On the Cauchy Problem for Von Neumann-Landau Wave Equation

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Abstract
In present paper we prove the local well-posedness for Von Neumann-Landau wave equation by the T. Kato’s method.

Keywords
Von Neumann-Landau Wave Equation, Strichartz Estimate, Cauchy Problem

1. Introduction
For the stationary Von Neumann-Landau wave equation, Chen investigated the Dirichlet problems [1], where the generalized solution is studied by Function-analytic method. The present paper is related to the Cauchy problem: the Von Neumann-Landau wave equation

\[
\begin{align*}
    i\partial_t u &= (-\Delta_x + \Delta_y) u + f(u), \\
    u(0,x,y) &= u_0(x,y) 
\end{align*}
\] (1)

where \( \Delta_x = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \) for \( x = (x_1, \cdots, x_n) \in \mathbb{R}^n \), \( u(t,x,y) \) is an unknown complex valued function on \( \mathbb{R}^{1+2n} \) and \( f \) is a nonlinear complex valued function.

If the plus “+” is replaced by the minus “-” on right hand in Equation (1), then the resulted equation is the Schrödinger equation. For the Schrödinger equation, the well-posedness problem is investigated for various nonlinear terms \( f \). In terms of the nonlinear terms \( f \), the problem (1) can be divided into the subcritical case and the critical case for \( H^1 \) solutions. We are concerned with the subcritical case and obtain a local well-
posedness result by the T. Kato’s method.

The paper is organized as follows. Section 2 contains the list of assumptions on the interaction term $f$ and the main result is presented. Section 3 is concerned with the Strichartz estimates. Finally, in Section 4, the main result is proved.

\section{Statement of the Main Result}

In this section we list the assumptions on the interaction term $f$ and state the main result. Firstly, we recall that the definition of admissible pair \[2\].

\textbf{Definition 2.1.} Fix $d = 2n$, $n \geq 1$. We say that a pair $(q, r)$ of exponents is admissible if
\[
\frac{2}{q} = d \left( 1 - \frac{1}{r} \right),
\] (2)

and
\[
2 \leq r \leq \frac{2d}{d-2} \quad (2 \leq r < \infty \text{ if } d = 2).
\] (3)

\textbf{Remark 2.1.} The pairs $(\infty, 2)$ is always admissible, so is the $\left( 2, \frac{2d}{d-2} \right)$ if $d > 2$. The two pairs are called the endpoint cases.

Secondly, let $f \in C(C, \mathbb{C})$ satisfy
\[
f(0) = 0,
\] (4)

and
\[
|f(u) - f(v)| \leq K(M)|u - v|,
\] (5)

for all $u, v \in \mathbb{C}$ such that $|u|, |v| \leq M$, with
\[
K(t) \leq C_1 (1 + t^\alpha), \quad 0 < \alpha < \frac{4}{d-2},
\] (6)

where $C_1$ is a constant independent of $t$. Set
\[
f(u)(x) = f(u(x)),
\] (7)

for all measurable function $u$ and a.e. $x \in \mathbb{R}^{1+2n}$.

Finally, let us make the notion of solution more precise.

\textbf{Definition 2.2.} Let $I$ be an interval such that $0 \in I$. We say that $u$ is a strong $H^1$-solution of (1) on $I$ if $u \in C(I, H^1(\mathbb{R}^d))$ satisfies the integral equation
\[
u(t) = e^{-i\Delta_s} u_0 - i \int_0^t e^{-i(t-s)\Delta_s} f(u(s)) \, ds,
\] (8)

for all $t \in I$, where $L = -\Delta_s + \Delta_s$.

