Application and Generalization of Eigenvalues Perturbation Bounds for Hermitian Block Tridiagonal Matrices

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ABSTRACT
The paper contains two parts. First, by applying the results about the eigenvalue perturbation bounds for Hermitian block tridiagonal matrices in paper [1], we obtain a new efficient method to estimate the perturbation bounds for singular values of block tridiagonal matrix. Second, we consider the perturbation bounds for eigenvalues of Hermitian matrix with block tridiagonal structure when its two adjacent blocks are perturbed simultaneously. In this case, when the eigenvalues of the perturbed matrix are well-separated from the spectrum of the diagonal blocks, our eigenvalues perturbation bounds are very sharp. The numerical examples illustrate the efficiency of our methods.

KEYWORDS
Singular Value; Eigenvalue Perturbation; Hermitian Matrix; Block Tridiagonal Matrix; Eigenvector

1. Introduction
There are many known results about eigenvalue perturbation bounds of Hermitian matrices. See example [2-5]. Among them, one well-known theory is the following result.

Theorem 2.1 [2]. Let $A$ and $A + E$ be n-by-n Hermitian matrices. Let $\lambda_i$ and $\hat{\lambda}_i$ denote the ith smallest eigenvalues of $A$ and $A + E$, respectively. Then for $i = 1, 2, \cdots, n$, we have

$$\lambda_i + \lambda_{\min}(E) \leq \hat{\lambda}_i \leq \lambda_i + \lambda_{\max}(E),$$

where all the eigenvalues of $A$ and $A + E$ are indexed in ascending order.

This is the Weyl’s theorem, which is one of the most classic eigenvalue perturbation theories. When the perturbation matrix $E$ is an arbitrary Hermitian matrix, the bounds obtained by Weyl’s theorem can be very small. However, for Hermitian matrices with special sparse structures such as block tridiagonal Hermitian matrix, the Weyl’s theorem may not be the best choice. For this reason, [1] considered the difference between eigenvalues of the block tridiagonal Hermitian matrices $A$ and $A + E_s$, where

$$A = \begin{pmatrix} A_1 & B_1^H \\ B_1 & \ddots & B_{n-1}^H \\ B_{n-1} & \cdots & A_n \end{pmatrix}$$

and

$$E_s = \begin{pmatrix} 0 & 0 \\ 0 & \Delta A_s & \Delta B_s^H \\ \Delta B_s & 0 & 0 \end{pmatrix},$$

where $\Delta A_s$ and $\Delta B_s$ are small perturbations.

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in which \( A_j \in \mathbb{C}^{n_j \times n_j} \), \( j = 1, 2, \ldots, n \), are Hermitian matrices and \( B_k \in \mathbb{C}^{n_k \times n_k} \), \( j = 1, 2, \ldots, n - 1 \), are arbitrary matrices, the perturbation matrices \( \Delta A_j \) and \( \Delta B_j \) have the same order with the matrices \( A_j \) and \( B_j \), respectively. Let \( \lambda_i \) and \( \hat{\lambda}_i \) denote the \( i \)-th smallest eigenvalues of matrices \( A \) and \( A + E_s \), respectively. Let \( \lambda(X) \) denote the set of all the eigenvalues of the Hermitian matrix \( X \). By defining \( \text{gap}_j = \min \{ \tilde{\lambda} - \lambda \} \), and assuming that there exists an integer \( \ell > 0 \) such that \( \text{gap}_j > \|B_j\|_2 + \|B_{j+1}\|_2 + \|E_s\|_2 \), \( j = 1, 2, \ldots, s + \ell \), the paper [1] obtained the sharper eigenvalue perturbation bounds

\[
||\lambda_i - \hat{\lambda}_i|| \leq ||\Delta A||_2 \left( \sum_{j=0}^{\ell} \delta_j \right)^2 + 2 ||\Delta B||_2 \left( \sum_{j=0}^{\ell} \delta_j \right)^2.
\]  

(1.3)

in which

\[
\delta_0 = \frac{||B||_2 + ||\Delta B||_2}{\text{gap}_s}, \quad \delta_j = \frac{||B_{s+j}||_2}{\text{gap}_{s+j} - ||E_s||_2 - ||B_j||_2 - ||\Delta B||_2},
\]

and

\[
\delta_j = \frac{||B_{s+j}||_2}{\text{gap}_{s+j} - ||E_s||_2 - ||B_{s+j-1}||_2}, \quad j = 2, \ldots, \ell.
\]

The natural questions are that whether the above results can be used to estimate the perturbation bounds for singular values of a block tridiagonal matrix, and how to get the eigenvalues perturbation bounds when two adjacent blocks of the matrix \( A \) in the formula (1.2) are perturbed simultaneously. If we apply the results above repeatedly, we can obtain a weaker upper bounds. Inspired by these questions, in this paper, we expect to obtain the eigenvalues perturbation bounds by directly using the bounds of eigenvector elements rather than applying the results in [1] repeatedly.

