Impulsive Synchronization of Hyperchaotic Lü Systems with Two Methods

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Abstract

In the paper, impulsive synchronization of two hyperchaotic Lü systems with different initial conditions is studied. The sufficient conditions on feedback strength and impulsive distances are established from two different angles to guarantee the synchronization. The relevant theoretical proofs are presented. Numerical simulations show the effectiveness of the methods.

Keywords

Impulsive Synchronization, Impulsive Distance, Hyperchaotic Lü System, Largest Lyapunov Exponent

1. Introduction

In 1990, Pecora and Corroll proposed the conception of chaotic synchronization and they presented a chaos synchronization method through investigating synchronization of Newcomb circuit [1] [2]. Since chaos control and synchronization have great potential applications in many areas such as information science, medicine, biology and Engineering, they have received a great deal of attention. Numerous researches have been done theoretically and experimentally [3]-[6]. Many approaches have been proposed for chaos synchronization, including feedback method, adaptive synchronization, impulsive synchronization and fuzzy synchronization method [7]-[12]. Because of transmitting signals in discrete times, impulsive synchronization demands less energy. Besides, it has faster synchronous speed than other methods. It is more practical in practical applications. Recently, many efforts have been devoted to impulsive synchronization. Ren et al. proposed impulsive synchronization of coupled chaotic systems via adaptive-feedback approach [13]. Chen et al. proposed a synchronization method of a class of chaotic systems using small impulsive signal [14]. Xi et al. presented adaptive impulsive synchronization for a class of fractional-order chaotic and hyperchaotic systems [15]. Xu et al. studied a new chaotic system without linear term and its impulsive synchronization [16]. In the paper, impulsive synchronization of hyper-
chaotic Lü system is studied. Based on the boundedness and the largest Lyapunov exponent, two different sufficient conditions are established to guarantee the synchronization. These two methods are analyzed and compared. Numerical simulations show the effectiveness of these methods.

2. Impulsive Synchronization Theory

Suppose a $n$-dimensional chaotic system as

$$\dot{X} = F(t,X)$$

choose system (1) as drive system, response system is as follows

$$\begin{aligned}
\dot{Y} &= F(t,Y) \ (t \neq t_i) \\
\Delta Y &= Y(t_i^+) - Y(t_i^-) = Y(t_i^+) - Y(t_i^-) = BE \ (t = t_i, i = 1, 2, 3, \cdots) \\
Y(t_i^+) &= Y(0)
\end{aligned}$$

$B$ is a matrix which stands for a linear combination of $Y - X$, let $B = \text{diag}(b_1, b_2, \cdots, b_n)$; The error vector is $E = Y - X$; $t_i$ is the discrete time at which the impulse is transmitted. According to system (1) and system (2), we can get the error system

$$\begin{aligned}
\dot{E} &= F(t,Y) - F(t,X) \ (t \neq t_i) \\
\Delta E &= BE \ (t = t_i)
\end{aligned}$$

Suppose the impulsive distance $\eta$ is invariable, $\eta = t_{i+1} - t_i$, if we can obtain $\lim_{t \to \infty} ||E(t)|| = 0$ under some conditions, system (1) and system (2) can be synchronized by impulses. Next we will take hyperchaotic Lü system as example for detailed description.

3. Implement of Impulsive Synchronization

3.1. Description of Hyperchaotic Lü System

Hyperchaotic Lü system [17] is described as

$$\begin{aligned}
\dot{x} &= a(y - x) + w \\
\dot{y} &= -xz + cy \\
\dot{z} &= xy - bz \\
\dot{w} &= xz + dw
\end{aligned}$$

in this paper choose $a = 36$, $b = 3$, $c = 20$ and $d = 1$ so that system (4) exhibits a hyperchaotic behavior [17], Figure 1 shows the projections of hyperchaotic Lü system’s attractor.

