The Sound and Complete $R$-Calculi with Respect to Pseudo-Revision and Pre-Revision

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ABSTRACT

The AGM postulates ([1]) are for the belief revision (revision by a single belief), and the DP postulates ([2]) are for the iterated revision (revision by a finite sequence of beliefs). Li [3] gave an $R$-calculus for $R$-configurations $\Delta|\Gamma$, where $\Delta$ is a set of literals, and $\Gamma$ is a finite set of formulas. We shall give two $R$-calculi such that for any consistent set $\Gamma$ and finite consistent set $\Delta$ of formulas in the propositional logic, in one calculus, there is a pseudo-revision $\Theta$ of $\Gamma$ by $\Delta$ such that $\Delta|\Gamma \Rightarrow \Theta$ is provable and $\Theta \subseteq \Delta \cup \Gamma$; and in another calculus, there is a pre-revision $\Xi$ of $\Gamma$ by $\Delta$ such that $\Delta|\Gamma \Rightarrow \Xi$ is provable, $\Xi \vdash \Delta$ and $\Delta, \Theta \vdash \Xi$ for some pseudo-revision $\Theta$; and prove that the deduction systems for both the $R$-calculi are sound and complete with the pseudo-revision and the pre-revision, respectively.

Keywords: Belief Revision; $R$-Calculus; Maximal Consistent Set; Pseudo-Revision; Pre-Revision

1. Introduction

The AGM postulates ([1],[4-6]) are for the revision $K \circ \phi$ of a theory $K$ by a formula $\phi$; and the DP postulates ([2]) are for the iterated revision $\cdot (\cdot (K \circ \phi)) \circ \phi$.

The $R$-calculus ([3]) gave a Gentzen-type deduction system to deduce a consistent theory $\Gamma \cup \Delta$ from any theory $\Gamma \cup \Delta$, where $\Gamma \subseteq \Delta$ should be a maximal consistent subtheory of $\Gamma \cup \Delta$ which includes $\Delta$ as a subset, where $\Delta|\Gamma$ is an $R$-configuration, $\Gamma$ is a consistent set of formulas, and $\Delta$ is a consistent sets of literals (atomic formulas or the negation of atomic formulas). It was proved that if $\Delta|\Gamma \Rightarrow \Delta|\Gamma'$ is deducible and $\Delta|\Gamma'$ is an $R$-termination, i.e., there is no $R$-rule to reduce $\Delta|\Gamma'$ to another $R$-configuration $\Delta|\Gamma''$, then $\Delta \cup \Gamma'$ is a pseudo-revision of $\Gamma$ by $\Delta$.

The $R$-calculus has the following features:

- $\Delta$ is a finite set of literals (propositional variables or the negation of propositional variables);
- $\Gamma$ is a set of formulas;
- $R^-, R^+, R^\circ, R^\ast$ are not sufficient for pseudo-revision, and $R^\ast$ is introduced to deduce $\Delta|\Gamma'$ into a consistent set $\Theta$ of formulas including $\Delta$;
- the soundness theorem holds, that is, if $\Delta|\Gamma \Rightarrow \Theta$ is provable then $\Theta$ is a pseudo-revision of $\Gamma$ by $\Delta$; and
- the completeness theorem holds, that is, if $\Theta$ is a pseudo-revision of $\Gamma$ by $\Delta$ then $\Delta|\Gamma \Rightarrow \Theta$ is provable.

Because each rule in the $R$-calculus consists of the statements of form

$$\Delta|\phi, \Gamma \Rightarrow \Delta|\Gamma,$$

the $R$-calculus is based on pseudo-revision, i.e., to contract $\phi$ from $\Delta \cup \Gamma \cup \{\phi\}$ if $\Delta \cup \Gamma \cup \{\phi\}$ is inconsistent, which makes the $R$-calculus not preserve the minimal change principle.

Given two theories $\Delta$ and $\Gamma$, a pseudo-revision $\Theta$ of $\Gamma$ by $\Delta$ is a consistent subset of $\Delta \cup \Gamma$ including $\Delta$ (if $\Delta \cup \Gamma$ is inconsistent; otherwise, $\Theta = \Delta \cup \Gamma$).