The structure of this paper is organized as follows. In Section 2, we provide preliminaries to outline our basic idea of deriving eigenvalue perturbation bounds via bounding eigenvector components [1]. In Section 3, we present a new approach to estimate the perturbation bounds for the singular values of the block tridiagonal matrix via applying the ideas in paper [1]. In Section 4, we consider the case which the \( s \)-th block and \( (s+1) \)-th block of the matrix \( A \) are perturbed simultaneously and present a new perturbation bound of the \( i \)- smallest eigenvalue \( \lambda_i \). Further, we discuss the eigenvalue perturbation bounds when the first \( s \) blocks of \( A \) are perturbed simultaneously and provide an algorithm to estimate the bounds. In Section 5, we present a numerical example to show the efficiency of our approach.

Notations. Let \( || \cdot ||_2 \) denote the matrix spectrum norm.

2. Preliminaries

For simplicity, the eigenvalues that we mention in this paper are all simple eigenvalues. We need the following conclusion about the partial derivative of simple eigenvalue of \( A + tE \) for further discussion, where \( t \in [0,1] \).

Lemma 2.1 [1]. Let \( A \) and \( E \) be \( n \)-by-\( n \) Hermitian matrices. Denote by \( \lambda_i(t) \) the \( i \)-th eigenvalue of \( A + tE \), and define the vector-valued function \( x(t) \) such that \( (A + tE)x(t) = \lambda_i(t)x(t) \) where \( ||x(t)||_2 = 1 \) for some \( t \in [0,1] \). If \( \lambda_i(t) \) is simple, then

\[
\frac{\partial \lambda_i(t)}{\partial t} = x(t)^H E x(t).
\]  

(2.1)

Especially, the perturbation matrix \( E \) has the special structure. For example, the perturbation matrix \( E \) has the form as the matrix \( E_s \) whose block elements are zero except for the \( s \)-th block. Moreover if \( x(t) \) has small components in the positions corresponding to the nonzero elements of \( E \), then \( \frac{\partial \lambda_i(t)}{\partial t} \) is small. Hence if we know a bound for the components of \( x(t) \) that are in the position corresponding to the nonzero elements of \( E \), then we can obtain a bound for \( ||\lambda_i - \hat{\lambda}_i|| = ||\lambda_i(0) - \hat{\lambda}_i(1)|| \) via integrating the Equation (2.1) over \( 0 \leq t \leq 1 \).
Yuji Nakatsukasa [1] has derived the eigenvalues perturbation bounds for the case (1.2) with this idea. In the following, we shall describe in detail how this idea can be exploited to derive perturbation bounds of singular values for block tridiagonal matrix, and how this idea is expanded to derive eigenvalue perturbation bounds for our cases.

Note that the Lemma 2.1 holds under the condition that \( \lambda_t(\tau) \) is a simple eigenvalue of \( A + tE \). Similarly, we also assume that \( \lambda_t(\tau) \) is simple for all \( t \in [0,1] \). For multiple eigenvalues, we can discuss this case by referring to the method of the paper [1,6,7].

3. Singular Value Perturbation Bounds

In this section, we use the results in paper [1] to study the perturbation bounds of singular values for the block tridiagonal matrices. For the sake of convenience, we define the sequence of nonzero singular values of a complex \( p \times q \) matrix \( A \) by

\[
\sigma(A) = (\sigma_1(A), \ldots, \sigma_r(A)),
\]

where \( r = \text{rank}(A^H) \) and \( \sigma_1(A) \leq \cdots \leq \sigma_r(A) \). Similarly, for the perturbation matrix \( E \), we denote the rank of \( A + E \) by \( \tilde{r} \). Note that the nonzero eigenvalues of \( AA^H \) and \( A^H A \) are the same. Generally, the nonzero singular values of \( A \) have important applications in many filed, so it's necessary to study singular value perturbation bounds. Just as the discussion of the [1,8] we only consider the simple singular values perturbation bounds.

3.1. 2 \times 2 Case

Firstly, for the \( 2 \times 2 \) case, we have the following results concerning the nonzero singular values perturbation bounds.

**Theorem 3.1.** Let

\[
A_{2 \times 2} = \frac{m}{n} \begin{pmatrix} A_1 & B_1 \\ C_1 & A_2 \end{pmatrix} \quad \text{and} \quad E_{2 \times 2} = \frac{m}{n} \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}
\]

be two complex matrices, \( 0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_r \) and \( 0 < \tilde{\sigma}_1 \leq \tilde{\sigma}_2 \leq \cdots \leq \tilde{\sigma}_{\tilde{r}} \) be the nonzero singular values of \( A_{2 \times 2} \) and \( A_{2 \times 2} + E_{2 \times 2} \), respectively. Define \( \gamma = \max\left\{ \|B_1\|_2, \|C_1\|_2 \right\} \), \( \epsilon = \max\left\{ \|E_1\|_2, \|E_2\|_2 \right\} \) and

\[
\tau_i = \min \left\{ \frac{\gamma + \epsilon}{\|A_i\|_2} \right\} \quad \text{for} \quad i = 1, 2, \ldots, \min\{r, \tilde{r}\}, \quad \text{if} \quad \tau_i > 0 \quad \text{and the ith singular value} \quad \sigma_i \notin \sigma(A_2), \quad \text{then}
\]

we have \( |\sigma_i - \tilde{\sigma}_i| \leq \|E_1\|_2 + \|E_2\|_2 \tau_i^2 + 2\epsilon \tau_i \).