![Figure 1. The projections of hyperchaotic Lü system’s attractor.](image)
Choose hyperchaotic Lü system

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) + x_4 \\
\dot{x}_2 &= -x_1 x_3 + cx_2 \\
\dot{x}_3 &= x_1 x_2 - bx_3 \\
\dot{x}_4 &= x_1 x_3 + dx_4
\end{align*}
\]  
(5)

as drive system. System (5) can be described as

\[
\dot{X} = AX + \varphi(X)
\]  
(6)

where

\[
A = \begin{pmatrix} -a & a & 0 & 1 \\ 0 & c & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & d & 0 \end{pmatrix}, \quad \varphi(X) = \begin{pmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \\ x_1 x_3 \end{pmatrix}.
\]

The response system is described as

\[
\begin{align*}
\dot{Y} &= AY + \varphi(Y) \quad (t \neq t_i, i = 1, 2, 3, \cdots) \\
\Delta Y &= Y(t_i^+) - Y(t_i^-) = Y(t_i^-) - Y(t_i) = BE \quad (t = t_i, i = 1, 2, 3, \cdots) \\
Y(t_0) &= Y(0)
\end{align*}
\]  
(7)

\[
Y = [y_1, y_2, y_3, y_4]^T,
\]

where

\[
E = [e_1, e_2, e_3, e_4]^T = [y_1 - x_1, y_2 - x_2, y_3 - x_3, y_4 - x_4]^T,
\]

\[
B = \text{diag}(h_1, h_2, b_3, b_4).
\]

The error system is

\[
\begin{align*}
\dot{E} &= AE + \rho(X, Y) \quad (t \neq t_i) \\
\Delta E &= BE \quad (t = t_i)
\end{align*}
\]  
(8)

where

\[
\rho(X, Y) = \varphi(Y) - \varphi(X) = \begin{pmatrix} 0 \\ x_1 x_3 - y_1 y_3 \\ y_1 y_2 - x_1 x_2 \\ y_1 y_3 - x_1 x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -y_1 e_1 - y_3 e_3 \\ y_2 e_1 + y_3 e_2 \\ y_3 e_1 + y_1 e_3 \end{pmatrix}.
\]

Next the sufficient conditions on feedback strength and impulsive distances will be established from two different angles to guarantee the synchronization.

3.2. Based on the Boundedness of Chaotic System

**Theorem 1** Suppose \( M \geq \max\left([|y_1|, |y_2|, |y_3|]\right) \), \( \beta \) is the largest eigenvalue of \((I + B)^T(I + B)\)

\((I = \text{diag}(1,1,1,1), \lambda \) is the largest eigenvalue of \(0.5(A + A^T)\), the constant \( \varepsilon > 1 \), \( \eta \) is impulsive distance, if choose suitable \( \beta \) and \( \eta \) such that

\[
\ln(\varepsilon \beta) + (2\lambda + 3M)\eta \leq 0,
\]  
(9)

then system (6) and system (7) can be synchronized.

**Proof:** Choose Lyapunov function as

\[
V = 0.5E^T E.
\]  
(10)

Calculate the derivative of Equation (10), yield
\[ \dot{V} = 0.5 \left( AE + \rho(X,Y) \right)^T E + 0.5 E^T \left( AE + \rho(X,Y) \right) \]
\[ = 0.5E^T (A^T + A) E - y_i e_i e_i + y_i e_i e_i + y_i e_i e_i + y_i e_i e_i \]
\[ \leq 2\lambda V + M \left( e_i e_i + e_i e_i + e_i e_i + e_i e_i \right) \]
\[ \leq 2\lambda V + M \left( e_i e_i + e_i e_i + e_i e_i + e_i e_i + e_i e_i + e_i e_i + e_i e_i \right) \]
\[ \leq 2\lambda V + 3MV = (2\lambda + 3M)V. \]

When \( t \in (t_{i-1}, t_i) \) \((i = 1, 2, 3, \cdots)\), we have
\[ V(E(t)) \leq V(E(t_i)) e^{(2\lambda + 3M)(t-t_{i-1})}. \] (12)

