On the Average Errors of Multivariate Lagrange Interpolation*

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ABSTRACT

In this paper, we discuss the average errors of multivariate Lagrange interpolation based on the Chebyshev nodes of the first kind. The average errors of the interpolation sequence are determined on the multivariate Wiener space.

Keywords: Multivariate Lagrange Interpolation; Average Error; Chebyshev Polynomial; Wiener Sheet Measure

1. Introduction

Let \( F \) be a real separable Banach space equipped with a probability measure \( \mu \) on the Borel sets of \( F \). Let \( X \) be another normed space such that \( F \) is continuously embedded in \( X \). By \( \| \cdot \|_X \) we denote the norm in \( X \). Any \( T : F \to X \) such that \( f \mapsto \| f - T(f) \|_X \) is a measurable mapping is called an approximation operator. The average error of \( T \) is defined as

\[
e(T, F, \| \cdot \|_X) := \left( \int_F \left[ \int \| f - T(f) \|_X^2 \right] \mu(df) \right)^{1/2}.
\]

For \( d \geq 1 \), let

\[
C_{0,d} = \{ f \in C[0,1]^d \mid f(x_1, \ldots, x_d) = 0, \text{whenever } x_i = 0 \text{ for some } 1 \leq i \leq d \}.
\]

The space \( C_{0,d} \) equipped with the sup norm

\[
\| f \| = \sup_{i=0,1} \| f(t) \|
\]

The classical Wiener sheet measure \( w_d \) on \( \mathcal{B}(C_{0,d}) \) is Gaussian with mean zero and covariance kernel

\[
R_{w_d}(s,t) = \int_{C_{0,d}} f(s)f(t) w_d(df) = \left[ \min\{s,t\} \right], \quad s,t \in [0,1]^d.
\]

For more detailed discussion and properties of \( w_d \), we refer to [1].

In this paper, we let

\[
F_d = \{ f \in C[-1,1]^d \mid g(x_1, \ldots, x_d) = f(2x_1 - 1, \ldots, 2x_d - 1) \in C_{0,d} \}.
\]

For every measurable subset \( A \subseteq \mathcal{B}(F_d) \), we define the measure of \( A \) by

\[
\mu_d(A) := w_d \{ g(x_1, \ldots, x_d) = f(2x_1 - 1, \ldots, 2x_d - 1) \in A \}.
\]

Let \( \rho(x_1, \ldots, x_d) = \left( \prod_{i=1}^d \sqrt{1-x_i^2} \right) \), the weighted \( L_2 \) norm for \( f \in F_d \) is defined as

\[
\| f \|_{L_2} := \left( \int_{[0,1]^d} | f(t) |^2 \rho(t) \, dt \right)^{1/2}.
\]

Let

\[
\xi_k := \frac{2k-1}{2n}, \quad k = 1, \ldots, n
\]

is the zeros of \( T_n(x) = \cos n\theta \) \( (x = \cos \theta) \), the \( n \)th degree Chebyshev polynomial of the first kind. For \( f \in F_d \), the well-known Lagrange interpolation polynomial of \( f \) based on \( \{ x_1, \ldots, x_d \}_k \) is given by

\[
L_{n,d}(f, x) := \sum_{k=1}^n f(\xi_k) \prod_{i=1, i \neq k}^d (x_i - \xi_i) / \prod_{i=1, i \neq k}^d (x_i - \xi_k),
\]

where

\[
l_{j,j}(x_j) = \prod_{k=1, k \neq j}^d \frac{x_j - x_k}{\xi_j - \xi_k}, \quad j = 1, \ldots, d.
\]

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2. Main Result

Since the polynomial interpolation operators are impor-
tant approximation tool in the continuous functions space,
there are a number of papers studying the convergence
for interpolation polynomial, especially the interpolation
polynomial based on roots of orthogonal polynomials.
Xu Guiqiao [2] studied the average errors of univariate
Lagrange interpolation based on the Chebyshev nodes on
the Wiener space. Motivated by [2], we consider the av-
erage errors of multivariate Lagrange interpolation. We
first study the bivariate Lagrange interpolation, then the
general multivariate Lagrange interpolation. Our main
results are the following:

Theorem 1. Let

\[ x = (x_1, x_2) \in [-1, 1]^2, \]
\[ \rho(x_1, x_2) = \frac{1}{\sqrt{1 - x_1^2 \sqrt{1 - x_2^2}}}, \]

and

\[ L_{m,d}(f, x) = \sum_{i=1}^{m} \sum_{j=1}^{d} f(x_i, x_j) \delta_{x_i, x_j}(x_i, x_j), \]

where

\[ \delta_{x_i, x_j}(x_i, x_j) = f(x_i, x_j) \delta_{x_i, x_j}(x_i, x_j). \]

Then we have

\[ e^2(L_{m,d}, F_2; \| \cdot \|_{L_2}), = \int_{F_2} \| f(x) - L_{m,d}(f, x) \|_{L_2^2}^2 \mu_d(df) \]
\[ = \frac{\pi}{2} \left( \sin \frac{\pi}{m} \cos \frac{\pi}{2m} + \sin \frac{\pi}{n} \cos \frac{\pi}{2n} \right) \frac{m^2 (1 - \cos \frac{\pi}{m})}{n^2 (1 - \cos \frac{\pi}{n})}, \]

Then we have

\[ e^2(L_{m,d}, F_2; \| \cdot \|_{L_2}), = \int_{F_2} \| f(x) - L_{m,d}(f, x) \|_{L_2^2}^2 \mu_d(df) \]
\[ = \frac{1}{2^{d-k} \sum_{k=0}^{d-k} \binom{d}{k} (-1)^{d-k}} \left( \sin \frac{\pi}{n} \cos \frac{\pi}{2n} + \sin \frac{\pi}{2n} \left( 1 - \cos \frac{\pi}{n} \right) \right)^k \frac{n^2 (1 - \cos \frac{\pi}{n})}{\pi^{d-k}}. \]

Remark 1. Let us recall some fundamental notions
about the information-based complexity in the average
case setting. Let \( F \) be a set with a probability measure \( \nu \), and let \( G \) be a measurable mapping from \( F \) into \( \mathbb{R}^d \), and let \( \phi \) be a measurable mapping from \( \mathbb{R}^d \) into \( \mathbb{G} \) which are
called an information operator and an algorithm, respecti-
vely. The average error of the approximation \( \phi \circ N \)
with respect to the measure \( \nu \) is defined by

\[ e(S, N, \phi) := \int_{F_2} \| S(f) - \phi(N(f)) \|_{L_2^2} \nu(df) \frac{1}{2}, \]

and the average radius of information \( N \) with respect to
\( \nu \) is defined by

\[ r(S, N) := \inf_{\phi} e(S, N, \phi), \]

where \( \phi \) ranges over the set of all possible algorithms
that use information \( N \). Furthermore, let \( \Lambda_m \) be the
class of all deterministic information operators \( N \) with
_cardinality \( m \). Then, the \( m \)th minimal average radius
is defined by

\[ r(S, \Lambda_m) := \inf_{N \in \Lambda_m} r(S, N). \]

For \( F = C_{0, d}, \nu = w_{d, 1}, S = I \) (the identity mapping),
and \( \Lambda_m \) consisting of function values taken at grid
points, i.e.,

\[ N(f) = \{ f(h_1, \cdots, h_d), \cdots, f(i_1 h_1, \cdots, i_d h_d), \cdots, f(1 - h_1, \cdots, 1 - h_d) \} \]
for some $h_i, \cdots, h_j$, by [3,p.16] we know

$$r(S, A_n) \leq \frac{1}{m^{1/2}}.$$  

From Theorem 2 we have

$$e(L_n, F_n, \|\|_{2,\rho}) = \frac{1}{n^{1/2}}.$$  

Note that $m = n^d$, we can say that the average error of $L_{n,d}$ is weakly equivalent to the corresponding $n^d$ th minimal average radius.

