Fixed Point Theorems of Hegedus Contraction Mapping in Some Types of Distance Spaces

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ABSTRACT

In the present paper, we prove some fixed point theorems of Hegedus contraction in some types of distance spaces, dislocated metric space, left dislocated metric space, right dislocated metric space and dislocated quasi-metric metric space which are generalized metrics spaces where self-distances are not necessarily zero.

Keywords: Distance Space; Dislocated Metric Spaces; Quasi-Metric Spaces; Dislocated Quasi-Metric Spaces; Contraction

1. Introduction

Fixed point theorems on metric spaces and generalized types of metric spaces have applications in the area of logic programming semantics (see, e.g., [1-3]). Dislocated metrics are also known as metric domains in the context of domain theory [4]. The slightly less general notation of partial metrics was also studied in [5].

The definition of a distance space was introduced by P. Waszkiewick [6,7].

Definition 1.1 [6,7]. Let X be a set. A **distance** on X is a map $d{:}X \times X \to [0, \infty)$. A pair (X, d) is called a **distance space**.

Definition 1.2. Let (X, d) be a distance space. Consider the following conditions:

Following Waszkiewick [4], let X be a set. A *distance* on X is a map $d{:}X \times X \to [0, \infty)$. A pair (X, d) is called a **distance space**. If d satisfies the following conditions:

$$(DM_1) \quad d(x,x) = 0 \quad \forall x \in X$$

$$(DM_2) \quad d(x,y) = d(y,x) = 0 \Rightarrow x = y$$

$$(DM_3) \quad d(x,y) = d(y,x)$$

$$(DM_4) \quad d(x,y) \le d(x,z) + d(z,y)$$

$$(DM_5) \quad x = y \text{ if } f \; d(x,x) = d(x,y) = d(y,y)$$

$$(DM_6) \quad d(x,x) \le d(x,y)$$

$$(DM_7) \quad d(x,z) \le d(x,y) + d(y,z) - d(y,y)$$

for all $x, y, z \in X$. If d satisfies $(DM_1) - (DM_4)$, then it is called a *metric* on X. If it satisfies conditions $(DM_2) -$

(DM_4) it is called a **dislocated metric** [1,2] (or simply d-metric) on X. If it satisfies conditions (DM_1), (DM_2), and (DM_4), It is called a **quasi-metric** (or simply q-metric) on X [8]. If it satisfies conditions (DM_2)) and (DM_4), It is called a **dislocated quasi-metric** (or simply dq-metric) on X [1,2]. If it satisfies conditions (DM_3), (DM_5), (DM_6) and (DM_7), It is called a **partial metric** on X [5].

It is clear that any partial metric is a d-metric and any d-metric is dq-metric.

Hitzler and Seda gave an alternative proof of Matthews's Theorem [4] as follows.

Theorem 1.1 [1]. Let (X, d) be a complete d-metric space and let $f{:}X \to X$ be a Banach contraction function. Then f has a unique fixed point.

The plan of this paper is as follows. In Section 2, we introduce some definitions in distance spaces. In Section 3, we establish some fixed point theorems in some types of distance space. In Section 4, we establish some fixed point theorems in some types of dislocated metric space.

2. Some Basic Concepts and Results in Distance Spaces

In the following we proceed with definitions which are needed for our results in a distance space. As it turns out, these notions can be carried over directly from conventional metrics.

Definition 2.1. A sequence (x_n) in a distance space (X, d) **ld-converges** (resp. **rd-converges**, *d-converges*) to $x \in X$ if $\lim_{n \to \infty} d(x_n, x) = 0$ (resp. $\lim_{n \to \infty} d(x_n, x) = 0$, $\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0$). In this case, x is

called the **ld-limit** (resp. **rd-limit**, **d-limit**) of the sequence (x_n).

Is is obvious that a *d*-limit of a sequence (x_n) is *ld*-limit and *rd*-limit but the converse may not be true.