When \( t = t_i \), system (8) is a discrete system, according to Equation (8), we get
\[ V \left( E \left( t_i \right) \right) = 0.5 \left( (I + B) E(t_i) \right)^T \left( (I + B) E(t_i) \right) \leq 0.5 \beta E(t_i)^T E(t_i) = \beta V \left( E(t_i) \right) \] (13)

Let \( i = 1 \), according to Equation (12), when \( t \in (t_0, t_1) \),
\[ V \left( E(t) \right) \leq V \left( E(t_0) \right) e^{(2\lambda + 3M)(t-t_0)} \] (14)

When \( t = t_1 \), we obtain
\[ V \left( E(t_1) \right) \leq V \left( E(t_0) \right) e^{(2\lambda + 3M)(t_1-t_0)} \] (15)

From Equation (13) and Equation (15),
\[ V \left( E \left( t_i \right) \right) \leq \beta V \left( E \left( t_i \right) \right) \leq \beta V \left( E \left( t_0 \right) \right) e^{(2\lambda + 3M)(t_i-t_0)} \] (16)

According to Equation (12), when \( t \in (t_1, t_2) \),
\[ V \left( E(t) \right) \leq V \left( E \left( t_i \right) \right) e^{(2\lambda + 3M)(t-t_1)} \leq \beta V \left( E \left( t_i \right) \right) e^{(2\lambda + 3M)(t_i-t_0)} = \beta V \left( E \left( t_i \right) \right) e^{(2\lambda + 3M)(t-t_0)} \] (17)

In the same way, when \( t \in (t_{i-1}, t_i) \) \((i = 1, 2, 3, \cdots)\),
\[ V \left( E(t) \right) \leq \beta^{-1} V \left( E \left( t_i \right) \right) e^{(2\lambda + 3M)(t-t_0)} \] (18)

From Equation (9), yield
\[ \varepsilon \beta e^{(2\lambda + 3M)\eta} \leq 1 \] (19)

hence
\[ \beta^{-1} \leq \frac{1}{\varepsilon^{-1} e^{(2\lambda + 3M)(\eta)\eta}} \]
\[ = \frac{1}{e^{-1} e^{(2\lambda + 3M)(\eta)\eta}} \]
(20)

Substitute Equation (20) into Equation (18), we can obtain when \( t \in (t_{i-1}, t_i) \) \((i = 1, 2, 3, \cdots)\),
\[ V \left( E(t) \right) \leq \frac{1}{e^{-1} e^{(2\lambda + 3M)(\eta)\eta}} V \left( E \left( t_i \right) \right) e^{(2\lambda + 3M)(t-t_0)} = \frac{1}{e^{-1} e^{(2\lambda + 3M)(\eta)\eta}} V \left( E \left( t_i \right) \right) e^{(2\lambda + 3M)(t-t_i)} \] (21)

Since \( t \in (t_{i-1}, t_i) \), \( t - t_{i-1} \leq \eta \). From the assumption given in Theorem 1, we have \( \lim_{t \to \infty} V \left( E \left( t \right) \right) = 0 \), Thus obtain \( \lim_{t \to \infty} \| E(t) \| = 0 \), i.e. system (6) and system (7) will be synchronized when Equation (9) is satisfied.