The main result is the following theorem:

\textbf{Theorem 1.} Suppose $n \geq 1$. Let $f \in C(C, \mathbb{C})$ satisfy (4)-(6). If $f$ (considered as a function $\mathbb{R}^2 \to \mathbb{R}^2$) is of class $C^1$, then the Cauchy problem (1) is locally well posed in $H^1(\mathbb{R}^d)$. More specially, the following properties hold:

(i) For any $R > 0$ there exists a time $T = T(d, \alpha, R) > 0$ and constant $c = c(d, \alpha)$ such that for each $u_0$ in the ball $B_R := \{ \varphi \in H^1(\mathbb{R}^d) : \| \varphi \|_{H^1(\mathbb{R}^d)} \leq R \}$ there exists a unique strong $H^1$-solution $u$ to the Equation (1) in $C([-T, T], H^1(\mathbb{R}^d))$ such that
\[ \|u\|_{L^r_t\left(\{-T,T\},H^r(\mathbb{R}^d)\}} + \|\partial_t u\|_{L^r_t\left(\{-T,T\},H^{r-1}(\mathbb{R}^d)\}} \leq c \|u_0\|_{H^r(\mathbb{R}^d)}, \]  

where \( r = \alpha + 2 \), and \((q, r)\) is an admissible pair.

(ii) The map \( u_0 \mapsto u \) is continuous from \( B_r \) to \( C\left([-T, T], H^1(\mathbb{R}^d)\right)\);

(iii) For every \( u_0 \in H^1(\mathbb{R}^d) \), the unique solution \( u \) is defined on a maximal interval \((-T_{\text{min}}, T_{\text{max}})\), with \( T_{\text{max}} = T_{\text{max}}(u_0) \in (0, \infty) \) and \( T_{\text{min}} = T_{\text{min}}(u_0) \in (0, \infty); \)

(iv) There is the blowup alternative: If \( T_{\text{max}} < \infty \), then \( \|u(t)\|_{H^1(\mathbb{R}^d)} \to +\infty \) as \( t \nearrow T_{\text{max}} \) (respectively, if \( T_{\text{min}} < \infty \), then \( \|u(t)\|_{H^1(\mathbb{R}^d)} \to +\infty \) as \( t \searrow -T_{\text{min}} \)).

**Remark 2.2.** It follows from Strichartz estimates that

\[ u \in L^p_{\text{loc}}\left((-T_{\text{min}}, T_{\text{max}}), W^{1, q}(\mathbb{R}^d)\right), \]

for any admissible pair \((q, p)\).

**Remark 2.3.** For the Schrödinger equations, the similar results hold [2]. It implies a fact that the ellipticity of the operator \(-\Delta_x - \Delta_y\) is not the key point in the local well-posedness problem.

### 3. Strichartz Estimates

In this subsection, we recall that the Strichartz estimates. Let \((\xi, \eta)\) denote a general Fourier variable in \(\mathbb{R}^{2n}\), \(\xi = (\xi_1, \ldots, \xi_n)\), \(\eta = (\eta_1, \ldots, \eta_n)\). Let \( L = -\Delta_x - \Delta_y \), then by Fourier transform (denoting by \(\mathcal{F}\) or \(\hat{\cdot}\)) we have

\[ Lu = \mathcal{F}^{-1}\left(\left|\xi\right|^2 - \left|\eta\right|^2 \hat{u}\right), \]

for any \( u \in H^2(\mathbb{R}^{2n}) \). It is easy to verify that the \( L \) is a self-adjoint unbounded operator on \( L^2(\mathbb{R}^{2n}) \) with the domain \( H^2(\mathbb{R}^{2n}) \). Then, by Stone theorem we see that \( e^{itL} \) is an unitary group on \( L^2(\mathbb{R}^{2n}) \). Moreover, \( e^{itL} \) can be expressed explicitly by Fourier transform.

\[ e^{it\varphi} = \mathcal{F}^{-1}\left(e^{i[H\cdot\varphi]}\right), \]

for any \( \varphi \in L^2(\mathbb{R}^{2n}) \). By the direct compute, we conclude

\[ (e^{it\varphi})(x, y) = \frac{1}{(4\pi|t|)^n} \int_{\mathbb{R}^{2n}} e^{-i(x-y)\cdot\frac{\xi}{t} - \frac{\xi_0}{t}} e^{i\eta_0\cdot\frac{\eta}{t}} \varphi(x', y') dx' dy'. \]

The following result is the fundamental estimate for \( e^{it\varphi} \).

