The Global and Pullback Attractors for a Strongly Damped Wave Equation with Delays*

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ABSTRACT

In this paper, we study the global and pullback attractors for a strongly damped wave equation with delays when the force term belongs to different space. The results following from the solution generate a compact set.

Keywords: Strongly Damped; Pullback Attractor; Global Attractor; Delays

1. Introduction

Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain with smooth boundary \( \partial \Omega \), we study the following initial boundary value problem

\[
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} - \beta \Delta u - \Delta u + g(u) &= f(x) + h(t, u_t), \\
0 &= u|_{\partial \Omega}, \\
\frac{\partial u}{\partial t} &= u_{t-r}, \\
u(x, t) &= \phi(x, t-r) - \beta \Delta u(x, t-r) + \frac{\partial \phi}{\partial t}(x, t-r), \\
x \in \Omega, t \in [r-r, r].
\end{aligned}
\]

(1.1)

where \( f + h(t, u_t) \) is the source intensity which may depend on the history of the solution, \( \alpha, \beta \) are the positive constants, \( \phi \) is the initial value on the interval \( [r-r, r] \) where \( r > 0 \), and \( u_t \) is defined for \( \theta \in [-r, 0] \) as \( u_t(\theta) = u(t + \theta) \). The assumption on \( g(u) \) and \( f(x) \) will be specified later.

It is well known that the long time behavior of many dynamical system generated by evolution equations can be described naturally in term of attractors of corresponding semigroups. Attractor is a basic concept in the study of the asymptotic behavior of solutions for the nonlinear evolution equations with various dissipation.

There have been many researches on the long-time behavior of solutions to the nonlinear damped wave equations with delays. The existence of random attractors has been investigated by many authors, see, e.g., [1-4]. A new type of attractor, called a pullback attractor, was proposed and investigated for non-autonomous or these random dynamical systems. The pullback attractor describing this attractors to a component subset for a fixed parameter value is achieved by starting progressively earlier in time, that is, at parameter values that are carried forward to the fixed value. see [5-20]. However, to our knowledge, in the case of functional differential equations of second order in time, there is only partial results.

Recently, In [5], some results on pullback and forward attractor for the following strongly damped wave equation with delays

\[
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} - \Delta u &= f(x) + h(t, u_t), \\
0 &= u|_{\partial \Omega}, \\
\frac{\partial u}{\partial t} &= u_{t-r}, \\
u(x, t) &= \phi(x, t-r) - \beta \Delta u(x, t-r) + \frac{\partial \phi}{\partial t}(x, t-r), \\
x \in \Omega, t \in [r-r, r].
\end{aligned}
\]

have been analyzed.

In this work, first, we apply the means in [3] to provide the existence of global attractor, for the dynamical system generated by the initial value problem...
The key is to deal with the nonlinear terms and the delay term is difficult to be handled, so we aimed at showing that it is dissipative and the solution is bounded and continuous with respect to initial value. Hence we can discover the global attractor. Then, we aim to obtain the pullback attractor. The technology we use is introduced in [1], that is, we divide the semigroup into two: the one is asymptotically close to 0, while the other is uniformly compact, so we can get the pullback attractor.

Now, we state the general assumptions for problem (1.1) on \( g: \mathbb{R} \to \mathbb{R} \) and \( h: \mathbb{R} \times C_H \to H \).

Let \( G(u) = \int_u^0 g(s) \, ds \), then there exist positive constants \( C_i \) such that the followings hold true:

\( G(0) = H \), \( g(0) \leq |C| \), \( g^*(z) \leq C_1 (1 + |z|^p) \), \( \forall z \in \mathbb{R} \), \( \liminf_{|z| \to 0} g(z)/z \geq 0 \), \( (g^*/z + C_0 z^2, \forall z \in \mathbb{R} \), and \( \liminf_{|z| \to 0} (g^*/z^2 - C_0 G(z))/z^2 \geq 0 \).

For any \( u \in V \), set \( J(u) = \int_{\mathbb{R}} G(u) \, ds \), by \( G_1 - G_2 \), there are \( C_0 > 0, C_\gamma > 0 \), and \( \rho_\gamma > 0 \), for any \( \gamma > 0 \), we have

\[ (g(u)/u, u - C_\gamma I(u) \geq -\gamma |u|^p - C_\gamma ; \]
\[ J(u) \geq -\gamma |u|^p - C_\gamma ; \]
\[ (g(u)/u, u - \rho_\gamma |u|^p \geq -\gamma \|u\|^p - C_\gamma . \]

\( H_1 \). \( \forall \xi \in C_H, t \in \mathbb{R} \to h(t, \xi) \in H \) is continuous;

\( H_2 \). \( \forall t \in \mathbb{R}, h(t, 0) = 0 \);

\( H_3 \). \( \exists L_n > 0 \) such that \( \forall t \in \mathbb{R}, \forall \xi, \eta \in C_H \)

\[ \|h(t, \xi) - h(t, \eta)\| \leq L_n \| \xi - \eta \|_H . \]

\( H_4 \). \( \exists m_0 > 0, C_0 > 0 \) such that \( \forall m \in [0, m_0], t \leq \tau \), \( u, v \in C^0 ([t - \tau, t]; H) \)

\[ \int_s^{s_0} |h(s, u) - h(s, v)| \, ds \leq C \int_s^{s_0} |u(s) - v(s)| \, ds ; \]

\( H_5 \). \( h \in C(\mathbb{R} \times C_H; H) \), and there exists \( C > 0 \) such that, for any \( (t, \xi) \in \mathbb{R} \times C_H \), the Frechet derivative \( \delta h(t, \xi) \in L^1 (\mathbb{R} \times C_H; H) \)

\[ \| \delta h(t, \xi) \|_{L^1 (\mathbb{R} \times C_H; H)} \leq C (1 + 1 \| \xi \|_H . \]

The rest of this paper is organized as follows. In Section 2, we introduce basic concepts concerning global and pullback attractor. In Section 3, we obtain the existence of the global attractor. In Section 4, we obtain the existence of the pullback attractor.

2. Preliminaries

In this section, firstly, we recall some basic concepts about the global attractor.