Definition 2.2. A sequence (x_n) in a distance space (X, d) is called **Cauchy** if $\lim_{m,n\to\infty} d(x_m, x_n) = 0$.

Definition 2.3. A distance space (X, d) is called **ld-complete** (resp. **rd-complete, d-complete**) if every Cauchy sequence in it is ld-convergent (resp. *rd*-convergent, *d*-convergent).

Definition 2.4. Let (X, d_1) and (Y, d_2) be distance spaces. Then $f:X \to Y$ is **sequentially l-continuous** if $\forall x_0 \in X$, $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that

$$d_1(x, x_0) < \delta(\varepsilon) \Rightarrow d_2(f(x), f(x_0)) < \varepsilon.$$

Definition 2.5. Let (X, d_1) and (Y, d_2) be distance spaces. Then $f:X \to Y$ is **sequentially r-continuous** if $\forall x_0 \in X$, $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that

$$d_1(x_0, x) < \delta(\varepsilon) \Rightarrow d_2(f(x_0), f(x)) < \varepsilon.$$

We state the following lemmas without proof.

Lemma 2.1. Let (X, d_1) and (Y, d_2) be distance spaces. A function $f:X \to Y$ is sequentially *ld*-continuous (resp. *rd*-continuous, *d*-continuous) if and only if for each sequence (x_n) which is *ld*-convergent (resp. *rd*-convergent, *d*-convergent) to $x_0 \in X$, the sequence $(f(x_n))$ is *ld*-convergent (resp. *rd*-convergent, *d*-convergent) to $f(x_0) \in Y$.

Lemma 2.2. Let (X, d_1) and (Y, d_2) be distance spaces. Then $f:X \to Y$ is sequentially *d*-continuous if f is sequentially *ld*-continuous and sequentially *rd*-continuous.

The following counterexample illustrates the reverse of Lemma 2.2 need not be true.

Counterexample 2.1. Let $d:N \times N \to [0, \infty)$ be defined by:

$$d(x,1) = \frac{1}{x} \quad \forall x \in N - \{1\},$$

$d(x, y) = 2$, $\forall x \in N$, $y \in N - \{1\}$, $d(1, 1) = 2$.

Let $f:N \times N$ defined by $f(x) = x$, $\forall x \in N - \{1\}$, $f(1) = 2$. Since (x_n) ($x_n = n \forall n \in N - \{1\}$) converges to 1 but $(f(x_n))$ does not converges to $f(1)$, then f is not sequentially *l*-continuous. But f is sequentially *d*-continuous.

Now, we give the following lemma without proof.

Lemma 2.4. Every subsequence of *ld*-convergent (resp. *rd*-convergent, *d*-convergent) sequence to x_0 is *ld*-convergent (resp. *rd*-convergent, *d*-convergent) to x_0.

It is obvious that the converse of Lemma 2.4 may not be true.

Definition 2.6. Let (X, d) be a distance space. Let N denote the set of positive integers and $N_0 = N \cup \{0\}$. For any set $Y \subseteq X$ the diameter of the set Y is defined by $diamY = sup\{d(x, y : x, y \in Y\}$ and

$$O(x)\{f^k x : k \in N_0\}$$

is the *f*-orbit of x. The point $x \in X$ is called **regular** (**f-regular**) if $diamO(x) < +\infty$.

A function $f:(X, d) \to (X, d)$ is called a **Hegedus contraction** [9] if there exists $0 \le \lambda \le 1$ such that

$$d(fx, fy)$$
$$\le \lambda \operatorname{diam}\{x, y, fx, fy, f^2x, f^2y, \cdots\},$$
$$= \lambda \operatorname{diam}[O(x) \cup O(y)]$$

where $\forall x \in X$, x is *f*-regular.