**Lemma 1.** If \( p \in [1, 2] \) and \( t \neq 0 \), then \( e^{it\varphi} \) maps \( L^p(\mathbb{R}^d) \) continuously to \( L^{p'}(\mathbb{R}^d) \) and

\[ \left\| e^{it\varphi} \right\|_{L^{p'}(\mathbb{R}^d)} \leq \left(4\pi|t|\right)^{\frac{1}{p} - \frac{1}{p'}} \left\| \varphi \right\|_{L^p(\mathbb{R}^d)}, \]

where \( p' \) is the dual exponent of \( p \), defined by the formula \( \frac{1}{p} + \frac{1}{p'} = 1 \).

**Proof.** For the proof please see [3] or [4]. \( \square \)

The following estimates, known as Strichartz estimates, are key points in the method introduced by T. Kato [5].

**Lemma 2.** Let \((q, r)\) and \((\tilde{q}, \tilde{r})\) be any admissible exponents. Then, we have the homogeneous Strichartz
estimate
\[ \left\| e^{it\Delta} \phi \right\|_{L^2(I, H^s(\mathbb{R}^d))} \lesssim_{\alpha, I, r} \left\| \phi \right\|_{L^2(\mathbb{R}^d)}, \] (14)
the dual homogeneous Strichartz estimate
\[ \left\| \int_k e^{-it\phi} \phi(t) dt \right\|_{L^2(I, H^s(\mathbb{R}^d))} \lesssim_{\alpha, I, r} \left\| \phi \right\|_{L^2(I, H^s(\mathbb{R}^d))}, \] (15)
and the inhomogeneous Strichartz estimate
\[ \left\| \int_k e^{it\phi} \phi(s) ds \right\|_{L^2(I, H^s(\mathbb{R}^d))} \lesssim_{\alpha, I, r} \left\| \phi \right\|_{L^2(I, H^s(\mathbb{R}^d))}, \] (16)
for any interval \( J \) and real number \( t_0 \).

**Proof.** For the proof please see [3] or [4] in the non-endpoint case. On the other hand, the proof in the endpoint case follows from the theorem 1.2 in [6] and the lemma 1 in the present paper. \( \Box \)

### 4. The Proof of Theorem

**Proof.** Let \( \chi \in C_0^\infty(\mathbb{R}^d) \) be such that \( \chi(z) = 1 \) for \( |z| \leq 1 \) and \( \chi(z) = 0 \) for \( |z| \geq 2 \). Setting
\[
\begin{align*}
f_1(z) &= \chi(z) f(z), \\
f_2(z) &= (1 - \chi(z)) f(z),
\end{align*}
\]
one easily verifies that for any \( z, w \in \mathbb{C} \)
\[
\begin{align*}
|f_1(z) - f_1(w)| &\lesssim |z - w|, \\
|f_2(z) - f_2(w)| &\lesssim \left(|z|^\alpha + |w|^\alpha\right)|z - w|.
\end{align*}
\] (17)