**Definition 2.1** ([3]) Let \( X \) be a Banach space and \( \{ S(t) \}_{t \geq 0} \) be a family of operators on \( X \). We say that \( \{ S(t) \}_{t \geq 0} \) is norm-to-weak continuous semigroup on \( X \), if \( \{ S(t) \}_{t \geq 0} \) satisfies:

1) \( S(0) = I d (\text{identify}) \);
2) \( S(t) S(s) = S(t + s) \);
3) \( S(t) x \to S(t) x \) if \( t_n \to t \) and \( x_n \to x \) in \( X \).

Remark: The strong continuous semigroup and the weak semigroup are both the norm-to-weak continuous.

**Definition 2.2** ([3]) The semigroup \( S(t) \) is called satisfying Condition (C) in \( X \), if and only if for any bounded set \( B \) of \( X \) and for any \( \epsilon > 0 \), there exist a positive constant \( t_\epsilon \) and a finite dimensional subspace \( X_1 \) of \( X \), such that \( \{ P(s) x \}_{x \in B, t \geq t_\epsilon} \) is bounded and

\[ \| (I - P) S(t) x \| < \epsilon \] for any \( t \geq t_\epsilon \) and \( x \in B \), where \( P: X \to X_1 \) is the canonical projector.

**Lemma 2.1** ([3]) Let \( X \) be a Banach space and \( \{ S(t) \}_{t \geq 0} \) be a norm-to-weak continuous semigroup on \( X \). Then \( \{ S(t) \}_{t \geq 0} \) has a global attractor in \( X \) provided that the following conditions hold:

1) \( \{ S(t) \}_{t \geq 0} \) has a bounded absorbing set \( B_0 \) in \( X \);
2) \( \{ S(t) \}_{t \geq 0} \) satisfies Condition (C) in \( X \).

Then, we state the concepts and some result about the process and the pullback attractor.

Instead of a family of the one-parameter map \( S(t) \), we need to use a two-parameter semigroup or process \( U(t, \tau) \) on the complete metric space \( X \), \( u(t, \tau) \) denotes the value of the solution at time \( t \) which was equal to the initial value \( u_0 \) at time \( \tau \).

The semigroup property is replaced by the process composition property

\[ U(t, \tau) U(t, r) = U(t, r) \] for all \( t \geq \tau \), and

and, obviously, the initial condition implies

\[ U(t, \tau) = I d . \]

**Definition 2.3** Let \( U \) be the two-parameter semigroup or process on the complete metric space \( X \). A family of compact set \( \{ A(t) \}_{t \geq 0} \) is said to be a pullback attractor for \( U \) if, for all \( \tau \in \mathbb{R} \), it satisfies

1) \( U(t, \tau) A(\tau) = A(\tau) \) for all \( t \geq \tau \), and
2) \( \lim \inf_{t \to - \infty} \inf_{\tau \geq 0} \inf_{t \geq \tau} \text{dist}(U(t, t - s) \, A(t)) = 0 \), for all bounded \( \mathcal{D} \subset X \), and all \( t \in \mathbb{R} \).
Definition 2.4 The family \( \{ B(t) \}_{t \in \mathbb{R}} \) is said to be
1) pullback absorbing with respect to the process \( U \),
if for all \( t \in \mathbb{R} \) and all bounded \( D \subset X \), there exists \( T_D(t) > 0 \) such that \( U(t, t-s)D \subset B(t) \) for all \( s \geq T_D(t) \);
2) pullback attracting with respect to the process \( U \),
if for all \( t \in \mathbb{R} \), all bounded \( D \subset X \), and all \( \varepsilon > 0 \),
there exists \( T_{\varepsilon, D}(t) > 0 \) such that for all \( s \geq T_{\varepsilon, D}(t) \)
\[ \text{dist}_V(U(t, t-s)D, B(t)) < \varepsilon; \]
3) pullback uniformly absorbing (respectively uniformly attracting) if \( T_D(t) \) in part (a) (respectively \( T_{\varepsilon, D}(t) \) in part (b)) does not depend on the time \( t \).

Theorem 2.1 Let \( U(t, \tau) \) be a two-parameter process, and suppose \( U(t, \tau) : X \rightarrow X \) is continuous for all \( t \geq \tau \). If there exists a family of compact pullback attracting sets \( \{ B(t) \}_{t \in \mathbb{R}} \), then there exists a pullback attractor \( \{ A(t) \}_{t \in \mathbb{R}} \), such that \( A(t) \subset \{ B(t) \} \) for all \( t \in \mathbb{R} \), and which is given by
\[ A = \bigcup_{D \in X} A_D(t), \text{ where } A_D(t) = \bigcap_{s \geq t} U(t, t-s)D. \]

We set \( E = V \times H \), where \( V = H^1_0(\Omega), H = L^2(\Omega) \), which are Hilbert spaces for the usual inner product and associated norms. We denote by \( \lambda_1 \) the first eigenvalue of \(-\Delta\) in \( V \).

Our problem can be written as a second-order differential equation in \( H \):
\[ \begin{align*}
(\alpha u'') + \beta \Delta u' - \Delta u + g(u) & = f(x) + h(t, u), \quad t > \tau, \\
(\alpha u' + \beta u) + \beta u = & \phi(t-\tau), \quad u'(t-\tau) = \phi'(t-\tau), \quad t \in [\tau-r, \tau].
\end{align*} \tag{2.1} \]

3. Existence of the Global Attractor

In this section, our objection is to show that the well-posed of the solution and the existence of global attractor for the initial boundary value problem (1.1), we assume that \( f \in \mathcal{L}^2(\Omega) \).

Let \( 0 < \varepsilon \leq \min \{ \frac{1}{\beta'}, \frac{\lambda_1}{4}, \frac{\lambda_2}{2\alpha} \} \) and \( \alpha > 0, \beta > 0 \), then by the transformation \( v = u' + \varepsilon u \). The initial boundary value problem (2.1) is equivalent to
\[ \begin{align*}
& v' + (\alpha - \varepsilon)v + \varepsilon (\alpha - \alpha)u + \beta \Delta v + (\beta e - 1)\Delta u + g(u) \\
& = f(x) + h(t, u), \quad t > \tau,
\end{align*} \tag{3.1} \]

with the initial value conditions
\[ v(t) = \phi'(t-\tau) + \varepsilon \phi(t-\tau), \quad t \in [\tau-r, \tau]. \]

Theorem 3.1 Assume that the hypotheses on \( g \) and \( h \) hold for all \( \{ (u, v) \} \in \mathcal{E} \) and \( f \in \mathcal{L}^2(\Omega) \), \( \alpha, \beta \) are the positive constants. Then the initial boundary value problem (3.1) has the unique solution \( \{ u, v \} \in \mathcal{E} \) for all \( t > \tau \).