From Figure 1, we can choose \( M = 50 \). We calculate the eigenvalues of \( 0.5 \left( A + A^T \right) \), obtain \( \lambda = 25.287 \). Suppose Equation (9) can be satisfied, then \( \ln (\varepsilon \beta) < 0 \), i.e., \( 0 < \varepsilon \beta < 1 \). It is obvious that \( 0 < \beta < 1 \) and \( \beta \) is near zero. Choose \( B = \text{diag}(-1.1, -1.1, -1.1, -1.1) \) and \( \varepsilon = 5 \), we obtain \( \eta \leq 0.0149 \) by solving Equation (9). That is to say, when \( B = \text{diag}(-1.1, -1.1, -1.1, -1.1) \), \( \eta \leq 0.0149 \) (sec.), System (6) and system (7) can be synchronized.
In this numerical simulation, let \( B = \text{diag}(-1.1,-1.1,-1.1,-1.1) \), impulsive distance \( \eta = 0.01(\text{sec.}) \). A time step of size 0.0001(\text{sec.}) is employed and fourth-order Runge-Kutta method is used to solve Equation (6) and Equation (7). The initial states of the drive system (6) and the response system (7) are taken as \( X(0) = (-10,5,8,15) \) and \( Y(0) = (8,-12,-20,30) \). The error system (8) has the initial state \( E(0) = (18,-17,-28,15) \). Figure 2 shows the history of \( e_1(t) \), \( e_2(t) \), \( e_3(t) \), \( e_4(t) \) in the error system (8). From Figure 2, we can see that \( e_1(t) \), \( e_2(t) \), \( e_3(t) \), \( e_4(t) \) are steady near zero at last, i.e., system (6) and system (7) can be synchronized when \( B = \text{diag}(-1.1,-1.1,-1.1,-1.1) \) and \( \eta = 0.01(\text{sec.}) \).

### 3.3. Based on the Largest Lyapunov Exponent of Chaotic System

Theorem 1 provides sufficient condition for the synchronization of system (6) and system (7), but Equation (9) is not necessary condition. Through simulations we find that the above condition is too rigorous. In fact the qualified impulsive distance can be much larger than \( \eta \) solved from Equation (9). Next we will present a new condition based on the largest Lyapunov exponent, which is much looser than Equation (9).

![Figure 2. (Based on boundedness) Synchronization error system (8) states: \( e_1(t) \), \( e_2(t) \), \( e_3(t) \), \( e_4(t) \).](image-url)
Suppose the initial distance between system (6) and system (7) is \( |E(0)| \), the largest Lyapunov exponent of system(6) is \( \lambda_i \). Without control, the largest distance between system (6) and system (7) will not go beyond \( |E(0)|e^{\lambda_i \Delta t} \) after a short time \( \Delta t \) (considering the average case). Generally speaking, the longest predicted time of the chaotic system is \( 1/\lambda_i \) [18]. Based on the above theory, we can obtain the following conclusions.

**Theorem 2** Suppose \( \beta \) is the largest eigenvalue of \( (I + B)^T(I + B) \) \( (I = \text{diag}(1,1,\ldots,1)) \), \( \lambda_i \) is the largest Lyapunov exponent of system (6), the constant \( \epsilon > 1 \), \( \eta \) is impulsive distance. Let \( \eta < 1/\lambda_i \), if choose suitable \( \beta \) and \( \eta \) such that
\[
\ln \epsilon \beta + 2\lambda_i \eta \leq 0, 
\]
(22)
System (6) and system (7) can be synchronized

**Proof:** Suppose
\[
V(t) = E(t)^T E(t),
\]
then
\[
V(t_0) = V(t_0^*) = E(t_0)^T E(t_0),
\]
(24)
hence
\[
V(t_i) = E(t_i)^T E(t_i) \leq \left[ e^{\lambda_i \eta} E(t_0) \right]^T \left[ e^{\lambda_i \eta} E(t_0) \right] = e^{2\lambda_i \eta} V(t_0),
\]
(25)
\[
V(t_i^*) = \left[ (I + B) E(t_i) \right]^T (I + B) E(t_i) \leq \beta E(t_i)^T E(t_i) = \beta V(t_i) \leq \beta e^{2\lambda_i \eta} V(t_i).
\]
(26)
\[
V(t_i^*) = \left[ (I + B) E(t_i^*) \right]^T (I + B) E(t_i^*) \leq \beta E(t_i^*)^T E(t_i^*) = \beta V(t_i^*) \leq \beta e^{2\lambda_i \eta} V(t_i^*).
\]
(27)
In the same way, we have
\[
V(t_i) \leq (\beta e^{2\lambda_i \eta})^i V(t_0) \quad (i = 1, 2, 3, \ldots),
\]
(29)
According to the condition of Theorem 2: \( \ln \epsilon \beta + 2\lambda_i \eta \leq 0 \), we have
\[
\beta e^{2\lambda_i \eta} \leq 1/\epsilon < 1,
\]
(30)
According to Equation (29) and Equation (30), we obtain \( \lim_{i \to \infty} V(t_i^*) = 0 \), then \( \lim_{t \to \infty} |E(t)| = 0 \), i.e. system (6) and system (7) can be synchronized if the condition of Theorem 2 is satisfied.