**Proof.** Let

\[
\tilde{A}_{4 \times 4} = \begin{pmatrix} 0 & A_{2 \times 2} \\ A_{2 \times 2}^H & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & A_1 & B_1 \\ 0 & 0 & C_1 & A_2 \\ A_1^H & C_1^H & 0 & 0 \\ B_1^H & A_2^H & 0 & 0 \end{pmatrix}
\]

and

\[
\tilde{E}_{4 \times 4} = \begin{pmatrix} 0 & E_{2 \times 2} \\ E_{2 \times 2}^H & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & E_1 & E_2 \\ 0 & 0 & E_3 & E_4 \\ E_1^H & E_2^H & 0 & 0 \\ E_3^H & E_4^H & 0 & 0 \end{pmatrix}.
\]

By Jordan-Wielandt theorem[2-Theorem I.4.2], we know that the eigenvalues of the matrix \( \tilde{A}_{4 \times 4} \) are \( \pm \sigma_i \), where \( 1 \leq i \leq m + n + k + \ell \). The same statement holds for \( \tilde{E}_{4 \times 4} \). Permuting the rows and columns of the matrix \( A_{4 \times 4} \) appropriately, we can get that the matrix \( A_{4 \times 4} \) is similar to
\[ A_{4 \times 4} = \begin{pmatrix} 0 & A_1 & 0 & B_1 \\ A_1^H & 0 & C_1^H & 0 \\ 0 & C_1 & 0 & A_2 \\ B_1^H & 0 & A_2^H & 0 \end{pmatrix} \]

and the matrix \( \tilde{E}_{4 \times 4} \) is similar to
\[ \tilde{E}_{4 \times 4} = \begin{pmatrix} 0 & E_1 & 0 & E_2 \\ E_1^H & 0 & E_2^H & 0 \\ 0 & E_3 & 0 & E_4 \\ E_2^H & 0 & E_4^H & 0 \end{pmatrix} \]

Let
\[ A_1 = \begin{pmatrix} 0 & A_1 \\ A_1^H & 0 \end{pmatrix}, \quad A_2^H = \begin{pmatrix} 0 & B_1 \\ C_1^H & 0 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} 0 & A_2 \\ A_2^H & 0 \end{pmatrix} \quad \text{and} \quad E_{21} = \begin{pmatrix} 0 & E_2 \\ E_2^H & 0 \end{pmatrix} \]

Obviously, the matrix \( \tilde{A}_{4 \times 4} = \begin{pmatrix} A_1 \\ A_2 \\ A_{22} \end{pmatrix} \) is a \( 2 \times 2 \) block Hermitian matrix, so is \( \tilde{E}_{4 \times 4} \). Note that the \( \| A_i \| = \max \{ \| B_i \|, \| C_i \| \} \), the eigenvalue set of \( A_{22} \) is \( \sigma(A_2) \), and \( \| E_{21} \| = \max \{ \| E_2 \|, \| E_4 \| \} \). So it is natural that we can apply the result of [1-Theorem 3.2] to get the conclusion.

3.2. 3 \times 3 Case

Secondly, we study the perturbation bounds for singular values of \( 3 \times 3 \) case. Let
\[ A_{3 \times 3} = \begin{pmatrix} A_1 \\ B_1 \\ \end{pmatrix}, \quad E_{3 \times 3} = \begin{pmatrix} 0 & A_3 \\ A_3^H & 0 \end{pmatrix} \]

be two complex matrices, where \( A_i \in \mathbb{C}^{n \times k_i} (i = 1, 2, 3) \) and \( \Delta A_i \in \mathbb{C}^{n \times k_i} \), \( 0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_r \) and \( 0 < \tilde{\sigma}_1 \leq \tilde{\sigma}_2 \leq \cdots \leq \tilde{\sigma}_r \) be the singular values of \( A_{3 \times 3} \) and \( A_{3 \times 3} + E_{3 \times 3} \), respectively. Similar to the discussion above, by permuting the rows and columns of \( \begin{pmatrix} 0 & A_3 \\ A_3^H & 0 \end{pmatrix} \) appropriately, we can get that the matrix
\[ \begin{pmatrix} 0 & A_{3 \times 3} \\ A_{3 \times 3}^H & 0 \end{pmatrix} \]

is similar to
\[ \tilde{A}_{6 \times 6} = \begin{pmatrix} 0 & A_1 & B_1 & 0 & 0 & 0 \\ A_1^H & 0 & 0 & C_1^H & 0 & 0 \\ B_1^H & 0 & 0 & A_2^H & C_2^H & 0 \\ 0 & C_1 & A_2 & 0 & 0 & B_2 \\ 0 & 0 & C_2 & 0 & 0 & A_3 \\ 0 & 0 & 0 & B_2^H & A_3^H & 0 \end{pmatrix} \]

and the matrix \( \begin{pmatrix} 0 & E_{3 \times 3} \\ E_{3 \times 3}^H & 0 \end{pmatrix} \) is similar to
\[ \tilde{E}_{6 \times 6} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta A_1^H & \Delta A_2^H & 0 & 0 \\ 0 & 0 & \Delta A_1 & 0 & 0 & \Delta B_2 \\ 0 & 0 & \Delta A_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta B_2^H & 0 & 0 \end{pmatrix} \]
Obviously, both $\hat{A}_{s\times s}$ and $\hat{E}_{s\times s}$ are block tridiagonal Hermitian matrices. Applying [1-Theorem 4.2], we can get the following theorem.