Lemma 2.2. Let (X, d) be a distance space. If $f:(X, d) \to (X, d)$ is a Hegedus contraction function, then for any $n \in N$ and $x, y \in X$,

$$d(f^n x, f^n y) \le \lambda^n [O(x) \cup O(y)].$$

Proof. Since

$$\forall f^k x, f^h y \in (O(f^{n-1}x) \cup O(f^{n-1}y))$$

where $k \ge n-1, h \ge n-1$,

$$d(f^k x, f^h y) \le \lambda \operatorname{diam}[O(f^{n-2}x) \cup O(f^{n-2}y)],$$

then

$$d(f^n(x), f^n(y)) \le \lambda \operatorname{diam}[O(f^{n-1}x) \cup O(f^{n-1}y)]$$
$$\le \lambda^2 \operatorname{diam}[O(f^{n-2}x) \cup O(f^{n-2}y)]$$
$$\vdots$$
$$\le \lambda^n \operatorname{diam}[O(x) \cup O(y)].$$

3. Fixed Point Theorems in Distance Spaces

In this section, we introduce some fixed point theorems in distance space.

Definition 3.1. A distance space (X, d) is called an **ld-Hausdorff (resp. rd-Hausdorff, d-Hausdorff) space** iff the *ld*-limt (resp. *rd*-limt, *d*-limt) of *ld*-convergent (resp. *rd*-convergent, *d*-convergent) sequence is unique. Such that $\forall x \in X$, $diam(O(x)) < +\infty$.

Lemma 3.1. Let (X, d) be a distance space such that $\forall x \in X$, $diam(O(x)) < +\infty$. If a function $f:X \to X$ is a Hegedus contraction, then $(f^n(x_0)$ is a Cauchy sequence for each $x_0 \in X$.

Proof. Choose any $x_0 \in X$, for any integer $r \in N \cup \{0\}$. By lemma 2.2,

$$d(f^n(x_0), f^{n+r}(x_0))$$
$$\le \lambda^n \operatorname{diam}[O(x_0) \cup O(f^r(x_0))].$$
$$= \lambda^n \operatorname{diam}O(x_0)$$

The last term tends to zero as *n* tends to infinity. Also,

we proceed similarly as above and obtain

$$d\left(f^{n+r}(x_0), f^n(x_0)\right)$$

$$\leq \lambda^n \mathrm{diam}\left[O\left(f^r(x_0)\cup O(x_0)\right)\right].$$

$$= \lambda^n \mathrm{diam} O(x_0)$$

The last term tends to zero as *n* tends to infinity. Thus, $(f^n(x_0)$ is a Cauchy sequence.

Theorem 3.1. Let (X, d) be a *ld*-Hausdorff *ld*-complete distance space and a function $f:X \to X$ be a *ld*-continuous Hegedus contraction mapping such that $\forall x \in X$, x is *f*-regular. Then *f* has a fixed point.

Proof. From Lemma 3.1, $(f^n(x_0)$ is a Cauchy sequence. Since (X, d) is *ld*-complete distance space, then $(f^n(x_0)$ is *ld*-convergent, say to $x \in X$. From the *ld*-continuity of the mapping *f*, $(f^{n+1}(x_0))$ *ld*-converges to $f(x)$. From Lemma 2.1, $(f^{n+1}(x_0)$ *ld*-converges to *x*. Since (X, d) is *ld*-Haus- dorff, then $f(x) = x$.

In a similar manner of Theorem 3.1, we can prove the following theorems.

Theorem 3.2. Let (X, d) be a *rd*-Hausdorff *rd*-complete distance space and a function $f:X \to X$ be a *rd*-continuous Hegedus contraction mapping such that $\forall x \in X$, x is *f*-regular. Then *f* has a fixed point.

Theorem 3.3. Let (X, d) be a d-Hausdorff *d*-complete distance space and a function $f:X \to X$ be a *d*-continuous Hegedus contraction mapping such that $\forall x \in X$, x is *f*-regular. Then *f* has a fixed point.

4. Fixed Point Theorems in Types of Dislocated Metric Spaces

In this section, we introduce the concept of left dislocated metric spaces and right dislocated metric spaces. Also, we state and prove some fixed point theorems in these spaces. Furthermore, we prove fixed point theorem in dislocated quasi-metric spaces and in dislocated metric spaces.

