Assessment of Profit of a Two-Stage Deteriorating Linear Consecutive 2-out-of-3 Repairable System

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Abstract

Most of the researches on profit and cost evaluation of redundant system focus on the effect of failure and repair on revenue generated. However, as these systems continue to work, their strength gradually deteriorates. Where such deterioration occurs, minor and major maintenance is employed to remedy the deterioration. Little or no attention is paid on the effect of deterioration on the impact of deterioration and their maintenance on the revenue generated. In this paper, we study the profit generated of two-stage deteriorating linear consecutive 2-out-of-3 system. Failure, repair and deterioration time are assumed exponential. The explicit expressions of availability, busy period of a repairman and profit function are derived using Kolmogorov’s forward equations method. Various cases are analyzed graphically to investigate the effect of deterioration parameters such as slow deterioration, fast deterioration, and their maintenance such as minor and major minimal maintenance on profit generated.

Keywords: Reliability; Availability; Profit; Deterioration

1. Introduction

During operation, the strengths of systems are gradually deteriorated, until some point of deterioration failure, or other types of failures. Minor and major maintenance policies are vital in the analysis of deterioration and deteriorating systems as they help in improving reliability, availability and the overall revenue generated. Both minor and major minimal maintenance are employed to check the effect of slow and fast deterioration and return the system to its state prior to slow and fast deterioration. Maintenance models assume perfect repair (as good as new), minimal repair (as bad as old) and imperfect repair which is between perfect and minimal repair. Many research results have been reported on the reliability of 2-out-of-3 redundant systems. For example, [1], analyzed reliability models for 2-out-of-3 redundant system are subject to conditional arrival time of the server. Reference [2] presented reliability and economic analysis of 2-out-of-3 redundant system with priority to repair and [3] studied MTSF and cost effectiveness of 2-out-of-3 cold standby system with probability of repair and inspection, while [4] examined the cost benefit analysis of series systems with cold standby components and repairable service station. Reference [5,6] examined the cost analysis of two unit cold standby system involving preventive maintenance respectively. Reference [7] studied the cost and probabilistic analysis of series system with mixed standby components while [8] studied cost benefit analysis of series systems with warm standby components involving general repair time where the server is not subject to breakdowns. The failure time and repair time are assumed to have exponential distribution. Measures of system effectiveness such as MTSF, steady-state availability, busy period and profit function are obtained.

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contribution of this paper is two fold. The first is to develop the explicit expressions for system availability, busy period and profit function. The second is to perform a parametric investigation of various system parameters on profit function and capture their effect on the profit function.

The rest of the paper is organized as follows. Section 2 is the description and states of the system. Section 3 deals with models formulation. The results of our numerical simulations are presented and discussed in Section 4. The paper is concluded in Section 5.

2. Description and States of the System

We consider a 2-out-of-3 system with three modes: normal, deterioration and failure. The deterioration mode consists of two consecutive stages: slow and fast. It is assumed that the system transits from normal to slow and later to fast deterioration with rate $\delta_1$ and $\delta_2$ respectively. It is also assumed that the two consecutive units never fail simultaneously. Whenever the system deteriorate with rate $\delta_1$, minor minimal maintenance is invoke with rate $\mu_1$ to regain the system to its early stage prior to slow deterioration stage or the deterioration will be faster with rate $\delta_2$ where major minimal maintenance will be done with rate $\mu_2$. Unit I fail with rate $\beta_1$ and is under minimal repair with rate $\alpha_1$ and unit III is switch on. It is assumed that the switch from standby to operation is perfect. Similarly, unit II fails with rate $\beta_2$ and is minimally repaired with rate $\alpha_2$. The system failed when unit I and II have failed. The system is attended by one repair man.

States of the System

State $S_0$: Units I and II are in operation, unit III is in standby, the system is operational.

State $S_1$: The system is under slow deterioration and is receiving minor minimal maintenance.