Proof. Taking the inner product of the Equation (3.1) with \( v \) in \( H \), we find that
\[ \begin{align*}
& \frac{1}{2} \frac{d}{dt} |v|^2 + (\alpha - \varepsilon)(v, v) + \varepsilon (\alpha - \alpha) (u, v) + \beta \|v\|^2 \\
& + \frac{1}{2} \beta \varepsilon \frac{d}{dt} \|v\|^2 + \varepsilon (1 - \beta \varepsilon) \|v\|^2 + (g(u), v) \\
& \geq \frac{1}{2} \|v\|^2 + \frac{1}{\lambda_1} \|v\|^2 + (g(u), v).
\end{align*} \tag{3.2} \]

Since \( v = u(t) + \varepsilon u \) and \( 0 < \varepsilon \leq \min \{ \frac{1}{\beta'}, \frac{\lambda_1}{4}, \frac{\lambda_2}{2\alpha} \} \), we deal with the terms in (3.2) one by one as follows
\[ \begin{align*}
& (\alpha - \varepsilon)(v, v) = (\alpha - \varepsilon)|v|^2 \geq \frac{3\alpha}{4} |v|^2; \\
& \varepsilon (\alpha - \alpha) (u, v) \geq \frac{\varepsilon (\alpha - \alpha)}{\sqrt{\lambda_1}} \|u\| \|v\| \geq \frac{\varepsilon \alpha}{\sqrt{\lambda_1}} |v|^2; \\
& \geq -\frac{\alpha \varepsilon^2}{\lambda_1} |v|^2 - \frac{\varepsilon}{4} \|v\|^2 \geq -\frac{\varepsilon}{4} |v|^2 - \frac{\alpha}{2} |v|^2; \\
& \geq \left( \frac{1}{\alpha} - \frac{\varepsilon}{4} \right) |v|^2 + \frac{\varepsilon}{4} |v|^2; \tag{3.6} \\
& \geq \left( \frac{1}{\alpha} - \frac{\varepsilon}{4} \right) |v|^2 + \frac{\varepsilon}{4} |v|^2; \tag{3.7}
\end{align*} \]

By (3.3)-(3.7), it follows from that
\[ \begin{align*}
& \frac{d}{dt} \left( |v|^2 + 2 \|f\|^2 + 2 J(u) \right) \left( \frac{\alpha}{2}, \frac{\alpha \varepsilon^2}{\lambda_1} + \beta \lambda_1 \right) |v|^2 \\
& + 2 \left( \varepsilon (1 - \beta \varepsilon) - \frac{\varepsilon}{4} \right) |v|^2 + 2eC_0 J(u) \\
& \leq \left( \frac{2}{\alpha} |h|^2 + \frac{2}{\alpha} |f|^2 + 2eC_0 \right). \tag{3.8}
\end{align*} \]

Since \( \varepsilon \leq \min \{ \frac{1}{\beta'}, \frac{\lambda_1}{4}, \frac{\lambda_2}{2\alpha} \} \) and \( 0 < \gamma < \frac{1}{4} - \frac{\varepsilon}{2} \), this will imply \( 2 \left( \varepsilon (1 - \beta \varepsilon) - \frac{\varepsilon}{4} \right) > \varepsilon (1 - \beta \varepsilon) \), then we have
\[ \begin{align*}
& \frac{d}{dt} \left( |v|^2 + 2 J(u) \right) \left( \frac{2}{\alpha} |h|^2 + \frac{2}{\alpha} |f|^2 + 2eC_0 \right) \tag{3.8} \\
& \leq \left( \frac{2}{\alpha} |h|^2 + \frac{2}{\alpha} |f|^2 + 2eC_0 \right) \\
& \leq \left( \frac{2}{\alpha} |h|^2 + \frac{2}{\alpha} |f|^2 + 2eC_0 \right). \tag{3.9}
\end{align*} \]

Set \( C_0 = \min \{ 2\beta \lambda_1, \varepsilon, 2eC_0 \} \), then (3.9) can be writ-

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ten as following
\[
\frac{d}{dt} \left( |u|^2 + (1 - \beta \epsilon)\|u\|^2 + 2J(u) \right) + C_o \left( |u|^2 + (1 - \beta \epsilon)\|u\|^2 + 2J(u) \right) \\
\leq \frac{2}{\alpha} |f|^2 + 2eC_r.
\]
As our assumptions ensure that
\[
-C_o + \frac{2C_o^2 \lambda_{-1}^{-1}}{\alpha (1 - \beta \epsilon - 2\gamma)} < 0,
\]
then we can choose
\[
m \in (0, m_o) \quad \text{small enough such that}
\]
\[
m - C_o + \frac{2C_o^2 \lambda_{-1}^{-1}}{\alpha (1 - \beta \epsilon - 2\gamma)} < 0.
\]
For this choice, we have
\[
e^{\alpha t} \left( |u|^2 + (1 - \beta \epsilon)\|u\|^2 + 2J(u) \right) \leq e^{\alpha t} \left( |v|^2 + (1 - \beta \epsilon)\|v\|^2 + 2J(u) \right) + \left( m - C_o \right) \int_0^t e^{\alpha s} \left( |u|^2 + (1 - \beta \epsilon)\|u\|^2 + 2J(u) \right) ds \\
+ \frac{2}{\alpha} \int_0^t e^{\alpha s} |f|^2 ds + \frac{2}{\alpha} \int_0^t e^{\alpha s} \|f\|^2 ds + 2eC_r \int_0^t e^{\alpha s} ds
\]
\[
\leq e^{\alpha t} \left( |v|^2 + (1 - \beta \epsilon)\|v\|^2 + 2J(u) \right) + \left( m - C_o \right) \int_0^t e^{\alpha s} \left( |u|^2 + (1 - \beta \epsilon)\|u\|^2 + 2J(u) \right) ds \\
+ \frac{2C_o^2 \lambda_{-1}^{-1}}{\alpha} \int_0^t |f|^2 ds + \frac{2}{\alpha} \int_0^t e^{\alpha s} \|f\|^2 ds + 2eC_r \int_0^t e^{\alpha s} ds
\]
\[
= \frac{2C_o^2 \lambda_{-1}^{-1}}{\alpha} \int_0^t e^{\alpha s} |f|^2 ds
\]
\[
\leq \frac{2}{\alpha} \int_0^t e^{\alpha s} |f|^2 ds
\]
\[
\leq \frac{2C_o^2 \lambda_{-1}^{-1}}{\alpha (1 - \beta \epsilon - 2\gamma)} \int_0^t e^{\alpha s} \left( |v|^2 + (1 - \beta \epsilon)\|v\|^2 + 2J(u) \right) ds
\]
\[
+ \frac{4C_o^2 \lambda_{-1}^{-1} C_r}{\alpha (1 - \beta \epsilon - 2\gamma)} \left( e^{\alpha t} - e^{\alpha \tau} \right);
\]
\[
(3.10)
\]
\[
(3.11)
\]
\[
(3.12)
\]
\[
(3.13)
\]
(3.10)-(3.13) means that
\[
(\alpha t, \alpha \tau) \rightarrow \infty, \quad \text{for any} \ u \in D
\]
\[
\|u\|^2 \leq \frac{1}{1 - \beta \epsilon - 2\gamma} \left( |u|^2 + 2J(u) \right) + \frac{2C_o}{1 - \beta \epsilon - 2\gamma}.
\]
In the Bounded set \( D \subset C_{r, \gamma} \), for any \( u \in D \), there exists a constant \( d \) such that
\[
\|u\|^2 + |f|^2 \leq d^2;
\]
\[
|u|^2 + (1 - \beta \epsilon)\|u\|^2 + 2J(u) \leq d^2.
\]
Hence, by (3.12)-(3.14) and the choice of
then (3.17) yields that
\[
\left| v' \right|^2 + \left[ (1 - \beta \epsilon) \right]^2 \left| w' \right|^2 + 2J(u) \leq \rho_0^2 + \rho_0^2 \alpha^2 \epsilon^{n(r-1)}, \quad \forall t > \tau.
\]
which means that the initial boundary value problem (3.1) has the solution \((u, w') \in E\).