Set \( f_1(u)(x) = f_1(u(x)) \) for \( l = 1, 2 \). Using (17), we deduce from Hölder’s inequality that
\[
\begin{align*}
\left\| f_1(u) - f_1(v) \right\|_{L^2(\mathbb{R}^d)} &\lesssim \left\| u - v \right\|_{L^2(\mathbb{R}^d)}, \\
\left\| f_2(u) - f_2(v) \right\|_{L^2(\mathbb{R}^d)} &\lesssim \left( \left\| u \right\|_{L^r(\mathbb{R}^d)}^\alpha + \left\| v \right\|_{L^r(\mathbb{R}^d)}^\alpha \right) \left\| u - v \right\|_{L^r(\mathbb{R}^d)}.
\end{align*}
\] (18)
And it follows from Remark 1.3.1 (vii) in [2] that
\[
\begin{align*}
\left\| \nabla f_1(u) \right\|_{L^2(\mathbb{R}^d)} &\lesssim \left\| \nabla u \right\|_{L^2(\mathbb{R}^d)}, \\
\left\| \nabla f_2(u) \right\|_{L^2(\mathbb{R}^d)} &\lesssim \left\| u \right\|_{L^r(\mathbb{R}^d)}^\alpha \left\| \nabla u \right\|_{L^r(\mathbb{R}^d)}.
\end{align*}
\] (19)

We now proceed in four steps.

**Step 1. Proof of (i).** Fix \( A, T > 0 \), to be chosen later, and let \( r = \alpha + 2, q \) be such that \((q, r)\) is an admissible pair, and set \( I = (-T, T) \). Consider the set
\[
E = \left\{ u \in L^q(I, H^s(\mathbb{R}^d)) \cap L^r(I, W^{1,r}(\mathbb{R}^d)) : \left\| u \right\|_{L^q(I, H^s(\mathbb{R}^d))} \leq A, \left\| u \right\|_{L^r(I, W^{1,r}(\mathbb{R}^d))} \leq A \right\},
\] (20)
equipped with the distance
\[
d(u, v) = \left\| u - v \right\|_{L^q(I, H^s(\mathbb{R}^d))} + \left\| u - v \right\|_{L^r(I, W^{1,r}(\mathbb{R}^d))}.
\] (21)
We claim that \((E, d)\) is a complete metric space. Indeed, let \( \{u_k\}_{k \geq 1} \subset E \) be a Cauchy sequence. Clearly, \( \{u_k\}_{k \geq 1} \) is also a Cauchy sequence in \( L^q(I, L^q(\mathbb{R}^d)) \) and \( L^r(I, L^r(\mathbb{R}^d)) \). In particular, there exists a
function \( u \in L^r \left( I, L^2 \left( \mathbb{R}^d \right) \right) \cap L^{r'} \left( I, L^2 \left( \mathbb{R}^d \right) \right) \) such that \( u_k \to u \) in \( L^r \left( I, L^2 \left( \mathbb{R}^d \right) \right) \) and \( L^{r'} \left( I, L^2 \left( \mathbb{R}^d \right) \right) \) as \( k \to \infty \). Applying theorem 1.2.5 in [2] twice, we conclude that

\[
\| u \|_{L^r \left( I, H^1 \left( \mathbb{R}^d \right) \right)} \leq \liminf_{k \to \infty} \| u_k \|_{L^r \left( I, H^1 \left( \mathbb{R}^d \right) \right)} \leq A,
\]

and that

\[
\| u \|_{L^r \left( I, W^{1,r} \left( \mathbb{R}^d \right) \right)} \leq \liminf_{k \to \infty} \| u_k \|_{L^r \left( I, W^{1,r} \left( \mathbb{R}^d \right) \right)} \leq A;
\]
thus, \( u_k \to u \) in \( E \) as \( k \to \infty \).

Taking up any \( u, v \in E \). Since \( f_1 \) is continuous \( L^2 \left( \mathbb{R}^d \right) \to L^2 \left( \mathbb{R}^d \right) \), it follows that \( f_1 \left( u \right) : I \to L^2 \left( \mathbb{R}^d \right) \) is measurable, and we deduce easily that \( f_1 \left( u \right) \in L^r \left( I, L^2 \left( \mathbb{R}^d \right) \right) \). Similarly, since \( f_2 \) is continuous \( L^r \left( \mathbb{R}^d \right) \to L^r \left( \mathbb{R}^d \right) \), we see that \( f_2 \left( u \right) \in L^r \left( I, L^r \left( \mathbb{R}^d \right) \right) \). Using inequalities (18) and (19) and Remark 1.2.2 (iii) in [2], We deduce the following:

\[
f_1 \left( u \right) \in L^r \left( I, H^1 \left( \mathbb{R}^d \right) \right), \quad f_2 \left( u \right) \in L^r \left( I, W^{1,r} \left( \mathbb{R}^d \right) \right),
\]

and

\[
\| f_1 \left( u \right) - f_1 \left( v \right) \|_{L^r \left( I, H^1 \left( \mathbb{R}^d \right) \right)} \lesssim \| u - v \|_{L^r \left( I, L^2 \left( \mathbb{R}^d \right) \right)}.
\]

Using the embedding \( H^1 \left( \mathbb{R}^d \right) \to L^2 \left( \mathbb{R}^d \right) \) and Hölder’s inequality in time, we deduce from the above estimates that

\[
\| f_1 \left( u \right) \|_{L^r \left( I, H^1 \left( \mathbb{R}^d \right) \right)} + \| f_2 \left( u \right) \|_{L^r \left( I, W^{1,r} \left( \mathbb{R}^d \right) \right)} \lesssim_{d, \alpha} \left( T + T^\frac{q-2}{q} \right) \left( 1 + A^\alpha \right) A,
\]

and

\[
\| f_1 \left( u \right) - f_1 \left( v \right) \|_{L^r \left( I, H^1 \left( \mathbb{R}^d \right) \right)} + \| f_2 \left( u \right) - f_2 \left( v \right) \|_{L^r \left( I, W^{1,r} \left( \mathbb{R}^d \right) \right)} \lesssim_{d, \alpha} \left( T + T^\frac{q-2}{q} \right) \left( 1 + A^\alpha \right) d \left( u, v \right).
\]

Given \( u_0 \in H^1 \left( \mathbb{R}^d \right) \). For any \( u \in E \), let \( \mathcal{H} \left( u \right) \) be defined by

\[
\mathcal{H} \left( u \right) \left( t \right) = e^{-it \Delta} u_0 - i \int_0^t e^{-i \left( t - s \right) \Delta} f \left( u \left( s \right) \right) ds.
\]

It follows from (22) and Strichartz estimates (lemma 2) that \( \mathcal{H} \left( u \right) \in C \left( \left[ -T, T \right], H^1 \left( \mathbb{R}^d \right) \right) \cap L^r \left( \left[ -T, T \right], W^{1,r} \left( \mathbb{R}^d \right) \right) \),

\[
\| \mathcal{H} \left( u \right) \|_{L^r \left( I, H^1 \left( \mathbb{R}^d \right) \right)} + \| \mathcal{H} \left( u \right) \|_{L^r \left( I, W^{1,r} \left( \mathbb{R}^d \right) \right)} \leq C \left( d, \alpha \right) \left[ \| u_0 \|_{H^1 \left( \mathbb{R}^d \right)} + \left( T + T^\frac{q-2}{q} \right) \left( 1 + A^\alpha \right) A \right].
\]
Also, we deduce from (23) that
\[ \| \mathcal{P}(u) - \mathcal{P}(v) \|_{L^q(I; L^r(\mathbb{R}^d))} + \| \mathcal{P}(u) - \mathcal{P}(v) \|_{L^q(I; L^r(\mathbb{R}^d))} \leq C_1(d, \alpha)\left( T + T^{-q'q} \right)d(u, v). \] (27)

Finally, note that \( q > q' \). We now proceed as follows. For any \( R \geq \|u_0\|_{L^q(\mathbb{R}^d)} \), we set \( A = 2C_1(d, \alpha R) \), and we let \( T = T(d, \alpha, R) \) be the unique positive number so that
\[ C_1(d, \alpha)\left( T + T^{-q'q} \right)d(u_0, v_0) = \frac{1}{2}. \] (28)