State $S_2$: The system is under fast deterioration and is receiving major minimal maintenance.

$$T = \begin{bmatrix}
- (\beta_1 + \delta_1) & \mu_1 & 0 \\
\delta_1 & - (\beta_1 + \mu_1 + \delta_2) & \mu_2 \\
0 & \delta_2 & - (\beta_1 + \mu_2) \\
\beta_1 & \beta_1 & 0 \\
0 & 0 & \beta_1 \\
0 & 0 & 0
\end{bmatrix}$$

3. Models Formulation

Let $P(t)$ be the probability row vector at time $t$, then the initial conditions for this problem are as follows:

$$P(0) = \begin{bmatrix} P_0(0), P_1(0), P_2(0), P_3(0), P_4(0), P_5(0) \end{bmatrix} = \begin{bmatrix} 1, 0, 0, 0, 0 \end{bmatrix}$$

we obtain the following system of differential equations:

$$P'_1(t) = - (\beta_1 + \delta_1) P_1(t) + \mu_1 P_0(t) + \alpha_1 P_3(t)$$

$$P'_2(t) = - (\beta_2 + \delta_2 + 2\alpha_1) P_2(t) + \beta_1 P_0(t) + \beta P_1(t) + \alpha_2 P_3(t)$$

$$P'_3(t) = - (\beta_2 + \alpha_1 + \mu_2) P_3(t) + \beta P_2(t) + \delta P_1(t) + \alpha_2 P_4(t)$$

$$P'_4(t) = - 2\alpha_2 P_4(t) + \beta_2 P_3(t) + \beta_2 P_4(t)$$

$$P'_5(t) = - 2\alpha_2 P_5(t) + \beta_2 P_4(t) + \beta_2 P_5(t)$$

(1)

The differential equations in (1) above is transformed into matrix as

$$P' = TP$$

(2)

where

$$P(0) = \begin{bmatrix} P_0(0), P_1(0), P_2(0), P_3(0), P_4(0), P_5(0) \end{bmatrix} = \begin{bmatrix} 1, 0, 0, 0, 0 \end{bmatrix}$$

3.1. System Availability Analysis

For the availability case of Figure 1 using the initial condition in section 3 for this system,
The system of differential equations in (1) for the system above can be expressed in matrix form as:

\[ \begin{bmatrix}
\begin{array}{cccccc}
P_0' \\
\vdots \\
P_n'
\end{array}
\end{bmatrix}
= \begin{bmatrix}
-\left(\beta_1 + \delta_1\right) & \mu_1 & 0 & \alpha_1 & 0 & 0 \\
\delta_1 & -\left(\beta_1 + \mu_1 + \delta_2\right) & \mu_2 & \alpha_1 & 0 & 0 \\
0 & \delta_2 & -\left(\beta_1 + \mu_2\right) & \alpha_1 & 0 & 0 \\
\beta_1 & \beta_1 & 0 & -\left(\beta_2 + \delta_2 + 2\alpha_1\right) & \mu_2 & \alpha_2 \\
0 & 0 & \beta_1 & \delta_2 & -\left(\beta_2 + \alpha_1 + \mu_2\right) & \alpha_2 \\
0 & 0 & 0 & \beta_2 & \beta_2 & -2\alpha_2
\end{bmatrix}
\begin{bmatrix}
P_0 \\
\vdots \\
P_n
\end{bmatrix}
\]

Let \( V \) be the time to failure of the system. The steady-state availability is given by

\[ A_v = P_0(\infty) + P_1(\infty) + P_2(\infty) + P_3(\infty) + P_4(\infty) \quad (3) \]