We shall give two $R$-calculi such that

- in one $R$-calculus, say $R_1$, for any consistent formula set $\Delta$ and finite formula set $\Gamma$, there is a
consistent formula set $\Theta \subseteq \Delta \cup \Gamma$ such that $\Delta \mid \Gamma \Rightarrow \Theta$ is provable and $\Theta$ is a pseudo-revision of $\Gamma$ by $\Delta$ (the soundness theorem); and conversely, given any pseudo-revision $\Theta$ of $\Gamma$ by $\Delta, \Delta \mid \Gamma \Rightarrow \Theta$ is provable (the completeness theorem);

- in another $R$-calculus, say $R_2$, for any consistent formula set $\Delta$ and finite formula set $\Gamma$, there are consistent formula sets $\Theta$ and $\Xi$ such that
  \begin{itemize}
  \item $\Delta \mid \Gamma \Rightarrow \Xi$ is provable,
  \item $\Theta$ is a pseudo-revision of $\Gamma$ by $\Delta$,
  \item $\Xi \vdash \Theta$ and
  \item there is no subformula $\xi$ of $\Xi$ contradictory to $\Delta$ (the soundness theorem);
  \end{itemize}
and conversely, given any pseudo-revision $\Theta$ of $\Gamma$ by $\Delta$, there is a consistent formula set $\Xi$ such that $\Delta \mid \Gamma \Rightarrow \Xi$ is provable, $\Theta \vdash \Xi$ and $\Xi$ is contradictory to no subformula $\xi$ of $\Xi$ (the completeness theorem).

The $R$-calculi are different from the $R$-calculus in [3] as follows:

- $\Delta$ is any set of formulas;
- The cut-rule in the $R$-calculus is eliminated in the $R$-calculi;
- Because $(\wedge)$-rule in the $R$-calculus is not sufficient for reducing $\Delta \mid \phi_1 \land \phi_2, \Gamma \Rightarrow \Delta \mid \Gamma$

   to either $\Delta \mid \phi_1, \Gamma \Rightarrow \Delta \mid \Gamma$ or $\Delta \mid \phi_2, \Gamma \Rightarrow \Delta \mid \Gamma$ the $R$-calculus is not complete with respect to the pseudo-revision of $\Gamma$ by $\Delta$. In the new $R$-calculi, we split $(\wedge)$ into two deduction rules $(R_1^\wedge)$ and $(R_2^\wedge)$ according to whether $\phi_1$ is consistent with $\Delta \cup \Gamma$ or not. The reason is given as follows.

Given a consistent theory $\Delta$ and formulas $\phi_1, \phi_2, \Delta \cup \{\phi_1 \lor \phi_2\}$ is inconsistent if and only if $\Delta \cup \{\phi_1\}$ and $\Delta \cup \{\phi_2\}$ are inconsistent; and if either $\Delta \cup \{\phi_1\}$ or $\Delta \cup \{\phi_2\}$ is inconsistent then $\Delta \cup \{\phi_1 \land \phi_2\}$ is inconsistent; and if $\Delta \cup \{\phi_1 \land \phi_2\}$ is inconsistent then we cannot deduce that either $\Delta \cup \{\phi_1\}$ or $\Delta \cup \{\phi_2\}$ is inconsistent, and what we have is that $\Delta \cup \{\phi_1 \lor \phi_2\}$ is inconsistent if and only if $\Delta \cup \{\phi_1\}$ is inconsistent or $\Delta \cup \{\phi_2\}$ is inconsistent. Formally,

\[
\text{incon}(\Delta, \phi_1) \text{ or incon}(\Delta, \phi_2) \quad \text{incon}(\Delta, \phi_1 \land \phi_2)
\]

\[
\text{incon}(\Delta, \phi_1) \text{ incon}(\Delta, \phi_2) \quad \text{incon}(\Delta, \phi_1 \lor \phi_2)
\]

\[
\text{incon}(\Delta, \phi_1) \text{ or incon}(\Delta \cup \{\phi_1\}, \phi_2) \quad \text{incon}(\Delta, \phi_1 \land \phi_2)
\]