Now, we prove the uniqueness of the solution. Assume that \(u(\cdot) = u(\cdot; \tau, \phi)\) and \(v(\cdot) = v(\cdot; \tau, \psi)\) are the two solutions of the initial boundary value problem (3.1), \(\phi, \psi\) are the corresponding initial value, we denote \(w(\cdot) = u(\cdot) - v(\cdot)\). Therefore we have
\[
w' + \alpha w' - \beta \Delta w - \Delta w + g(u) - g(v) = h(t, u) - h(t, v).
\]
we take the inner product of the above equation with \(w'\) and we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \left| w' \right|^2 + \left| w' \right|^2 \right) + \alpha \left| w' \right|^2 + \beta \left| w' \right|^2 + \left( g(u) - g(v), w' \right) = (h(t, u) - h(t, v), w').
\]
Since
\[
\left( g(u) - g(v), w' \right) \leq C_1 \left| w' \right|^2,
\]
we have
\[
2 \left( h(t, u) - h(t, v), w' \right) \leq \left| h(t, u) - h(t, v) \right|^2 + \left| w' \right|^2.
\]
So (3.20) can yields that
\[
\frac{d}{dt} \left( \left| w' \right|^2 + \left| w' \right|^2 \right) \leq 2C_1 \left| w' \right|^2 + \left| h(t, u) - h(t, v) \right|^2 + \left| w' \right|^2 \leq \left( \lambda_1^{-1} C_1 r \phi - \psi \right) \left| w' \right|^2 + \left| \lambda_1^{-1} C_1 r \phi - \psi \right| \left| w' \right|^2.
\]
Integrating (3.21) over the interval \([\tau, t]\), we can get
\[
\left| w'(t) \right|^2 + \left| w'(t) \right|^2 \leq \left( \lambda_1^{-1} C_1 r \phi - \psi \right) \left| w'(\tau) \right|^2 + \left( \lambda_1^{-1} C_1 r \phi - \psi \right) \left| w'(\tau) \right|^2.
\]
Set \(\gamma_1 = \max \left\{ \lambda_1^{-1} C_1 + C_2, C_1 \right\}\), then we have
\[
\left| w'(t) \right|^2 + \left| w'(t) \right|^2 \leq \left( \lambda_1^{-1} C_1 r \phi - \psi \right) \left| w'(\tau) \right|^2 + \gamma_1 \left( \left| w'(\tau) \right|^2 + \left| w'(\tau) \right|^2 \right).
\]
Combining the Gronwall Lemma, we get
\[
|w'(t)|^2 + \|w(t)|^2 \leq (1 + \lambda_i^{-1}C^2),
\]
\[
(\phi - \phi') e^{\lambda_i(t-\tau)}, \quad \text{for all } t \geq \tau.
\]
(3.22)
If \( \phi, \phi' \) stand for the same initial value, there has
\[
|w'(t)|^2 + \|w(t)|^2 \leq 0.
\]
that shows that
\[
|w'(t)|^2 = 0, \quad \|w(t)|^2 = 0.
\]
that is
\[
w(t) = 0.
\]
therefore
\[
u = w.
\]
we get the uniqueness of the solution. So the proof of the
theorem 3.1 has been completed.

By the theorem 3.1, we obtain the global smooth solution
\((u, u')\) continuously depends on the initial value
\((\phi, \phi')\), the initial boundary value problem (1.1)
generates a continuous semigroup
\[
\{S(t)\}_{t \geq 0}, S(t) : E \to E; (u, u') = S(t)(\phi, \phi').
\]
Then \( B_{\phi} = \left\{ (u, u') \middle| \|u, u'\| \leq \rho_0 \right\} \) is a bounded
absorbing set for the semigroup \( \{S(t)\}_{t \geq 0} \) generated by
(1.1).
Under the assumption on \( g \) and \( f \), we can get the nonlinear term \( g(u) \) compact and continuous, \( f(x) \) continuous. Next, our object is to show that the
\( C^0 \) semigroup \( \{S(t)\}_{t \geq 0} \) satisfies condition C.