**Theorem 3.2.** Let $\sigma_i$ and $\hat{\sigma}_i$ be the $i$th smallest nonzero singular values of $A_{s\times s}$ and $A_{s\times s} + E_{s\times s}$, respectively. Define $\gamma = \max \{\|B_2\|_2, \|C_2\|_2\}$, $\epsilon = \max \{\|\Delta B_{s}\|_2, \|\Delta C_{s}\|_2\}$, $\delta = \max \{\|B_0\|_2, \|C_{n-1}\|_2\}$ and

$$r_i = \min \{\sigma_i - \sigma(A_{2}) - \|\Delta A_{2}\|_2 - \delta\}.$$ 

For $i = 1, 2, \ldots, \min\{r, \hat{r}\}$, if $\sigma_i \not\in (\sigma(A_{r}) \cup \sigma(A_{\hat{r}}))$, then we have

$$|\sigma_i - \hat{\sigma}_i| \leq \|\Delta A_{i}\|_2 r_i^2 + 2 \|\Delta B_{i}\|_2 r_i.$$ 

### 3.3. $n \times n$ Case

Further, we gradually consider the general $n \times n$ case and extend above statements to the $n \times n$ block tridiagonal matrices. Let

$$A_{n\times n} = \begin{pmatrix} A_1 & B_1 & & & \cdots & B_{n-1} \\ C_1 & & & & \cdots & C_{n-1} \\ & & \ddots & & \ddots & \vdots \\ & & & & & A_n \end{pmatrix}$$

and

$$E_{n\times n} = \begin{pmatrix} 0 & 0 & & & \cdots & 0 \\ 0 & \Delta A_1 & & & \cdots & \Delta B_1 \\ & & \ddots & & \ddots & \vdots \\ & & & & & \Delta C_{n-1} \\ & & & & & 0 \\ & & & & & \cdots \\ & & & & & 0 \end{pmatrix},$$

where $A_i \in \mathbb{C}^{m_i \times m_i}$ ($i = 1, 2, \ldots, n$) and $\Delta A_i \in \mathbb{C}^{m_i \times m_i}$, $0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_s$ and $0 < \hat{\sigma}_1 \leq \hat{\sigma}_2 \leq \cdots \leq \hat{\sigma}_s$ be the nonzero singular values of $A_{n\times n}$ and $A_{n\times n} + E_{n\times n}$, respectively. The following conclusion can be demonstrated.

**Theorem 3.3.** Let $\sigma_i$ and $\hat{\sigma}_j$ be the $i$th smallest nonzero singular values of $A_{n\times n}$ and $A_{n\times n} + E_{n\times n}$, respectively. Define $\gamma_j = \min \{\sigma_i - \sigma(A_i)\}$, $\gamma_j = \max \{\|B_j\|_2, \|C_j\|_2\}$, $\epsilon_j = \max \{\|\Delta B_j\|_2, \|\Delta C_j\|_2\}$. If there exists a positive integer $\ell$ such that $\gamma_j > \gamma_j + \epsilon_j + \epsilon_j + \epsilon_j + \epsilon_j + \epsilon_j + \epsilon_j$, where $j = 1, \ldots, s + \ell$, and

$$\delta_0 = \frac{\gamma_j + \epsilon_j}{\text{gap}_j - \gamma_j - \epsilon_j - \epsilon_j - \epsilon_j - \epsilon_j},$$

and

$$\delta_k = \frac{\gamma_{s+k}}{\text{gap}_{s+k} - \gamma_{s+k} - \epsilon_k - \epsilon_k},$$

for $k = 1, \ldots, \ell$,

then we have

$$|\sigma_i - \hat{\sigma}_j| \leq \|\Delta A_{i}\|_2 \left( \prod_{j=0}^{\ell} \delta_j \right)^2 + 2 \epsilon \delta_0 \left( \prod_{j=1}^{\ell} \delta_j \right)^2.$$ 

In what follows, we give an example to illustrate the singular values perturbation bounds obtained by our results.

**Example 3.1.** Consider the $4 \times 4$ matrices $A$ and $E$ represented by

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

and

$$E = \begin{pmatrix} 0 & E_3^* \\ E_3 & 0 \end{pmatrix},$$

where

$$A_1 = \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 3 \times 10^{-4} & -2 \times 10^{-2} \\ 2 \times 10^{-4} & -10^{-2} \end{pmatrix}.$$ 

Obviously, the last two singular values of $A$ are $\sigma_5 = 5, \sigma_6 = 6$. By computing, we can get that the two singular values of $A + E$ are

$$\hat{\sigma}_5 = 5.00011663424745 \quad \text{and} \quad \hat{\sigma}_6 = 6.000000023715121.$$ 

Therefore, we can get
\[ |\tilde{\sigma}_3 - \sigma_3| = 0.000116663424745 \quad \text{and} \quad |\tilde{\sigma}_4 - \sigma_4| = 0.000000023715121. \] (3.5)

Through the Theorem 3.1 we know that
\[ |\tilde{\sigma}_3 - \sigma_3| \leq 2.535e - 4 \quad \text{and} \quad |\tilde{\sigma}_4 - \sigma_4| \leq 2.022e - 4. \]

By comparing the differences in the equation (3.5) with the bounds obtained by the Theorem 3.1, we can find that the singular values perturbation bounds obtained by the Theorem 3.1 are sharp and this estimating method is efficient.