In steady state, the derivatives of state probabilities become zero, thus (2) becomes

\[ TP(\infty) = 0 \quad (4) \]

which in matrix form is

\[ \begin{bmatrix}
-\left(\beta_1 + \delta_1\right) & \mu_1 & 0 & \alpha_1 & 0 & 0 \\
\delta_1 & -\left(\beta_1 + \mu_1 + \delta_2\right) & \mu_2 & \alpha_1 & 0 & 0 \\
0 & \delta_2 & -\left(\beta_1 + \mu_2\right) & \alpha_1 & 0 & 0 \\
\beta_1 & \beta_1 & 0 & -\left(\beta_2 + \delta_2 + 2\alpha_1\right) & \mu_2 & \alpha_2 \\
0 & 0 & \beta_1 & \delta_2 & -\left(\beta_2 + \alpha_1 + \mu_2\right) & \alpha_2 \\
0 & 0 & 0 & \beta_2 & \beta_2 & -2\alpha_2
\end{bmatrix}
\begin{bmatrix}
P_0 \\
\vdots \\
P_n
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

using the normalizing condition

\[ P_0(\infty) + P_1(\infty) + P_2(\infty) + P_3(\infty) + P_4(\infty) = 1 \quad (5) \]

we substitute (5) in the last row of (4) following [2,3,5]. The resulting matrix is

\[ \begin{bmatrix}
-\left(\beta_1 + \delta_1\right) & \mu_1 & 0 & \alpha_1 & 0 & 0 \\
\delta_1 & -\left(\beta_1 + \mu_1 + \delta_2\right) & \mu_2 & \alpha_1 & 0 & 0 \\
0 & \delta_2 & -\left(\beta_1 + \mu_2\right) & \alpha_1 & 0 & 0 \\
\beta_1 & \beta_1 & 0 & -\left(\beta_2 + \delta_2 + 2\alpha_1\right) & \mu_2 & \alpha_2 \\
0 & 0 & \beta_1 & \delta_2 & -\left(\beta_2 + \alpha_1 + \mu_2\right) & \alpha_2 \\
1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
P_0 \\
\vdots \\
P_n
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

Expression for \( A_v \) thus is:

\[ A_v = \frac{N_2}{D_2} \]

\[ N_2 = \alpha_1 \alpha_2 \left(4\alpha_1 \mu_1 \mu_2 + 4\mu_1 \mu_2 \right) + 2\beta_1 \left(\beta_1 \delta_2 + \beta_2 \right) + \alpha_1 \alpha_2 \left(4\alpha_1 \mu_1 \delta_1 + 2\alpha_1 \beta_1 \mu_2 + 4\mu_1 \delta_2 + 2\beta_1 \mu_2 \mu_2 + 2\alpha_1 \beta_1 \mu_2 + 2\beta_1 \mu_2 \mu_2 + 2\beta_1 \delta_2 + \beta_2 \right)
\]

\[ + 2\beta_2 \left(\beta_1 \delta_1 + \beta_2 \delta_2 + \beta_2 \delta_2 \right) + \alpha_1 \alpha_2 \left(4\alpha_1 \mu_1 \delta_1 + 2\alpha_1 \beta_1 \mu_2 + 4\mu_1 \delta_2 + 2\beta_1 \mu_2 \mu_2 + 2\alpha_1 \beta_1 \mu_2 + 2\beta_1 \mu_2 \mu_2 + 2\beta_1 \delta_2 + \beta_2 \right)
\]

\[ + 3\beta_1 \mu_2 \delta_1 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2
\]

\[ + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2
\]

\[ + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2
\]

\[ + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2
\]

\[ + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2 + 2\beta_1 \mu_2 \delta_2
\]
3.2. Busy Period Analysis

Using the same initial condition in section 3 above as for the reliability case

\[ P(0) = [P_1(0), P_2(0), P_3(0), P_4(0), P_5(0), P_6(0)] = [1, 0, 0, 0, 0, 0, 0] \]

and (4) and (5) the busy period is obtained as follows:

In the steady state, the derivatives of the state probabilities become zero and this will enable us to compute steady state busy period:

The system of differential equations in (1) for the system above can be expressed in matrix form as:

\[
\begin{bmatrix}
P_0' \
P_1' \
P_2' \
P_3' \
P_4' \
P_5'
\end{bmatrix} = \begin{bmatrix}
-(\beta_1 + \delta_1) & \mu_1 & 0 & \alpha_1 & 0 & 0 \\
\delta_1 & -(\beta_1 + \mu_1 + \delta_2) & \mu_2 & \alpha_1 & 0 & 0 \\
0 & \delta_2 & -(\beta_1 + \mu_2) & \alpha_1 & 0 & 0 \\
\beta_1 & 0 & 0 & (\beta_2 + \alpha_1 + \mu_2) & \alpha_2 & \alpha_2 \\
0 & \beta_1 & 0 & 0 & \beta_2 & 0 \\
0 & 0 & \beta_1 & 0 & 0 & \beta_2
\end{bmatrix}
\begin{bmatrix}
P_0 \\
P_1 \\
P_2 \\
P_3 \\
P_4 \\
P_5
\end{bmatrix}
\]

Let \( V \) be the time to failure of the system. The steady-state busy period is given by

\[ B_v = P_1(\infty) + P_2(\infty) + P_3(\infty) + P_4(\infty) + P_5(\infty) \]  

(6)

In steady state, the derivatives of state probabilities become zero, thus (2) becomes

\[ TP(\infty) = 0 \]  

(7)

which in matrix form is

\[
\begin{bmatrix}
-(\beta_1 + \delta_1) & \mu_1 & 0 & \alpha_1 & 0 & 0 \\
\delta_1 & -(\beta_1 + \mu_1 + \delta_2) & \mu_2 & \alpha_1 & 0 & 0 \\
0 & \delta_2 & -(\beta_1 + \mu_2) & \alpha_1 & 0 & 0 \\
\beta_1 & 0 & 0 & (\beta_2 + \alpha_1 + \mu_2) & \alpha_2 & \alpha_2 \\
0 & \beta_1 & 0 & 0 & \beta_2 & 0 \\
0 & 0 & \beta_1 & 0 & 0 & \beta_2
\end{bmatrix}
\begin{bmatrix}
P_0 \\
P_1 \\
P_2 \\
P_3 \\
P_4 \\
P_5
\end{bmatrix} = 0
\]

using the normalizing condition

\[ P_5(\infty) + P_1(\infty) + P_2(\infty) + P_3(\infty) + P_4(\infty) + P_5(\infty) = 1 \]  

(8)

The resulting matrix is

\[
\begin{bmatrix}
-(\beta_1 + \delta_1) & \mu_1 & 0 & \alpha_1 & 0 & 0 \\
\delta_1 & -(\beta_1 + \mu_1 + \delta_2) & \mu_2 & \alpha_1 & 0 & 0 \\
0 & \delta_2 & -(\beta_1 + \mu_2) & \alpha_1 & 0 & 0 \\
\beta_1 & 0 & 0 & (\beta_2 + \alpha_1 + \mu_2) & \alpha_2 & \alpha_2 \\
0 & \beta_1 & 0 & 0 & \beta_2 & 0 \\
1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
P_0 \\
P_1 \\
P_2 \\
P_3 \\
P_4 \\
P_5
\end{bmatrix} = 0
\]
In the steady state, the derivatives of the state probabilities become zero and this will enable us to compute steady state busy:
\[ B(\infty) = 1 - P_n(\infty) \]
\[ N_3 = \alpha_1 \alpha_2 (4 \alpha_1 \mu_2 \delta_1 + 2 \alpha_2 \beta_1 \mu_2 + 4 \mu_2^3 \delta_1 + 2 \beta_1 \mu_2 \delta_1 + 2 \beta_1 \mu_2 \delta_2 + 2 \beta_2 \delta_1 \delta_2 + 2 \beta_1 \delta_1 \delta_2 + 2 \beta_2 \delta_1 \delta_2 + 2 \beta_1 \delta_1 \delta_2 ) + \alpha_1 \delta_2 (2 \alpha_1 \mu_2 \mu_2 + 2 \beta_1 \mu_2 \mu_2 + 2 \beta_1 \mu_2 \mu_2 + 2 \alpha_1 \delta_2 \delta_1 + 2 \alpha_1 \delta_2 \delta_2 + 2 \alpha_1 \delta_2 \delta_3 + 2 \alpha_1 \delta_2 \delta_4 + 2 \alpha_1 \delta_2 \delta_5 + 2 \alpha_1 \delta_2 \delta_6 + 2 \alpha_1 \delta_2 \delta_7 ) \]