It then follows from (26) and (28) that for any \( u_0 \in B_R \)
\[ \| \mathcal{H}(u) \|_{L^q(I; L^r(\mathbb{R}^d))} + \| \mathcal{H}(u) \|_{L^q(I; L^r(\mathbb{R}^d))} \leq C_1(d, \alpha)\|u_0\|_{L^q(\mathbb{R}^d)} + \frac{1}{2}A \leq C_1(d, \alpha)R + \frac{1}{2}A = A. \] (29)

Thus, \( \mathcal{H}(u) \in E \) and by (27) we obtain
\[ d(\mathcal{H}(u), \mathcal{H}(v)) \leq \frac{1}{2}d(u, v). \] (30)

In particular, \( \mathcal{H} \) is a strict contraction on \( E \). By Banach's fixed-point theorem, \( \mathcal{H} \) has a unique fixed point \( u \in E \); that is \( u \) satisfies \( (8) \). By (25), \( u = \mathcal{H}(u) \in C\left([-T, T], H^1(\mathbb{R}^d)\right) \). By the definition 2.2, we conclude that \( u \) is a strong \( H^1 \)-solution of \( (1) \) on \([-T, T] \). Note that \( T(d, \alpha, R) \) is decreasing on \( R \), then the estimate \( (9) \) holds for \( c = 2C_1(d, \alpha) \) by letting \( R = \|u_0\|_{L^q(\mathbb{R}^d)} \) in (29).

For uniqueness, assume that \( u, v \) are two strong \( H^1 \)-solution of \( (1) \) on \([-T, T] \) with the same initial value \( u_0 \). Then, we have
\[ u(t) - v(t) = -i\int_0^t e^{-i(s-t)T} \left[ f(u(s)) - f(v(s)) \right] ds. \] (31)

For simplicity, we set
\[ w_1(t) = -i\int_0^t e^{-i(s-t)T} \left[ f_1(u(s)) - f_1(v(s)) \right] ds, \]
for \( l = 1, 2 \), and \( w = u - v \). For any interval \( J \subset (-T, T) \), by (18) and Strichartz estimates (16), then we obtain
\[ \|w_1\|_{L^q(I; L^r(\mathbb{R}^d))} + \|w_2\|_{L^q(I; L^r(\mathbb{R}^d))} \lesssim_{d, \alpha} \|f_1(u) - f_1(v)\|_{L^r(I; L^q(\mathbb{R}^d))} \lesssim_{d, \alpha} \|w\|_{L^q(I; L^r(\mathbb{R}^d))}. \] (32)

Similarly, for \( w_2 \) we have
\[ \|w_2\|_{L^q(I; L^r(\mathbb{R}^d))} + \|w_2\|_{L^q(I; L^r(\mathbb{R}^d))} \lesssim_{d, \alpha} \|f_2(u) - f_2(v)\|_{L^r(I; L^q(\mathbb{R}^d))} \lesssim_{d, \alpha} \|w\|_{L^q(I; L^r(\mathbb{R}^d))}. \] (33)

Note that \( w = w_1 + w_2 \). Then, it follows from that
\[ \|w\|_{L^q(I; L^r(\mathbb{R}^d))} + \|w\|_{L^q(I; L^r(\mathbb{R}^d))} \leq C_2(1 + B)\left( \|w\|_{L^q(I; L^r(\mathbb{R}^d))} + \|w\|_{L^q(I; L^r(\mathbb{R}^d))} \right), \] (34)
where the constant \( B = \|w\|_{L^q(I; L^r(\mathbb{R}^d))} + \|w\|_{L^q(I; L^r(\mathbb{R}^d))} \) and the constant \( C_2 \) is independent of \( J \) by above inequalities. Note that \( q' < q \), we conclude that \( w = 0 \) by the lemma 4.2.2 in [2]. So \( u = v \).