4. Eigenvalue Perturbation Bounds

On the basis of conclusions of the paper [1], in this section we study eigenvalue perturbation bounds of block tridiagonal matrix for the cases where two adjacent blocks of \( A \) are perturbed and the first \( s \) blocks of \( A \) are perturbed by the perturbation matrix \( E_{1,\cdots,s} \).

4.1. Two Adjacent Blocks of \( A \) Being Perturbed

In this subsection, we discuss eigenvalue perturbation bounds when two adjacent blocks of \( A \) are perturbed. In other words, we consider the matrices in the following form

\[
A = \begin{pmatrix}
A_1 & B_1^H \\
B_1 & \ddots & B_{s-1}^H \\
& \ddots & \ddots & B_{s-1}^H \\
& & \ddots & A_s \\
B_{n-1} & & B_{n-1} & A_n
\end{pmatrix}
\]

and

\[
E_{s+1} = \begin{pmatrix}
\delta_1 & \Delta A_j & 0 & \Delta B_{s-1}^H & 0 \\
\Delta B_j & \Delta A_{s+1} & \Delta B_{s-2}^H & 0 & \Delta B_{s+1} \\
0 & \Delta B_{s-1}^H & \Delta A_s & \Delta B_{s-3}^H & 0 \\
\Delta B_j & \Delta B_{s-2}^H & \Delta B_{s-3}^H & \Delta A_{s+1} & \Delta B_{s+1} \\
0 & \Delta B_{s-1}^H & \Delta B_{s-3}^H & \Delta B_{s+1} & \delta_1
\end{pmatrix}
\]

Similar to discussion of the paper [1], we need the following assumption.

Assumption 1. There exists an integer \( \ell > 0 \) such that \( \lambda_i \notin \lambda_{\min}(A_j) - \eta_j, \lambda_{\max}(A_j) + \eta_j \), where

\[ \eta_j = \|B_j\|_2 + \|B_{j-1}\|_2 + \|E_{s+1}\|_2 + \|\Delta A_j\|_2 + \|\Delta B_j\|_2 + \|\Delta B_{j-1}\|_2, \quad j = 1, \cdots, s + \ell. \]

Roughly, the assumption demands that \( \lambda_i \) is far away from the eigenvalues of \( A_1, \cdots, A_{s+1} \), respectively, and the norms of \( E_{s+1} \) and \( B_1, \cdots, B_{s+1} \) are not too large.

Now, on the basis of the Assumption 1, we first discuss upper bounds for the eigenvector components of the matrix \( A + tE_{s+1} \).

Lemma 4.1. Let \( A \) and \( E_{s+1} \) be Hermitian block-tridiagonal matrices in (4.1), \( \lambda_i \) be the ith smallest eigenvalue of \( A \). For \( t \in [0,1] \), let \( (A + tE_{s+1})x(t) = \lambda_i(t)x(t) \), where \( \|x(t)\|_2 = 1 \) and \( x(t)^* = [x_1(t)^*, x_2(t)^*, \cdots, x_n(t)^*]^* \) satisfying that \( x_j(t) \) and \( A_j \) have the same number of rows. Define

\[ \delta_0 = \frac{\|B_j\|_2 + \|\Delta B_j\|_2}{\text{gap}_{s+1} - \|E_{s+1}\|_2 - \|\Delta A_j\|_2 - \|B_{s+1}\|_2}, \]

\[ \delta_1 = \frac{\|B_{s+1}\|_2 + \|\Delta B_{s+1}\|_2}{\text{gap}_{s+1} - \|E_{s+1}\|_2 - \|B_j\|_2 - \|\Delta B_j\|_2 - \|\Delta A_{s+1}\|_2}, \]

and for \( j = 3, \cdots, \ell \),

\[ \delta_j = \frac{\|B_{s+1}\|_2}{\text{gap}_{s+j} - \|E_{s+1}\|_2 - \|B_{s+j}\|_2}. \] (4.2)

If \( \lambda_i \) satisfies Assumption 1, then, for all \( t \in [0,1] \) we have
\[
\left\| x(t) \right\|_2 \leq \prod_{j=2}^{l} \delta_j, \quad \left\| x_{s+1}(t) \right\|_2 \leq \prod_{j=4}^{l} \delta_j \quad \text{and} \quad \left\| x_{s+2}(t) \right\|_2 \leq \prod_{j=2}^{l} \delta_j. \tag{4.3}
\]