Definition 4.1. A left dislocated metric (simply *ld*-metric) is a distance *d* on a set *X* satisfying the following axioms, for all $x, y, z \in X$,

(LD_1) if $d(x, y) = d(y, x) = 0$ then $x = y$;

(LD_2) $d(x, y) \leq d(z, x) + d(z, y)$.

Lemma 4.1. *ld*-limits in *ld*-metric spaces are unique.
Proof. Let *x* and *y* be *ld*-limits of the sequence (x_n). By property (LD_2), it follows that

$$d(x, y) \leq d(x_n, x) + d(x_n, y) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, $d(x, y) = 0$. In a similar way, one can deduce that $d(y, x) = 0$. So, we obtain from Property (LD_1) that $x = y$.

Theorem 4.1. Let (X, d) be a *ld*-complete *ld*-metric space and a function $f:X \to X$ be *ld*-continuous Hegedus

contraction function such that $\forall x \in X$, x is *f*-regular. Then *f* has a fixed point.

Proof. Existence. From Lema 3.1, $(f^n(x_0))$ is a Cauchy sequence. Since (X, d) is *ld*-complete *ld*-metric then $(f^n(x_0))$ *ld*-convergent, say to *x*, *f* is *ld*-continuous and so from Lemmas 2.1 and 4.1,

$$f(x) = f\left(\lim_{n\to\infty} f^n(x_0)\right)$$

$$= \lim_{n\to\infty} f^{n+1}(x_0) \quad .$$

$$= x$$

Uniqueness. Suppose that there are two fixed points *x* and *y*. Then $f(x) = x$, $f(y) = y$ and

$$d(x, y) = d\left(f(x), (y)\right)$$

$$\leq \lambda \, \mathrm{diam}\left[O(x) \cup O(y)\right]$$

$$= \lambda \, \mathrm{diam}\left\{x, y, fx, fy, f^2x, f^2y, f^3x, f^3y, \cdots\right\}$$

$$= \mathrm{diam}\left\{x, y, x, y, x, y, x, y, \cdots\right\}$$

$$= \lambda \max\left\{d(x, y), d(y, x), d(x, x), d(y, y)\right\}.$$

If $\max\{d(x, y), d(y, x), d(x, x), d(y, y)\}$ is $d(x, y)$ or $d(y, x)$ or $d(x, x)$ or $d(y, y)$, then one can deduce that $d(x, y) = d(y, x) = d(x, x) = d(y, y) = 0$. Hence from (LD_1), $x = y$.

Definition 4.2. A right dislocated metric (simply rd-metric) is a distance *d* on a set *X* satisfying the following axioms, for all $x, y, z \in X$,

(RD_1) if $d(x, y) = d(y, x) = 0$ then $x = y$;

(RD_2) $d(x, y) \leq d(x, z) + d(y, z)$.

In a similar proof of Lemmas 4.1, we give the following lemmas:
Lemma 4.2. *rd*-limits in *rd*-metric spaces are unique.
Lemma 4.3. *d*-limits in *dq*-metric spaces are unique.
Lemma 4.4. *d*-limits in *d*-metric spaces are unique.
In a similar manner of Theorem 4.1, one can have the following theorems.

Theorem 4.2. Let (X, d) be a *rd*-complete *rd*-metric space and a function $f:X \to X$ be rd-continuous Hegedus contraction function such that $\forall x \in X$, x is *f*-regular. Then *f* has a fixed point.

Theorem 4.3. Let (X, d) be a *d*-complete *dq*-metric space and a function $f:X \to X$ be *d*-continuous Hegedus contraction function such that $\forall x \in X$, x is *f*-regular. Then *f* has a fixed point.

Theorem 4.4. Let (X, d) be a *d*-complete *d*-metric space and a function $f:X \to X$ be d-continuous Hegedus contraction function such that $\forall x \in X$, x is *f*-regular. Then *f* has a fixed point.

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