Spectrum of Signals on the Quaternion Fourier Transform Domain

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ABSTRACT
The quaternion Fourier transform plays a vital role in the representation of two-dimensional signals. This paper characterizes spectrum of quaternion-valued signals on the quaternion Fourier transform domain by the partial derivative.

Keywords: Spectrum; Quaternion Fourier Transform; Partial Derivative

1. Introduction
The quaternion Fourier transform (QFT) is a nontrivial generalization of the real and complex Fourier transform to quaternion cases. The four QFT components separate four cases of symmetry in real signals instead of only two in the complex FT. The QFT plays a vital role in the representation of signals and transforms a quaternion 2D signal into a quaternion-valued frequency domain signal. Many efforts had been devoted to some important properties and applications of the QFT [1-7].

In the last few years, there has been a great interest to the study of the spectrum of signals, i.e. the support of the convolution of spectrums of signals relatively to certain integral transforms [8-15].

Motivated by the treatment of the QFT in quaternion algebra, in this paper we will characterize the quaternion-valued signals whose QFT has compact support. The main difficulty lies in the fact that the quaternion algebra is non-commutative, so one cannot directly extend the results for the Fourier transform to those for the QFT.

This paper is organized as follows: Section 2 is devoted to reviewing some necessary results about the quaternion algebra. In Section 3, based on the definition and some properties of the QFT, we get a result to describe the spectrum for the QFT.

2. Preliminaries
The quaternion algebra \( \mathbb{H} \) is an extension of the algebra of complex numbers to a four dimensional real algebra. It is given by

\[ \mathbb{H} = \{ q \mid q_0 + iq_1 + jq_2 + kq_3, q_0, q_1, q_2, q_3 \in \mathbb{R} \} , \]

where the elements \( i, j, k \) obey Hamilton’s multiplication rules

\[ ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j \]
\[ i^2 = j^2 = k^2 = ijk = -1. \]

The conjugate of a quaternion \( q \) is obtained by changing the sign of the pure quaternion part, i.e. the support of the transform of these signals relatively to certain integral transforms [8-15].

Motivated by the treatment of the QFT in quaternion algebra, in this paper we will characterize the quaternion-valued signals whose QFT has compact support. The main difficulty lies in the fact that the quaternion algebra is non-commutative, so one cannot directly extend the results for the Fourier transform to those for the QFT.

This paper is organized as follows: Section 2 is devoted to reviewing some necessary results about the quaternion algebra. In Section 3, based on the definition and some properties of the QFT, we get a result to describe the spectrum for the QFT.

3. Main Results
Note that \( L^1(\mathbb{R}^2; \mathbb{H}) \cap L^2(\mathbb{R}^2; \mathbb{H}) \) is dense in
Hence, standard density arguments allow us to extend the definition of the QFT of \( f \in L^2 (\mathbb{R}^2 ; \mathbb{H}) \) in a unique way to the whole of \( L^2 (\mathbb{R}^2 ; \mathbb{H}) \). We give the following definition of the QFT as an operator from \( L^2 (\mathbb{R}^2 ; \mathbb{H}) \) into \( L^2 (\mathbb{R}^2 ; \mathbb{H}) \) \[3\].

**Definition 1**

The two-sided QFT of \( f \in L^2 (\mathbb{R}^2 ; \mathbb{H}) \) is the function \( F_q f \) defined by

\[
F_q f (\omega) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} e^{-i\omega \cdot x} f(x) e^{-i\omega x} d^2 x \tag{3.1}
\]

with arbitrary frequency \( \omega = (\omega_1, \omega_2) \).

The QFT can be inverted by

\[
f(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} e^{i\omega \cdot x} F_q f(\omega) e^{i\omega x} d^2 \omega \]

with \( d^2 \omega = d\omega_1 d\omega_2 \).

In what follows, we review some properties of the QFT, such as the Parseval theorem and the partial derivative. For more details, we refer to \[3\].

**Lemma 2**

For \( f \in L^2 (\mathbb{R}^2 ; \mathbb{H}) \) we have

\[
\|f\| = \frac{1}{2\pi} \|F_q f\|.
\]

where the norm \( \|\cdot\| \) is defined by Equation (2.2).

**Lemma 3**

If \( \partial_{\omega_1,\omega_2}^m f(x) \in L^2 (\mathbb{R}^2 ; \mathbb{H}) \), \( m,n \in \mathbb{N}_0 \) and \( f \in L^2 (\mathbb{R}^2 ; \mathbb{H}) \). Then we have

\[
F_q \left\{ \partial_{\omega_1,\omega_2}^m f(x) \right\}(\omega) = i^m \omega_1^m \omega_2^n F_q f(\omega) \omega_1^m \omega_2^n f^*,
\]

where the QFT \( F_q f \) is defined by Equation (3.1).

Given a multi-index \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2 \), we write as usual \( |\alpha| = \alpha_1 + \alpha_2 \), \( D^\alpha = \partial_{\omega_1,\omega_2}^\alpha \) for the partial derivative.

Moreover, we denote by \( \text{supp} F_q f \) the support of \( F_q f \) describing the smallest closed set in \( \mathbb{R}^2 \) outside which \( F_q f \) vanishes almost everywhere. The following theorem describes the spectrum of signals for the QFT, i.e. the compactness of the support of \( F_q f \) by means of the norm of its partial derivative on \( \mathbb{R}^2 \).