**Proof.** The first block component of \( (A + tE_{s+1}) x(t) = \lambda_i(t) x(t) \) is
\[
A_i x_i(t) + B_i^{\mu} x_i(t) = \lambda_i(t) x_i(t).
\]
Since \( \text{gap}_j > B_j \left\| E_{s+1} \right\|_2 + \left\| E_{s+1} \right\|_2 + \left\| \Delta J \right\|_2 + \left\| \Delta B \right\|_2 + \left\| \Delta B_j \right\|_2 \) for \( j = 1, \ldots, s + \ell \), by Weyl’s theorem, we have \( \lambda_i(t) \notin \lambda(A_j) \). Therefore, we have
\[
x_i(t) = (\lambda_i(t) I - A_i)^{-1} B_i^{\mu} x_i(t).
\]
Further, by applying the Theorem 2.1[2], we know \( \lambda_i(t) \in \left[ \lambda_i - \left\| E_{s+1} \right\|_2, A_i + \left\| E_{s+1} \right\|_2 \right] \) and\[
\left\| \lambda_i(t) I - A_i \right\|_2 \leq \frac{1}{\text{gap}_i - \left\| E_{s+1} \right\|_2}.
\]
So we can bound \( \left\| x_i(t) \right\|_2 / \left\| x_i(t) \right\|_2 \) by
\[
\left\| x_i(t) \right\|_2 \leq \left\| x_i(t) \right\|_2 \leq \frac{1}{\text{gap}_i - \left\| E_{s+1} \right\|_2} \leq 1,
\]
where the right inequality follows from Assumption 1. Continuously, the second block component of \( (A + tE_{s+1}) x(t) = \lambda_i(t) x(t) \) is
\[
B_i x_i(t) + A_i x_2(t) + B_i^{\mu} x_3(t) = \lambda_i(t) x_2(t).
\]
So,
\[
x_2(t) = (\lambda_i(t) I - A_2)^{-1} (B_i x_i(t) + B_i^{\mu} x_3(t)).
\]
Similarly, by Weyl's theorem, we have \( \left\| (\lambda_i(t) I - A_2)^{-1} \right\|_2 \leq \frac{1}{\text{gap}_2 - \left\| E_{s+1} \right\|_2} \). Combining this inequality with (4.4), we can get
\[
\left\| x_2(t) \right\|_2 \leq \left\| x_2(t) \right\|_2 \leq \frac{\left\| B_i \right\|_2}{\text{gap}_2 - \left\| E_{s+1} \right\|_2} \leq 1.
\]
Hence, \( \left\| x_2(t) \right\|_2 < \left\| x_1(t) \right\|_2 \).

By the same argument, we can prove \( \left\| x_1(t) \right\|_2 < \left\| x_2(t) \right\|_2 < \cdots < \left\| x_{s+1}(t) \right\|_2 \) for all \( t \in [0,1] \).

To consider the \( s \)th block component of \( (A + tE_{s+1}) x(t) = \lambda_i(t) x(t) \), we have
\[
B_{s-1} x_{s-1}(t) + (A_i + t \Delta A_i) x_s(t) + (B_i^{\mu} + t \Delta B_s^{\mu}) x_{s+1}(t) = \lambda_i(t) x_s(t),
\]
thus,
\[
x_s(t) = (\lambda_i(t) I - A_i - t \Delta A_i)^{-1} (B_{s-1} x_{s-1}(t) + (B_i^{\mu} + t \Delta B_s^{\mu}) x_{s+1}(t)).
\]
By using the results of the Assumption 1 and Theorem 2.1[2], we know that \( (\lambda_i I - A_i - t \Delta A_i)^{-1} \) is invertible and \( \left\| (\lambda_i I - A_i - t \Delta A_i)^{-1} \right\|_2 \leq \frac{1}{\text{gap}_s - \left\| E_{s+1} \right\|_2 - \left\| \Delta A_i \right\|_2} \). Since \( \left\| x_{s-1}(t) \right\|_2 < \left\| x_{s+1}(t) \right\|_2 \), we can get
\[
\left\| x_s(t) \right\|_2 \leq \frac{\left\| B_{s-1} \right\|_2 \left\| x_{s-1}(t) \right\|_2 + \left\| B_i^{\mu} + t \Delta B_s^{\mu} \right\|_2 \left\| x_{s+1}(t) \right\|_2}{\text{gap}_s - \left\| E_{s+1} \right\|_2 - \left\| \Delta A_i \right\|_2}.
\]
Therefore, for all \( t \in [0,1] \) we can obtain the following result.
Continuously, considering the \(s+1\)th block of \((A + tE_{s+1})x(t) = \lambda(t)x(t)\),
\[
(B_{s+1} + t\Delta B_{s+1})x_{s+1}(t) + (A_{s+1} + t\Delta A_{s+1})x_{s+1}(t) + (B_{s+1} + t\Delta B_{s+1}''')x_{s+2}(t) = \lambda(t)x_{s+1}(t),
\]
we have
\[
x_{s+1}(t) = (\lambda(t)I - A_{s+1} + t\Delta A_{s+1})^{-1}((B_{s+1} + t\Delta B_{s+1})x_{s}(t) + (B_{s+1} + t\Delta B_{s+1}''')x_{s+2}(t)).
\]