Theorem 3.2 Assume that the hypotheses on \( g \) and \( h \) hold for all \( (u, u') \in E \), \( \alpha, \beta \) are positive constants.
Then the \( C^0 \) semigroup \( \{S(t)\}_{t \geq 0} \) associated with initial value
problem (3.1) satisfies condition C, that is, there exists \( m \in \mathbb{N} \) and \( T = T(B, R) \), for any
\( N \geq m, t \geq T \) such that
\[
|v|^{2} + (1 - \beta)\|u\|^{2} \leq C, \quad C \text{ is the positive constant.}
\]
Proof. Let \( \lambda_j \) be the eigenvalues of \(-\Delta u\) and \( w_j \)
be the corresponding eigenvectors, \( j = 1, 2, \ldots \), without
loss of generality, we can assume that \( \lambda_1 < \lambda_2 < \cdots \), and
\( \lim_{m \to \infty} \lambda_m = \infty \).
It is well known that \( \{w_j\}^\infty_{j=1} \) form an orthogonal basis
of \( H_0^1 \). Write
\[
H_m = \text{span}\{w_1, w_2, \cdots, w_m\}
\]
Since \( f \in H_0^1 \) and \( f \) is compact, for any \( \epsilon > 0 \),
there exists some \( m \in \mathbb{N} \) such that
\[
\left\| (I - P_m) f \right\| \leq \frac{\epsilon}{2}, \quad (3.23)
\]
where \( P_m : H_0^1 \to H_m \) is orthogonal projection and \( R \)
is the radius of the absorbing set. For any \((u, u') \in E\),
we write
\[
(u, u) = (P_m u, P_m u) + ((I - P_m) u, (I - P_m) u)
\]
\[
= (u_1, u_2) + (u_2, u_3).
\]
We note that
\[
h_2 = (I - P_m) h, g_2 = (I - P_m) g, f_2 = (I - P_m) f,
\]
Taking the inner product of the second equation of
(3.1) with \( v \) in \( L^2(D) \). After a computation like in
the proof of Theorem 3.1, we can yield that
\[
\frac{1}{2} \frac{d}{dt} \left( |v^2 + (1 - \beta)\|u\|^{2} + (\alpha - \beta)\|v\|^2 \right) + \epsilon \left( |v^2 + (1 - \beta)\|u\|^{2} + (\alpha - \beta)\|v\|^2 \right)
\]
\[
= f_2(x, v) + (h_2(t, u, v_2).
\]
This the same as in the proof of the Theorem 3.1, except for a replacement of \( \lambda_j \) with \( \lambda_{m+1} \). Combined
with (3.23), (3.24) and (3.4), then we have
\[
\frac{1}{2} \frac{d}{dt} \left( |v^2 + (1 - \beta)\|u\|^{2} + \left( \frac{3\alpha}{4} + \beta \lambda_{m+1} - \frac{5\epsilon}{2} \right)\|v\|^2 \right)
\]
\[
+ \epsilon \left( |v^2 + (1 - \beta)\|u\|^{2} + \left( \frac{3\alpha}{4} + \beta \lambda_{m+1} - \frac{5\epsilon}{2} \right)\|v\|^2 \right)
\]
\[
= \frac{L^2}{2\epsilon} + \epsilon^2.
\]
Choose \( k_{\epsilon} = \min \left\{ \frac{3\alpha}{4} + \beta \lambda_{m+1} - \frac{5\epsilon}{2}, 1 \right\} \), we can get
\[
\frac{1}{2} \frac{d}{dt} \left( |v^2 + (1 - \beta)\|u\|^{2} + k_{\epsilon} (|v^2 + \|u\|^2) \right) \leq \frac{L^2}{2\epsilon} + \epsilon^2.
\]
By Gronwall lemma, we can obtain
\[
|v|^2 + (1 - \beta)\|u\|^2 \leq \frac{L^2 + \epsilon^2}{2\epsilon k_{\epsilon}}
\]
for all \( t \geq \tau, N \geq m \) and \( (u, u') \in E \). This shows that
Condition C is satisfied, and the proof is completed.
Due to Lemma 2.1, Theorem 3.1 and Theorem 3.2, we
obtain the following Theorem

Theorem 3.3 Assume that the hypotheses on \( g \) and \( h \) hold for all \( (u, u') \in E \), \( \alpha, \beta \) are positive constants.
Then the \( C^0 \) semigroup \( \{S(t)\}_{t \geq 0} \) associated with initial value
problem (3.1) has a global attractor in \( E \).

4. Existence of the Pullback Attractor

In this subsection, we assume that \( f \in H \), we aim to study the pullback attractor for the initial value problem
From Theorem 3.1, the initial value problem (1.1) generates a family two-parameter semigroup \( U(\cdot, \cdot) \) in \( C_{r,H} \), which can be defined by

\[
U(t, \tau)(\phi) = u_\phi(t; \tau, \phi), \quad t \geq \tau, \phi \in C_{r,H}
\]

**Lemma 4.1** Let \( \phi, \psi \) be the two initial values for the problem (1.1), \( \tau \in \mathbb{R} \) is the initial time, Denote by \( u(\cdot) = u(\cdot; \tau, \phi) \) and \( v(\cdot) = v(\cdot; \tau, \psi) \) the corresponding solutions to (1.1). Then, there exists a constant \( \gamma_1 > 0 \) which is independent of initial value and time, such that the following estimates hold:

\[
\begin{align*}
\|u'(t) - v'(t)\|^2 + \|u(t) - v(t)\|^2 &
\leq (1 + \lambda_1^t C^2_r) \|\phi - \psi\|^2_{C_{r,H}} e^{\gamma_1(t-\tau)}, \quad \text{for all } t \geq \tau; \\
\|u(t) - v(t)\|^2 &
\leq (1 + \lambda_1^t C^2_r) \|\phi - \psi\|^2_{C_{r,H}} e^{\gamma_1(t-\tau)}, \quad \text{for all } t \geq \tau + r.
\end{align*}
\]

**Proof.** We denote \( w = u - v \), by (3.22), we can get (4.1) easily.

If we consider \( t \geq \tau + r \), then \( t + \theta \geq \tau \) for any \( \theta \in [-r,0) \), and

\[
\begin{align*}
\|w'(t + \theta)\|^2 + \|w(t + \theta)\|^2 &
\leq (1 + \lambda_1^t C^2_r) \|\phi - \psi\|^2_{C_{r,H}} e^{\gamma_1(t-\tau + \theta)} \\
&
\leq (1 + \lambda_1^t C^2_r) \|\phi - \psi\|^2_{C_{r,H}} e^{\gamma_1(t-\tau)}.
\end{align*}
\]

Thus, \( \|w(t)\|^2 \leq (1 + \lambda_1^t C^2_r) \|\phi - \psi\|^2_{C_{r,H}} e^{\gamma_1(t-\tau)}, \forall t \geq \tau + r \).