Step 2. Proof of (ii). Suppose that \( u^{(k)}_0 \rightarrow u_0 \) in \( B_R \) as \( k \rightarrow \infty \). By the part (i), we denote \( u_1 \) and \( u \) by
the unique solution of (1) corresponding to the initial value $u_0^{(k)}$ and $u$, respectively. We will show that $u_k \to u$ in $C([-T,T],H^1(\mathbb{R}^d))$ as $k \to \infty$. Note that
\begin{equation}
  u_k(t) - u(t) = e^{-it\Delta} (u_0^{(k)} - u_0) + \mathcal{H}(u_k) - \mathcal{H}(u),
\end{equation}
and the estimate (29) which implies that (27) holds for $\nu = u_k$. Note that the choosing of the time $T$ in (28), it follows from (27) with (30) that
\begin{equation}
  d(u_k,u) \leq d(u_0^{(k)} - u_0) + \frac{1}{2} d(u_k,u).
\end{equation}
Hence, we have
\begin{equation}
  \left\| u_k - u \right\|_{\mathcal{L}(\mathbb{C}([-T,T],\mathbb{R}^d))} + \left\| u_k - u \right\|_{\mathcal{L}(\mathbb{C}([-T,T],\mathbb{R}^d))} \leq \left\| u_0^{(k)} - u_0 \right\|_{\mathbb{C}([T,T],\mathbb{R}^d)}.
\end{equation}
Next, we need to estimate $\nabla(u_k - u)(t)$. Note that $\nabla$ commutes with $e^{-it\Delta}$, and so
\begin{equation}
  \nabla(u_k - u)(t) = e^{-it\Delta} \nabla u_0 - i \int_0^t e^{-i(t-s)\Delta} \nabla f(u(s)) ds.
\end{equation}
A similar identity holds for $u_k$. We use the assumption $f \in C^1(\mathbb{R}^2,\mathbb{R}^2)$, which implies that $\nabla f(u) = f'(u) \nabla u$, where $f'(u)$ is a $2 \times 2$ real matrix. Therefore, we may write
\begin{equation}
  \nabla(u_k - u)(t) = e^{-it\Delta} \nabla u_0 - i \int_0^t e^{-i(t-s)\Delta} f'(u_k) \nabla(u_k - u) ds
  - i \int_0^t e^{-i(t-s)\Delta} (f'(u_k) - f'(u)) \nabla u ds.
\end{equation}
Note that $f_1$ and $f_2$ are also $C^1$, so that $f' = f'_1 + f'_2$, and from (17) we deduce that $\left| f'_1(z) \right| \leq C_3$ and $\left| f'_2(z) \right| \lesssim \left| z \right|^\alpha$ for any $z \in \mathbb{C}$ and some constant $C_3$. Therefore, arguing as in Step 1, we obtain the estimate
\begin{equation}
  \left\| \nabla(u_k - u) \right\|_{\mathcal{L}(\mathbb{C}([-T,T],\mathbb{R}^d))} + \left\| \nabla(u_k - u) \right\|_{\mathcal{L}(\mathbb{C}([-T,T],\mathbb{R}^d))} \leq \left\| u_0^{(k)} - u_0 \right\|_{\mathbb{C}([T,T],\mathbb{R}^d)}.
\end{equation}
By choosing $T = T(d,\alpha,R)$ as (28) and noting that $u_k \in B_R$, from (40) we obtain that
\begin{equation}
  \left\| \nabla(u_k - u) \right\|_{\mathcal{L}(\mathbb{C}([-T,T],\mathbb{R}^d))} + \left\| \nabla(u_k - u) \right\|_{\mathcal{L}(\mathbb{C}([-T,T],\mathbb{R}^d))} \leq \left\| u_0^{(k)} - u_0 \right\|_{\mathbb{C}([T,T],\mathbb{R}^d)}.
\end{equation}
There, if we prove that
\[ \left\| (f'_u(u_k) - f'_u(u)) \nabla u \right\|_{L^p(T, L^2([\mathbb{R}^d]))} + \left\| (f''_u(u_k) - f''_u(u)) \nabla u \right\|_{L^p(T, L^2([\mathbb{R}^d]))} \to 0, \tag{42} \]