3. Proof of Theorem 1

Proof of Theorem 1. By a simple computation, we have

$$e^2(L_n, F_n, \|\|_{2,\rho}) = \int_{[0,1]^d} f(x) - L_n(f, x) \rho(x) dx \mu_2(df)$$

$$= \int_{[0,1]^d} \left( \int_{[0,1]^d} f(x) - 2f(x)L_n(f, x) + L_n^2(f, x) \right) \rho(x) dx \mu_2(df)$$

$$= \int_{[0,1]^d} \left( f(x) \mu_2(df) - 2 \int_{[0,1]^d} f(x)L_n(f, x) \mu_2(df) \right) \rho(x) dx + \int_{[0,1]^d} \{ L_n(f, x) \mu_2(df) \} \rho(x) dx$$

$$= I_1 - 2I_2 + I_3.$$  

On using (1) and (2), we obtain

$$I_1 = \int_{[0,1]^d} f(x) \mu_2(df) \rho(x) dx = \int_{[0,1]^d} g(x) \left( \frac{x_1+1}{2}, \frac{x_2+1}{2} \right) \mu_2(df) \rho(x) dx$$

$$= \int_{0}^{1} \frac{x_1+1}{2} dx \int_{0}^{1} \frac{x_2+1}{2} dx = \frac{\pi^2}{4}. \tag{5}$$

From [2], we have

$$I_2 = \sum_{i=1}^{m} \sum_{j=1}^{m} \int_{[0,1]^d} f(x)L_n(f, x) \mu_2(df) \rho(x) dx$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \int_{[0,1]^d} g(x_1, x_2) \left( \xi_{i,m}^{x_1}, \xi_{j,m}^{x_2} + 1 \right) \left( \frac{x_1+1}{2}, \frac{x_2+1}{2} \right) \mu_2(df) \rho(x_1, x_2) dx_1 dx_2$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \int_{[0,1]^d} \left( \frac{x_1+1}{2}, \frac{x_2+1}{2} \right) \mu_2(df) \rho(x_1, x_2) dx_1 dx_2$$

$$= \int_{0}^{1} \frac{x_1+1}{2} dx \int_{0}^{1} \frac{x_2+1}{2} dx$$

$$= \frac{\pi^2}{4}. \tag{6}$$

and

$$I_3 = \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{l=1}^{m} \int_{[0,1]^d} L_n(f, x) \mu_2(df) \rho(x) dx$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{l=1}^{m} \int_{[0,1]^d} g(x_1, x_2) \left( \xi_{i,m}^{x_1}, \xi_{j,m}^{x_2} + 1 \right) \left( \xi_{j,m}^{x_1}, \xi_{l,m}^{x_2} + 1 \right) \mu_2(df) \rho(x_1, x_2) dx_1 dx_2$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{l=1}^{m} \int_{[0,1]^d} \left( \frac{x_1+1}{2}, \frac{x_2+1}{2} \right) \mu_2(df) \rho(x_1, x_2) dx_1 dx_2$$

$$= \int_{0}^{1} \frac{x_1+1}{2} dx \int_{0}^{1} \frac{x_2+1}{2} dx$$

$$= \frac{\pi^2}{4}. \tag{7}$$

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On combining (4)-(7), we obtain

\[
e^2(L_{min}, F_2, \|\cdot\|_{l_2, \rho}) = \frac{\pi^2}{4} \left[ \frac{\sin \frac{\pi}{m} \cos \frac{\pi}{2m} + \sin \frac{\pi}{2m} (1 - \cos \frac{\pi}{m}) \cdot (\pi - \frac{\sin \frac{\pi}{n} \cos \frac{\pi}{2n} + \sin \frac{\pi}{2n} (1 - \cos \frac{\pi}{n})}{n^2 (1 - \cos \frac{\pi}{n})} \right] + \frac{\pi^2}{4} \frac{1}{m^2 (1 - \cos \frac{\pi}{m})} \]

we complete the proof of Theorem 1.