**Theorem 4.1** The mapping \( U(t, \tau): C_{r,H} \to C_{r,H} \) is continuous for any \( t \geq \tau \).

**Proof.** Let \( \phi, \psi \in C_{r,H} \) be the initial value for the problem (1.1) and \( t \geq \tau \). Denote by \( u(\cdot) = u(\cdot; \tau, \phi) \) and \( v(\cdot) = v(\cdot; \tau, \psi) \) the corresponding solutions to (1.1). Then, writing again \( w = u - v \) we obtain the following. If \( t \in [\tau - r, \tau] \), then \( w(t) = \phi(t - \tau) - \psi(t - \tau) \) and

\[
\begin{align*}
\|w(t)\|^2 &
\leq (1 + \lambda_1^t C^2_r) \|\phi - \psi\|^2_{C_{r,H}} e^{\gamma_1(t-\tau)}.
\end{align*}
\]

Thus, we have

\[
\begin{align*}
\|w(t)\|^2 &
\leq (1 + \lambda_1^t C^2_r) \|\phi - \psi\|^2_{C_{r,H}} e^{\gamma_1(t-\tau)}, \forall t \geq \tau - r,
\end{align*}
\]

whence

\[
\begin{align*}
\|w(t)\|^2 &
\leq (1 + \lambda_1^t C^2_r) \|\phi - \psi\|^2_{C_{r,H}} e^{\gamma_1(t-\tau)}, \forall t \geq \tau,
\end{align*}
\]

which implies the continuity of \( U(t, \tau) \).

**Theorem 4.2** Assume that the hypotheses on \( g \) and \( h \) hold with \( m_0 > 0 \), \( \alpha, \beta \) are the positive constants.

Suppose in addition that \( \sqrt{2C_\alpha} \sqrt{\lambda_1^t} < \alpha, \beta - 2\gamma \).

Then exists a family \( \{B(t)\}_{t \geq 3} \) of bounded sets in \( C_{r,H} \) which is uniformly pullback absorbing for the process \( U(\cdot, \cdot) \). Moreover, \( B(t) = B^0 \) for all \( t \in \mathbb{R} \), where \( B^0 \) is the bounded set in \( C_{r,H} \).

**Proof.** By (3.18), we can have

\[
\|u(t, \tau, \phi)\|^2 + \|u'(t, \tau, \phi)\|^2 \leq \rho_0^2 + \rho_0^2 d^2 e^{\alpha(t-\tau)}, \forall t \geq \tau.
\]

and, in particular,

\[
\|u(t, \tau, \phi)\|^2 + \|u'(t, \tau, \phi)\|^2 \leq \rho_0^2 + \rho_0^2 d^2, \forall t \geq \tau.
\]

Moreover, as \( u(t, \tau, \phi) = \phi(t - \tau) \) and \( u'(t, \tau, \phi) = \phi'(t - \tau) \) for \( t \in [\tau - r, \tau] \), then inequality (4.3) holds true for \( t \geq \tau - r \).

If we take now \( t \geq \tau + r \), then for all \( \theta \in [-r,0] \) we have \( t + \theta \geq \tau \) and so

\[
\|u(t, \tau, \phi)\|^2 + \|u'(t, \tau, \phi)\|^2 \leq \rho_0^2 + \rho_0^2 d^2 e^{\alpha(t-\tau)},
\]

or, in other words,

\[
\|U(t, \tau)\|_{C_{r,H}}^2 \leq \rho_0^2, \forall \tau \in \mathbb{R}.
\]

Therefore, there exists \( T_D \geq \tau \) such that

\[
\|U(t, t - s)\|_{C_{r,H}}^2 \leq \rho_0^2, \forall t \in \mathbb{R}, s \geq T_D, \phi \in D.
\]

which means that the ball \( B_D(0, \rho_0) = B^0 \subset C_{r,H} \) is uniformly pullback absorbing for the process \( U(\cdot, \cdot) \).

**Remark:** On the one hand, observe that if \( t_0 \in \mathbb{R} \) and \( t \geq t_0 \), then

\[
\phi(t, t_0 - s, \phi) = u(t, t_0 - s, \phi), \quad u'(t, t_0 - s, \phi) = u'(t, t_0 - s, \phi).
\]

And

\[
\|u(t, t_0 - s, \phi)\|^2 + \|u'(t, t_0 - s, \phi)\|^2 \leq \rho_0^2.
\]

On the other hand, (4.3) implies,

\[
\|u(t, t_0 - s, \phi)\|^2 + \|u'(t, t_0 - s, \phi)\|^2 \leq \rho_0^2 + \rho_0^2 d^2, \forall t \geq \tau.
\]

**Theorem 4.3** Under the assumption in Theorem 4.1. Then there exists a compact set \( B^0 \subset C_{r,H} \) which is uniformly pullback attracting for the process \( U(\cdot, \cdot) \), and consequently, there exists the pullback attractor.
\[ \{A(t)\}_{t \in \mathbb{R}}. \] Moreover, \( \{A(t)\}_{t \in \mathbb{R}} \subseteq C_{\phi(t)}, \) for all \( t \in \mathbb{R}. \)

**Proof.** For each \( \epsilon \in \mathbb{R}, \) the norm
\[
\| \phi \|_{t}^{2} = \| \phi' + e \phi \|_{t}^{2}, \quad \phi \in C_{\phi(t)},
\]
is equivalent to
\[
\| \phi \|_{t}^{2} = \| \phi' + e \phi \|_{t}^{2}. \]
This allows us to obtain absorbing ball for the original norm by proving the existence of absorbing balls for this new norm for some suitable value of \( \epsilon. \)

Indeed, let us denote \( B_{\epsilon}(0, \rho) = \{ \phi \in C_{\phi(t)} : \| \phi \|_{t} < \rho \}. \)

Noticing that for \( c_{1} = \max \{2 + 2e^{2} \lambda^{-1} \} \) it follows that
\[
\| \phi \|_{t}^{2} = \| \phi' + e \phi \|_{t}^{2} \leq \| \phi' \|_{t}^{2} + e \| \phi \|_{t}^{2} \leq c_{1} \| \phi \|_{t}^{2},
\]
we then have \( B_{\epsilon}(0, \rho) \subset B_{\epsilon}(0, c_{1}^{2} \rho). \)