**Theorem 4**

Let \( f \in L^2 (\mathbb{R}^2 ; \mathbb{H}) \). Then the QFT \( F_q f(\omega) \) is compactly supported in \([-\sigma, \sigma]^2 \) if and only if partial derivatives \( D^\alpha f \in L^2 (\mathbb{R}^2 ; \mathbb{H}) \), \( \omega_1^\alpha \omega_2^{\alpha_2} F_q f \in L^2 (\mathbb{R}^2 ; \mathbb{H}) \) for all \( \alpha \in \mathbb{Z}_+^2 \) and

\[
\lim_{|\alpha| \to \infty} \left\| D^\alpha f \right\| = \sigma,
\]

where \( \sigma = \sup \{ |\alpha|, k = 1,2 : F_q f(\omega) \neq 0, \omega \in \mathbb{R}^2 \} \).

**Proof.** Firstly, we prove the necessity. Suppose that \( \text{supp} F_q f(\omega) = [-\sigma, \sigma]^2 \). The compactness of the support of \( F_q f \) and \( f \in L^2 (\mathbb{R}^2 ; \mathbb{H}) \) imply that \( \omega_1^\alpha \omega_2^{\alpha_2} F_q f \) belongs to \( L^2 (\mathbb{R}^2 ; \mathbb{H}) \) and \( \text{supp} F_q f(\omega) \) in a unique way to the whole of \( L^2 (\mathbb{R}^2 ; \mathbb{H}) \), thus partial derivatives \( D^\alpha f \) exist and belong to \( L^2 (\mathbb{R}^2 ; \mathbb{H}) \) for all \( \alpha \in \mathbb{Z}_+^2 \). Moreover, by Lemma 3 we have

\[
F_q \left\{ D^\alpha f \right\}(\omega) = i^m \omega_1^m \omega_2^n F_q f(\omega) \omega_1^m \omega_2^n f^*.
\]

Applying Lemma 2, it follows

\[
\left\| D^\alpha f \right\|^2 = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} |\omega_1^m \omega_2^\alpha F_q f(\omega) \omega_1^m \omega_2^\alpha f^*|^2 d^2 \omega,
\]

that is,

\[
\left\| D^\alpha f \right\|^2 = \frac{1}{(2\pi)^3} \int_{[-\sigma, \sigma]^2} |\omega_1^m \omega_2^\alpha F_q f(\omega)|^2 d^2 \omega \tag{3.2}
\]

which leads to

\[
\left\| D^\alpha f \right\| \leq C |\alpha| \sigma
\]

for \( \alpha \in (0, \sigma/2) \). On the other hand, using Equation (3.2) again, for \( \varepsilon \in (0, \sigma/2) \) it holds

\[
\left\| D^\alpha f \right\|^2 \geq \frac{1}{(2\pi)^3} \int_{[-\sigma-2\varepsilon, \sigma-2\varepsilon]^2} |\omega_1^m \omega_2^\alpha F_q f(\omega)|^2 d^2 \omega
\]

which leads to

\[
\liminf_{|\alpha| \to \infty} \left\| D^\alpha f \right\| \geq \sigma - 2\varepsilon.
\]

The arbitrariness of \( \varepsilon \) implies

\[
\lim_{|\alpha| \to \infty} \left\| D^\alpha f \right\| = \sigma.
\]

Therefore, we can conclude that

\[
\lim_{|\alpha| \to \infty} \left\| D^\alpha f \right\| = \sigma.
\]

Secondly, we prove the sufficiency. Suppose that partial derivatives \( D^\alpha f(x) \in L^2 (\mathbb{R}^2 ; \mathbb{H}) \), \( \omega_1^\alpha \omega_2^{\alpha_2} F_q f \in L^2 (\mathbb{R}^2 ; \mathbb{H}) \) for all \( \alpha \in \mathbb{Z}_+^2 \) and

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We need to prove that
\[
\sigma = \sup \left\{ |\omega_k|, k = 1, 2 : F_q f (\omega) \neq 0 \right\}.
\]
Otherwise, \( F_q f (\omega) \neq 0 \) holds for almost everywhere \( \omega \in \mathbb{R}^2 \) and thus, associated with Equation (3.2) we obtain that for arbitrary \( M \) it holds
\[
D^{\sigma} f \int |H| = \frac{1}{(2\pi)^2} \int |E \omega|^{2\sigma} \frac{d^2 \omega}{2^{\sigma}M^{\sigma}} \leq \frac{1}{(2\pi)^2} \int E |\omega|^{2\sigma} \frac{d^2 \omega}{2^{\sigma}M^{\sigma}} \geq CM|H|,
\]
where \( E = \{ \omega \in \mathbb{R}^2 : |\omega_k| \geq M, k = 1, 2 \} \) and \( C \) is some positive constant independent of \( |\sigma| \), that is to say,
\[
D^{\sigma} f \int |H| = \frac{1}{(2\pi)^2} \int |E \omega|^{2\sigma} \frac{d^2 \omega}{2^{\sigma}M^{\sigma}} \geq CM|H|.
\]
(3.4)
The above inequality (3.4) implies
\[
\lim_{|H| \to \infty} D^{\sigma} f \int |H| = \infty,
\]
which contradicts the assumption (3.3). Thus, we have
\[
\sigma = \sup \left\{ |\omega_k|, k = 1, 2 : F_q f (\omega) \neq 0 \right\} < \infty,
\]
which means \( F_q f (\omega) \) is compactly supported in \( [-\sigma, \sigma] \). Finally, the same technique as the part of the proof for the necessity yields that \( d = \sigma \). Thus, the proof is complete.

REFERENCES