4. Proof of Theorem 2

Proof of Theorem 2. Similar to the proof of Theorem 1,

\[
e^2(L_{n,d}, F_d, \|\cdot\|_{l_2, \rho}) = \left[ \int \int f(x) - L_{n,d}(f, x) \right] \mu_d(df) = \int f(x) \mu_d(df) \cdot \rho(x) dx = \int f(x) L_{n,d}(f, x) \mu_d(df) \cdot \rho(x) dx + \int L_{n,d}(f, x) \mu_d(df) \cdot \rho(x) dx = J_1 + J_2.
\]

Form (1) and (2),

\[
J_1 = \left[ \int \int f(x) \mu_d(df) \cdot \rho(x) dx = \int f(x) \mu_d(df) \cdot \rho(x) dx = \left[ \int f(x) L_{n,d}(f, x) \mu_d(df) \cdot \rho(x) dx + \int L_{n,d}(f, x) \mu_d(df) \cdot \rho(x) dx = J_1 + J_2.
\right]
\]

By a simple computation similar to (6)-(7) we obtain

\[
J_2 = \frac{\pi^2}{4} \left[ \int f(x) L_{n,d}(f, x) \mu_d(df) \cdot \rho(x) dx + \int L_{n,d}(f, x) \mu_d(df) \cdot \rho(x) dx = \left[ \int f(x) L_{n,d}(f, x) \mu_d(df) \cdot \rho(x) dx + \int L_{n,d}(f, x) \mu_d(df) \cdot \rho(x) dx = J_1 + J_2.
\right]
\]
and

\[ J_3 = \int_{[-1,1]^d} \{ \int_{[-1,1]} l_{n,d}^2 (f, x) \mu_d (df) \} \rho(x) dx \]

\[ = \sum_{k_1, \ldots, k_d=1}^{n} \int_{[-1,1]} \left\{ \int_{[0,1]} \left( \int_{[0,1]} \left( \prod_{i=1}^{d} l_{k_i} (x_i) \right) \prod_{i=1}^{d} l_{k_i} (x_i) \rho(x_1, \ldots, x_d) dx_1 \right) \prod_{i=1}^{d} l_{k_i} (x_i) \rho(x_1, \ldots, x_d) dx_1 \right\} dx_1 \cdots dx_d \]

\[ = \sum_{k_1, \ldots, k_d=1}^{n} \int_{[-1,1]} \left\{ \int_{[0,1]} \left( \prod_{i=1}^{d} l_{k_i} (x_i) \right) \prod_{i=1}^{d} l_{k_i} (x_i) \rho(x_1, \ldots, x_d) dx_1 \right\} dx_1 \cdots dx_d \]

(11)

On combining (8)-(11), we obtain

\[ e^2 (L_{n,d}, F_2, \|F_2\|_2) = \left( \frac{\pi}{2} \right) ^d - \frac{1}{2^{d+1}} \left( \pi - \frac{\sin \frac{\pi}{n} \cos \frac{\pi}{2n} + \sin \frac{\pi}{2n} (1 - \cos \frac{\pi}{n})}{n^2 (1 - \cos \frac{\pi}{n})} \right)^d + \left( \frac{\pi}{2} \right) ^d \]

\[ = \frac{1}{2^{d+1}} \sum_{k=1}^{d} \binom{d}{k} (-1)^{k-1} \left( \frac{\sin \frac{\pi}{n} \cos \frac{\pi}{2n} + \sin \frac{\pi}{2n} (1 - \cos \frac{\pi}{n})}{n^2 (1 - \cos \frac{\pi}{n})} \right)^k n^{d-k} \]

We complete the proof of Theorem 2.

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