Let \( D \subset C_{\phi(t)} \) be a bounded set, i.e. there exists \( d > 0 \) such that for any \( \phi \in \) it holds
\[
\| \phi \|_{t}^{2} = \| \phi' + e \phi \|_{t}^{2} \leq c_{d} \| \phi \|_{t}^{2}.
\]
Denote by \( u(\cdot) = u(\cdot; t) \) the solution of the problem (2.1), and consider the problems:
\[
\begin{cases}
v'' + \alpha v' - \beta \Delta v' - \Delta v + g(u) = f(x) + h(t, u), & t > \tau, \\
v(t) = 0, & t \in [0, \tau]. 
\end{cases}
\]  
\[ (4.5) \]
\[
\begin{cases}
w'' + \alpha w' - \beta \Delta w' - \Delta w \equiv 0, & t > \tau, \\
w(t) = \phi(t - \tau), & t \in [0, \tau]. 
\end{cases}
\]  
\[ (4.6) \]

From the uniqueness of the solution of problems (2.1), (4.5) and (4.6) it follows that
\[ u(\cdot) = v(\cdot) + w(\cdot), \quad \forall t \in \mathbb{R}, \text{ and } \forall t \geq \tau. \]

Consequently, \( U(t, \tau) \) can be written as
\[ U(t, \tau)(\phi) = U_{1}(t, \tau)(\phi) + U_{2}(t, \tau)(\phi), \]
\[ \forall \phi \in C_{\phi(t)}, t \geq \tau, \]
where \( U_{1}(t, \tau)(\phi) = v(\cdot; \tau, \phi) \) and \( U_{2}(t, \tau)(\phi) \) are the solutions of (4.5) and (4.6) respectively.

First, thanks to (4.4), but with \( g = f = h = 0, \) it follows that
\[
\| v_{1} \|_{t}^{2} + \| v_{2} \|_{t}^{2} \leq c_{d} \| \phi \|_{t}^{2}, \quad \forall t \geq t_{0} + s, \phi \in D.
\]  
\[ (4.7) \]

Furthermore, for \( t_{0} \in \mathbb{R}, t \geq t_{0} \) and \( s \geq T_{D} \geq r, \)
\[ w(t; t_{0} - s, \phi) = w(t; t_{0} - s, t), \phi. \]

with \( s + t - t_{0} \geq s \geq T_{D} \geq r. \) Thus, Equation (4.7) implies in particular
\[ w(t; t_{0} - s, \phi) \leq c_{d} \| \phi \|_{t}^{2} e^{n(s + t_{0} - s - r)}, \quad \forall t_{0} \in \mathbb{R}, t \geq t_{0}, s \geq T_{D}, \phi \in D. \]

Then we can obtain that
\[ \| U_{2}^{2} (t, t - s) \phi \|_{t}^{2} \leq c_{d} \| \phi \|_{t}^{2} e^{n(s - s - r)}, \quad \forall t \in \mathbb{R}, t \geq t_{0}, \phi \in D. \]

Thus, we have
\[ \limsup_{t \rightarrow \infty} \| U_{2}^{2} (t, t - s) \phi \|_{t}^{2} = 0. \]

Next, fix \( t_{0} \in \mathbb{R}, s \geq T_{D}, \phi \in D \) and denote
\[ u(t) = u(t; t_{0} - s, \phi), \]
\[ v(t) = v(t; t_{0} - s, \phi), \]
\[ F(t) = f + h(t, u), \quad t \geq t_{0} - s. \]

Then, for \( t \geq t_{0}, \)
\[ F(t) \leq |f| + |g| + L_{n} \| u \| \]
\[ \leq |f| + |g| + L_{n} \lambda_{2}^{2} \rho_{0} = K_{1}, \quad (4.8) \]

and for \( t \geq t_{0} - s, \) we have
\[ F(t) \leq |f| + |g| + L_{n} \| u \| \]
\[ \leq |f| + |g| + L_{n} \lambda_{2}^{2} \left( \rho_{0}^{2} + \rho_{0}^{2} \right)^{\frac{1}{2}} \]
\[ \leq K_{1} + L_{n} \lambda_{2}^{2} \rho_{0}. \quad (4.9) \]

Then, we deduce from the assumption on \( h \) that
\[ F''(t) = \{ \delta h(t, u), (u' + u') \} \leq \{ \alpha \delta h, u' \} \text{ and} \]
\[ F''(t) \leq K \left[ (1 + \| u \|_{0})^{2} \right] \left[ (1 + \| u' \|_{0})^{2} \right] + C_{s} \| u' \| \text{.} \]

Arguing as we did in order to obtain (4.8) and (4.9), we have
\[ F''(t) \leq K \left[ (1 + \lambda_{2}^{2} \rho_{0})^{2} + C_{s} \| u' \| \right. \text{ and} \]
\[ \text{and} \]
\[ F''(t) \leq K \left[ (1 + \lambda_{2}^{2} \rho_{0})^{2} + C_{s} \| u' \| \right. \text{ and} \]
\[ \forall t \geq t_{0}, \]
\[ (4.10) \]

and
\[ F''(t) \leq K \left[ (1 + \lambda_{2}^{2} \rho_{0})^{2} + C_{s} \| u' \| \right. \text{ and} \]
\[ \forall t \geq t_{0} - s. \]
\[ (4.11) \]