as \( k \to \infty \), then we have
\[ \left\| \nabla (u_k - u) \right\|_{L^p(T, L^2([\mathbb{R}^d]))} + \left\| \nabla (u_k - u) \right\|_{L^p(T, L^2([\mathbb{R}^d]))} \to 0, \tag{43} \]

as \( k \to \infty \), which, combined with (37), yields the desired convergence. We prove (42) by contradiction, and we assume that there exists \( \varepsilon_0 > 0 \) and a subsequence, which we still denote by \( \{u_k\}_{k \geq 1} \) such that
\[ \left\| (f'_u(u_k) - f'_u(u)) \nabla u \right\|_{L^p(T, L^2([\mathbb{R}^d]))} + \left\| (f''_u(u_k) - f''_u(u)) \nabla u \right\|_{L^p(T, L^2([\mathbb{R}^d]))} \geq \varepsilon_0. \tag{44} \]

By using (37) and possibly extracting a subsequence, we may assume that \( u_k \to u \) a.e. on \( (-T, T) \times \mathbb{R}^d \) and that there exists \( v \in L^p\left((-T, T), L^p\left(\mathbb{R}^d\right)\right) \) such that \( |u_k| \leq v \) a.e. on \( (-T, T) \times \mathbb{R}^d \). In particular, both \( (f'_u(u_k) - f'_u(u)) \nabla u \) and \( (f''_u(u_k) - f''_u(u)) \nabla u \) converge to 0 a.e. on \( (-T, T) \times \mathbb{R}^d \). Since
\[ \left\| (f'_u(u_k) - f'_u(u)) \nabla u \right\|_{L^p(T, L^2([\mathbb{R}^d]))} \leq 2C \|u\| \in L^p\left((-T, T), L^2\left(\mathbb{R}^d\right)\right), \]

and
\[ \left\| (f''_u(u_k) - f''_u(u)) \nabla u \right\|_{L^p(T, L^2([\mathbb{R}^d]))} \leq \alpha \left( |u_k| + |u| \right) \|u\| \leq \alpha \left( |v| + |u| \right) \|u\| \in L^p\left((-T, T), L^p\left(\mathbb{R}^d\right)\right), \]

we obtain from the dominated convergence a contradiction with (44).

Step 3. Proof of (iii). Consider \( u_0 \in H^1\left(\mathbb{R}^d\right) \) and let
\[ T_{\max}(u_0) = \sup \{T > 0 : \text{there exists a solution of (1) on } [0, T] \}, \]
\[ T_{\min}(u_0) = \sup \{T > 0 : \text{there exists a solution of (1) on } [-T, 0] \}. \]

It follows from part (i) there exists a solution \( u \in C\left((-T_{\min}, T_{\max}), H^1\left(\mathbb{R}^d\right)\right), \) of (1).

Step 4. Proof of (iv). Suppose now that \( T_{\max} < \infty \), and assume that there exist \( M < \infty \) and a sequence \( t_j \to T_{\max} \) such that \( \left\| u(t_j) \right\|_{L^p\left(\mathbb{R}^d\right)} \leq M \). Let \( k \) be such that \( t_k + T(d, \alpha, M) > T_{\max}(u_0) \). By part (i), from the initial data \( u(t_k) \), one can extend \( u \) up to \( t_k + T(d, \alpha, M) \), which contradicts maximality. Therefore,
\[ \left\| u(t) \right\|_{L^p\left(\mathbb{R}^d\right)} \to \infty, \text{ as } t > T_{\max}. \]

One shows by the same argument that if \( T_{\min} < \infty \), then
\[ \left\| u(t) \right\|_{L^p\left(\mathbb{R}^d\right)} \to \infty, \text{ as } t < -T_{\min}. \]

This completes the proof. \( \square \)

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