Similarly, by using the results of the Assumption 1 and Theorem 2.1[2], we know that \(\lambda(t)I - A_{s+1} + t\Delta A_{s+1}\) is invertible and
\[
\| (\lambda(t)I - A_{s+1} + t\Delta A_{s+1})^{-1} \| \leq \frac{1}{\text{gap}_{s+1} - \|E_{s+1}\|_2 - \|\Delta A_{s+1}\|_2 - \|B_{s+1}\| - \|\Delta B_{s+1}\|_2}.
\]
Since \(\|x_s(t)\|_2 < \|x_{s+1}(t)\|_2\), for all \(t \in [0,1]\), we can get
\[
\|x_{s+1}(t)\|_2 \leq \frac{\|B_{s+1}\| + \|\Delta B_{s+1}\|_2}{\text{gap}_{s+1} - \|E_{s+1}\|_2 - \|\Delta A_{s+1}\|_2} \|x_s(t)\|_2 + \|x_{s+2}(t)\|_2.
\]

Hence,
\[
\|x_{s+1}(t)\|_2 \leq \frac{\|B_{s+1}\| + \|\Delta B_{s+1}\|_2}{\text{gap}_{s+1} - \|E_{s+1}\|_2 - \|\Delta A_{s+1}\|_2} \|x_s(t)\|_2 + \|x_{s+2}(t)\|_2 = \delta_{s+1}.
\]

Similar to the discussion above, we also have
\[
(B_{s+1} + \Delta B_{s+1})x_{s+1}(t) + A_{s+2}x_{s+2}(t) + B_{s+2}'x_{s+3}(t) = \lambda(t)x_{s+2}(t),
\]
\[
x_{s+2}(t) = (\lambda(t)I - A_{s+2})^{-1}((B_{s+1} + \Delta B_{s+1})x_{s+1}(t) + B_{s+2}'x_{s+3}(t)).
\]

By \(\| (\lambda(t)I - A_{s+2})^{-1} \| \leq \frac{1}{\text{gap}_{s+2} - \|E_{s+2}\|_2}\), we have
\[
\|x_{s+2}(t)\|_2 \leq \frac{\|B_{s+1}\| + \|\Delta B_{s+1}\|_2}{\text{gap}_{s+2} - \|E_{s+2}\|_2} \|x_{s+1}(t)\|_2 + \|B_{s+2}'\| \|x_{s+3}(t)\|_2.
\]

Consequently, for all \(t \in [0,1]\),
\[
\|x_{s+2}(t)\|_2 \leq \frac{\|B_{s+1}\| + \|\Delta B_{s+1}\|_2}{\text{gap}_{s+2} - \|E_{s+2}\|_2} \|x_{s+1}(t)\|_2 + \|B_{s+2}'\| \|x_{s+3}(t)\|_2 - \|\Delta A_{s+1}\|_2 = \delta_{s+2}.
\]

Similar to the discussion above, we can prove \(\|x_{s+3}(t)\|_2 \leq \delta_{s+3}, j = 1, \ldots, \ell\). In addition, \(\|x_{s+4}(t)\|_2 \leq \|x(t)\|_2 \leq 1\).

Based on the discussion above, we conclude that for all \(t \in [0,1]\),
\[
\|x(t)\|_2 \leq \sum_{j=1}^{\ell} \delta_j, \quad \|x_{s+1}(t)\|_2 \leq \sum_{j=1}^{\ell} \delta_j\text{ and } \|x_{s+2}(t)\|_2 \leq \sum_{j=1}^{\ell} \delta_j.
\]

The following Theorem 4.1 is aiming to present perturbation bounds for \(\lambda\).

**Theorem 4.1.** Let \(\lambda_i\) and \(\hat{\lambda}_i\) be the \(i\)th smallest eigenvalues of matrix \(A\) and \(A + E_{s+1}\), respectively, and \(\delta_i\) be defined as in (4.2). If \(\lambda_i\) satisfies the Assumption 1, we have
\[
|\lambda_i - \hat{\lambda}_i| \leq \|\Delta A\|_2 \left( \sum_{j=1}^{\ell} \delta_j \right)^2 + \left( 2 \|\Delta B\|_2 \delta_0 + \|\Delta A_{s+1}\|_2 \right) \left( \sum_{j=1}^{\ell} \delta_j \right)^2 + 2 \|B_{s+1}\|_2 \delta_0 \left( \sum_{j=1}^{\ell} \delta_j \right)^2.
\]

**Proof.** Integrating (2.1) over \(0 \leq t \leq 1\) we get
Together with (4.3), it follows that

\[
|\lambda - \hat{\lambda}| \leq \left\| \Delta A \right\|_2 \left\| \frac{1}{0} \left( \prod_{j=0}^{t} \delta_j \right)^2 \right\| + 2 \left\| \Delta B \right\|_2 \left\| \frac{1}{0} \left( \prod_{j=0}^{t} \delta_j \right)^2 \right\| dr
+ \left\| \Delta A_{s+1} \right\|_2 \left\| \frac{1}{0} \left( \prod_{j=0}^{t} \delta_j \right)^2 \right\| \left\| \frac{1}{0} \left( \prod_{j=0}^{t} \delta_j \right)^2 \right\| dr
+ 2 \left\| \Delta B_{s+1} \right\|_2 \left\| \frac{1}{0} \left( \prod_{j=0}^{t} \delta_j \right)^2 \right\| .
\]

\[\square\]