Let us denote
\[ y(t) = \| v' \|_{t}^{2} + \frac{\alpha}{2} v(t) \]
\[ + 4 \| v(t) - F(t) \|^{2} \text{ and make use of the estimates in Theorem 4.2. On the one hand, for all} \]
\[ t \geq t_{0}, \]
\[
\frac{d}{dt}(y(t)) + \frac{\alpha}{2} y(t) \leq \alpha |F(t)|^2 + \frac{4}{\alpha}[F'(t)]^2 + \frac{\alpha^2}{8} \|v(t)\|^2 \\
\leq \alpha \left( K_1 + L_\alpha \lambda_\alpha^{-\frac{1}{2}} \rho_0 d \right)^2 + \frac{4}{\alpha} K_5 (d)^2 + \frac{\alpha^2}{8} \|v(t)\|^2.
\]
but, as (4.4) and (4.7) ensure
\[
\|v(t)\|^2 \leq 2 \|v(t)\|^2 + 2 \|w(t)\|^2 \leq 2 \rho_0^2 + 2 \rho_0^2 d^2 + 2c_0 d^2.
\]
if we denote by
\[
K_4 (d) = \left( K_1 + L_\alpha \lambda_\alpha^{-\frac{1}{2}} \rho_0 d \right)^2 + \frac{4}{\alpha} K_5 (d)^2 + \frac{\alpha^2}{4} (\rho_0^2 + \rho_0^2 d^2 + c_0 d^2).
\]
then, in particular,
\[
y'(t) + \frac{\alpha}{2} y(t) \leq K_4 d, \quad \forall t \in [t_0 - s, t_0].
\]
Noticing that \(y(t_0 - s) = |F(t_0 - s)|^2\), the Gronwall lemma leads us to
\[
y(t_0) \leq \frac{2}{\alpha} K_4 (d) + \left( K_1 + L_\alpha \lambda_\alpha^{-\frac{1}{2}} \rho_0 d \right)^2 = K_5 (d).
\]
On the other hand, if \(t \geq t_0\), we deduce that
\[
\|v(t)\|^2 \leq 2 \|v(t)\|^2 + 2 \|w(t)\|^2 \leq 2 \rho_0^2 + 2c_0 d^2 e^{m(r-s)},
\]
and, from (4.8) and (4.10),
\[
y'(t) + \frac{\alpha}{2} y(t) \leq \alpha K_1 + \frac{4}{\alpha} K_2 + \frac{\alpha^2}{8} \|v(t)\|^2 \\
\leq \alpha K_1 + \frac{4}{\alpha} K_2 + \frac{\alpha^2}{4} \rho_0^2 + \frac{\alpha^2}{4} c_0 d^2 e^{m(r-s)} \\
= K_6 + K_7 d^2 e^{m(r-s)}, \quad \forall t \geq t_0.
\]
Once again, the Gronwall lemma implies that
\[
y(t) \leq y(t_0) e^{\frac{\alpha}{2} (t-t_0)} + \frac{2}{\alpha} K_6 + \frac{2}{\alpha} K_7 d^2 e^{m(r-s)} \\
\leq K_5(d) e^{\frac{\alpha}{2} (t-t_0)} + \frac{2}{\alpha} K_6 + \frac{2}{\alpha} K_7 d^2 e^{m(r-s)}, \quad \forall t \geq t_0.
\]
Then, there exists \(T' > T_0\) such that, if \(s \geq T'\),
\[
y(t) \leq K_5(d) e^{\frac{\alpha}{2} (t-t_0)} + \frac{3}{\alpha} K_6, \quad \forall t \in \mathbb{R}, t \geq t_0.
\]
Recalling that \(y(t) = y(t; t_0 - s, \phi)\), if we fix \(t \geq t_0\), take \(s = T_0\) and denote \(s = t - t_0 + T_0\) we have, provided \(t - t_0\) is large enough, that
\[
y(t; t_0 - T_0, \phi) = y(t; t - (t_0 + T_0), \phi) \\
= y(t; t - \tilde{s}, \phi) \leq \frac{4}{\alpha} K_6.
\]
In conclusion, there exists \(T_D > 0\) such that for all \(t \in \mathbb{R}\), and all \(s \geq T_D + T_D\),
\[
y(t; t - s, \phi) \leq \frac{4}{\alpha} K_6, \quad \forall \phi \in D.
\]
Denoting \(\tilde{T}_D = T_D + T_D + r\), we have for all \(\phi \in D, t \in \mathbb{R}, s \geq \tilde{T}_D\),
\[
\|v'(t; t - s, \phi) + \frac{\alpha}{2} v(t; t - s, \phi)\|^2 \\
+ \|Av(t; t - s, \phi) - F(t; t - s, \phi)\|^2 \leq \frac{4}{\alpha} K_6,
\]
where
\[
F(t; t - s, \phi) = f + h(u(t; t - s, \phi)) - g(u(t; t - s, \phi)).
\]
But as for all \(\phi \in D, t \in \mathbb{R}\) and \(s \geq \tilde{T}_D\), we get
\[
\|v'(t; t - s, \phi)\|^2 \leq \rho_0^2 \quad \text{and} \quad \|F(t; t - s, \phi)\|^2 \leq K_2^2 = 2|f|^2 + 2C_2 \rho_0^2 + 2L_\lambda \lambda_\alpha^{-1} \rho_0^2,
\]
and, consequently, for all \(\phi \in D, t \in \mathbb{R}\) and \(s \geq \tilde{T}_D\),
\[
\|v'(t; t - s, \phi)\|^2 + \|Av(t; t - s, \phi) - F(t; t - s, \phi)\|^2 \leq \frac{4}{\alpha} K_6 \leq \frac{8}{\alpha} K_6 + \frac{\alpha^2}{2} \rho_0^2 + 2K_7^2,
\]
which shows that
\[
\|y(t; t - s, \phi)\|^2 \leq \rho_0^2 + \frac{8}{\alpha} K_6 + \frac{\alpha^2}{2} \rho_0^2 + 2K_7^2,
\]
for all \(\phi \in D, t \in \mathbb{R}\) and \(s \geq \tilde{T}_D\). This means that the all \(B^1 = B_{\alpha, \rho_0} \setminus \{0, \rho_1\}\) is the bounded set in \(C_{\mathbb{R}}\),
which, in addition, is uniformly absorbing for the family of operators \(U(t, \cdot)\). As \(B^1\) is the bounded set in \(C_{\mathbb{R}}\), then there exists \(T_D \geq r\) such that
\[
U_1(t; t - s) B^1 \subset B^1, \forall t \in \mathbb{R}, s \geq T_D,
\]
and, therefore, the bounded set \(B^2 \subset C_{\mathbb{R}}\) given
\[
B^2 = \bigcup_{t \in \mathbb{R}} U_1(t; t - s) B^1 \subset B^1,
\]
is uniformly pullback absorbing for \(U_1(t, \cdot)\) in \(C_{\mathbb{R}}\).
By Ascoli-Arzelà theorem, we can prove that \(B^2\) is compact, so \(B(t) = B^2_{\mathbb{R}}\) is a family of compact subsets in \(C_{\mathbb{R}}\), which is also uniformly pullback attracting for \(U(t, \cdot)\), and the proof has been completed.

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