\[ N_3 = \alpha_1 \alpha_2 (4 \alpha_1 \mu_2 \delta_1 + 2 \alpha_2 \beta_1 \mu_2 + 4 \mu_2^3 \delta_1 + 2 \beta_1 \mu_2 \delta_1 + 2 \beta_1 \mu_2 \delta_2 + 2 \beta_2 \delta_1 \delta_2 + 2 \beta_1 \delta_1 \delta_2 + 2 \beta_2 \delta_1 \delta_2 + 2 \beta_1 \delta_1 \delta_2 ) + \alpha_1 \delta_2 (2 \alpha_1 \mu_2 \mu_2 + 2 \beta_1 \mu_2 \mu_2 + 2 \beta_1 \mu_2 \mu_2 + 2 \alpha_1 \delta_2 \delta_1 + 2 \alpha_1 \delta_2 \delta_2 + 2 \alpha_1 \delta_2 \delta_3 + 2 \alpha_1 \delta_2 \delta_4 + 2 \alpha_1 \delta_2 \delta_5 + 2 \alpha_1 \delta_2 \delta_6 + 2 \alpha_1 \delta_2 \delta_7 ) \]

The steady state busy period \( B(\infty) \) is therefore:
\[ B(\infty) = \frac{N_3}{D_2} \]

### 3.3. Profit Analysis

The system/units are subjected to minor and major minimal maintenance and corrective maintenance at failure as can be observed in states 1, 2, 3, 4, and 5. From Figure 1, the repairman is busy performing corrective maintenance action to the units/system at failure in states 1, 2, 3, 4 and 5. According to [1-3], the expected profit per unit time incurred to the system in the steady-state is given by:

\[ \text{Profit} = \text{total revenue generated} - \text{accumulated cost incurred due maintenance/repairing the failed units} \]

\[ PF = C_o A(\infty) - C_i B(\infty) \]

where \( PF \) is the profit incurred to the system.

\( C_o \) is the revenue per unit up time of the system.

\( C_i \) is the cost per unit time which the system is under repair.

### 4. Results and Discussions

In this section, we numerically obtained the results for mean time to system failure, system availability, busy period and profit function for all the developed models. For the model analysis, the following set of parameters values are fixed throughout the simulations for consistency: \( \beta_1 = 0.1, \beta_2 = 0.2, \alpha_1 = 0.4, \alpha_2 = 0.1, \delta_1 = 0.1, \delta_2 = 0.1, \mu_1 = 0.3, \mu_2 = 0.4, C_o = 50,000, C_i = 10,000 \)

The impact of \( \delta_1 \) on profit can be observed in Figure 2. From this figure it is evident that the profit decreases as \( \delta_1 \) increases while in Figure 3, the increases with increase in \( \mu_1 \). Similar results can be observed in Figures 4 and 5 of profit with respect to \( \delta_2 \) and \( \mu_2 \). From these figures, the profit decreases as \( \delta_2 \) increases and increases with increase in \( \mu_2 \). Results of profit with respect to \( \beta_1 \) is given in Figure 6. It is evident from Figure 6 that as \( \beta_1 \) increases, the profit decreases while from Figure 7, the profit increases with increase in \( \alpha_1 \).

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**Figure 1.** Transition diagram of the system.

**Figure 2.** Effect of \( \delta_1 \) on Profit.
5. Conclusion

In this paper, we constructed a two-stage linear consecutive 2-out-of-3 system to study the impact of deterioration and maintenance on the generated profit. Explicit expressions of steady-state availability, busy period and profit function were derived. We performed numerical investigation to see the effect of slow deterioration, fast deterioration, minor minimal maintenance, major minimal maintenance, failure and repair rates on the generated profit. It is evident from the results obtained that repair rate, minor minimal maintenance rate and major minimal maintenance rates increase the profit generated while slow deterioration, fast deterioration and failure rate decrease the profit. It is evident from the results obtained that deterioration makes a tremendous effect on the generated revenue (profit).
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