4.2. The First \(s\) Blocks of \(A\) Being Perturbed

In this subsection, we gradually consider the bounds of eigenvalues of the matrix \(A\), whose the first \(s\) blocks are perturbed simultaneously. In other words, we consider the perturbation matrix

\[
E_{t,2,\ldots,s} = \begin{pmatrix}
\Delta A_1 & \Delta B_1^H & \cdots & \\
\Delta A_2 & \cdots & \Delta B_s^H & \\
\cdots & \cdots & \cdots & \\
\Delta A_s & 0 & 0 & \\
\Delta B_1 & 0 & 0 & \\
\end{pmatrix},
\]

where \(s\) is a positive integer.Let \((\hat{\lambda}_i(t), x(t))\) denote the \(i\)th eigenpair of \(A + tE_{t,2,\ldots,s}\) satisfying

\[\left( A + tE_{t,2,\ldots,s} \right) x(t) = \hat{\lambda}_i x(t), \]

and the partition of \(x(t)^H = \left[ x_1(t)^H, x_2(t)^H, \ldots, x_n(t)^H \right]^H\) satisfies \(x_j(t)\) and \(A_j\) have the same number of rows, where \(t \in [0,1]\).

If \(\hat{\lambda}_i\) satisfies the Assumption 1, through the similar discussion as above, we can derive a similar conclusion for calculating the eigenvalue perturbation bounds. For simplicity, we don't repeat the proof here. The Algorithm 1 below shows the calculation in detail, where \(B_0 = 0\), \(B_s = 0\) and \(\Delta B_0 = 0\).

5. Numerical Example

In this section, we use the following example to illustrate the validity of our method and to show the advantage of the our method over the method proposed in [1].

Example 5.1 [1]. Let \(A + E\) be the \(1000 \times 1000\) tridiagonal matrix

\[
A + E = \text{tridiag} \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1000 & 999 & \cdots & 2 \\
1 & 1 & \cdots & 1
\end{pmatrix}
\]

(5.1)
Algorithm 1. Eigenvalue perturbation bound algorithm for the first \( s \) blocks of \( A \) being perturbed.

**Input:** \( A, \ A+E_{i_{r-1}}, \) and \( s; \)

**step 1:** Compute the \( i \)th eigenvalues \( \lambda_i \) and \( \hat{\lambda}_i \) of \( A \) and \( A+E_{i_{r-1}}, \) respectively;

Choose an integer \( \ell > 0 \) such that \( \lambda_i \) satisfies the Assumption 1;

**step 2:**

for \( j = 1,2,\ldots,s+1 \) do

Compute the \( \mu_j = \| B_j + \| \Delta B_j \| ; \)

\( \nu_j = \text{gap}_j - \| E_{i_{r-1}} \| - \| B_j \| - \| \Delta E_{i_{r-1}} \| - \| \Delta B_j \| ; \)

\( \delta_j = \mu_j / \nu_j ; \)

end for

**step 3:**

for \( k = 1 \) to \( s + \ell - 1 \) do

\( \omega_k = \delta_k ; \)

for \( j = k \) to \( s + \ell - 1 \) do

\( \omega_j = \omega_k \delta_{n_{k-1}} ; \)

end for

end for

**step 4:**

\( \Delta \lambda_i = \omega_i \| \Delta A \| + 2 \| \Delta B \| \omega_i \omega_i ; \quad \omega_{n_{k-1}} = 0 ; \)

**step 5:**

for \( k = 2,3,\ldots,s+\ell \) do

\( \Delta \lambda_i = \Delta \lambda_{i_{k-1}} + \omega_k \| \Delta E_{i_{r-1}} \| + 2 \| \Delta B \| \omega_k \omega_k ; \)

end for

**Output:**

the eigenvalue perturbation bounds \( \Delta \lambda_i \);
Table 1. The eigenvalue perturbation bounds and CUP times.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>bound</td>
<td>$8.5e-292$</td>
<td>$6.8e-289$</td>
<td>$1.5e-286$</td>
<td>$2.1e-284$</td>
<td>$3.3e-282$</td>
</tr>
<tr>
<td>time (s)</td>
<td>160.81</td>
<td>164.94</td>
<td>165.11</td>
<td>168.75</td>
<td>171.84</td>
</tr>
</tbody>
</table>

Table 2. The eigenvalue perturbation bounds and CUP times.

<table>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>bound</td>
<td>$1.05e-290$</td>
<td>$9.7e-288$</td>
<td>$2.4e-285$</td>
<td>$3.96e-283$</td>
<td>$7.2e-281$</td>
</tr>
<tr>
<td>time (s)</td>
<td>330.93</td>
<td>372.81</td>
<td>316.51</td>
<td>383.42</td>
<td>378.66</td>
</tr>
</tbody>
</table>

Table 3. The eigenvalue perturbation bounds and CUP times.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
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<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>bound</td>
<td>$1.1e-289$</td>
<td>$1.1e-286$</td>
<td>$3.1e-284$</td>
<td>$5.7e-282$</td>
<td>$1.2e-279$</td>
</tr>
<tr>
<td>time (s)</td>
<td>157.95</td>
<td>154.78</td>
<td>156.89</td>
<td>156.57</td>
<td>157.30</td>
</tr>
</tbody>
</table>

provide an algorithm for the general case when the first $s$ blocks of $A$ are perturbed simultaneously. Number examples are presented to show the effectiveness of our methods.

REFERENCES