Here, we still choose \( a = 36, \quad b = 3, \quad c = 20, \quad d = 1 \) so that system (5) exhibits hyperchaotic behavior [17]. For this system, the largest Lyapunov exponent \( \lambda_i = 1.065 \). Suppose system (5) is described as system (6), choose system (6) as drive system, system (7) is the relevant response system, system (8) is the error system, we have
\[
X = [\begin{array}{c} x_1, x_2, x_3, x_4 \\ y_1, y_2, y_3, y_4 \end{array}], \quad Y = [\begin{array}{c} y_1, y_2, y_3, y_4 \end{array}], \quad E = [e_i, e_2, e_3, e_4]^T = [y_i - x_i, y_i - x_i, x_i - x_i, y_i - x_i]^T.
\]
Choose
\[
B = \text{diag}(-1.1, -1.1, -1.1, -1.1), \quad \epsilon = 5, \quad \eta = 0.3\text{(sec.)}.\]
A time step of size 0.0001(sec.) is employed and fourth-order Runge-Kutta method is used to solve Equation (6) and Equation (7). The initial states of the drive system (6) and the response system (7) are taken as \( X(0) = (-10, 5, 8, 15) \) and \( Y(0) = (8, -12, -20, 30) \). The error system (8) has the initial state \( E(0) = (18, -17, -28, 15) \). Figure 3 shows the history of \( e_i(t) \), \( e_2(t) \), \( e_3(t) \), \( e_4(t) \) in the error system (8). From Figure 3, we can see that \( e_i(t) \), \( e_2(t) \), \( e_3(t) \), \( e_4(t) \) are steady near zero at last, i.e., system (6) and system (7) can be synchronized when \( B = \text{diag}(-1.1, -1.1, -1.1, -1.1) \) and \( \eta = 0.3\text{(sec.)}.\)
3.4. Comparison of Two Methods

If system (6) and system (7) achieve synchronization, system (8) will be steady at zero. Suppose \( B = \text{diag}(k, k, k) \), \( \eta \) stands for impulsive distance. Figure 4 shows the boundaries of the stable region for Theorem 1, Figure 5 shows the boundaries of the stable region for Theorem 2. (Taking \( \varepsilon = 1, 3, 5 \) as examples, the region below the boundary is stable, not considering \( \eta < 1/\lambda \) in Figure 5).

From Figure 4 and Figure 5, we can see that the requirement of impulsive distance in Theorem 1 is more rigorous than Theorem 2. Comparing the methods of Theorem 1 and Theorem 2, the former is based on the boundedness of chaotic system, it considers the extreme case all the time, while the latter is based on the largest Lyapunov exponent of chaotic system, it represents the average case. Therefore, the sufficient condition in Theorem 1 is a very small part of the condition in Theorem 2. Of course, in view of the requirements of synchronous time and quality, it is not suitable to choose very large impulsive distance in practical applications.

4. Conclusion

In the paper, impulsive synchronization of hyperchaotic Lü systems is studied. We use two methods to achieve
the sufficient conditions for synchronization and relevant analysis and comparison are presented. Mohammad et al.
adopted the first method to study impulsive synchronization of hyperchaotic Chen systems [19]. The second
method has not been reported before. Obviously it is more compatible than the first one. Numerical simulations
show the effectiveness of the methods.

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