > 0, \forall t > 0 \). □

Then the proof of Theorem 3.1 is complete.
Remark that by continuity, we can show that if \( a_0 \geq 0 \), then \( a(t) \geq 0 \) along the flow (2.1).

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**References**