where $\text{con}(\Delta, \phi)$ and $\text{incon}(\Delta, \phi)$ denote that $\Delta \cup \{\phi\}$ is consistent and inconsistent, respectively. Therefore, we use

\[
\begin{align*}
(R_1^\land) & \quad \Delta \mid \phi_1, \Gamma \Rightarrow \Delta \mid \Gamma, \\
(R_2^\land) & \quad \Delta \mid \phi_1 \land \phi_2, \Gamma \Rightarrow \Delta \mid \Gamma.
\end{align*}
\]

in $R_1$ and $R_2$ instead of

\[
\begin{align*}
\Delta \mid \phi_1, \Gamma \Rightarrow \Delta \mid \Gamma, \\
\Delta \mid \phi_2, \Gamma \Rightarrow \Delta \mid \Gamma
\end{align*}
\]

in the $R$-calculus.

In $R_1$, we use a rule

\[
(R^{\lor}) \quad \Delta \cup \Gamma \not\Rightarrow \phi \\
\Delta \mid \phi, \Gamma \Rightarrow \Delta \mid \Gamma
\]

deduce $\Delta \mid \phi, \Gamma$ to $\Delta, \phi, \Gamma$ if $\Delta, \phi, \Gamma$ are consistent. In $R_2$, we shall give a deduction rule to reduce $\Delta \mid \phi, \Gamma$ to the atomic cases where

\[
\begin{align*}
\Delta \cup \Gamma \not\Rightarrow \phi \quad \Delta \cup \Gamma \not\Rightarrow \phi \\
\Delta \mid p, \Gamma \Rightarrow \Delta \mid \Gamma', \\
\Delta \mid \Gamma \not\Rightarrow \phi \quad \Delta \mid \Gamma \not\Rightarrow \phi
\end{align*}
\]

with a cost that we cannot prove that if $\Delta \mid \Gamma \Rightarrow \Xi$ is provable then $\Xi$ is a pseudo-revision of $\Gamma$ by $\Delta$. Instead we shall prove that if $\Delta \mid \Gamma \Rightarrow \Xi$ is provable then $\Xi$ is a pre-revision of $\Gamma$ by $\Delta$, that is, there is a consistent theory $\Theta \subseteq \Delta \cup \Gamma$ such that 1) $\Theta \supseteq \Delta$; 2) $\Theta \vdash \Xi$; and 3) no subformula $\xi$ of $\Xi$ is contradictory to $\Delta$.

The paper is organized as follows: the next section gives the $R$-calculus in [3] and basic definitions; the third section defines an $R$-calculus $R_1$ for the pseudo-revision and proves that $R_1$ is sound and complete with respect to the pseudo-revision; the fourth section defines another $R$-calculus $R_2$ for the pre-revision and prove that $R_2$ is sound and complete with respect to the pseudo-revision, and the last section concludes the whole paper.

### 2. The $R$-Calculus

The $R$-calculus is defined on a first-order logical language. Let $L'$ be a logical language of the first-order logic; $\phi_1, \phi_2, \phi_3$ formulas and $\Gamma, \Delta$ sets of formulas (theories), where $\Delta$ is a set of atomic formulas or the negations of atomic formulas, and $\Delta \mid \Gamma$ is called an $R$-configuration.

The $R$-calculus consists of the following axiom and inference rules:
\[
\begin{align*}
(A^-) & \quad \Delta, \phi \vdash \neg \phi, \Gamma \Rightarrow \phi, \Delta \mid \Gamma \\
(R \text{cut}) & \quad \Delta \mid \phi_1 \neg \phi_2, \Gamma \Rightarrow \phi_2
\end{align*}
\]

where \( R \) -cut abbreviates \( \neg \phi \in \phi \) means \( \phi \) occurs in the proof tree \( T \) of \( \phi \), from \( \Gamma \) and \( \phi \); and in \( R \), \( \Gamma \) is a term, and is free in \( \phi \) for \( x \).

The \( R \) -calculus is in the first-order logic. In the following we discuss the \( R \) -calculi in the propositional logic.

Let \( L \) be a logical language of the propositional logic which contains the following symbols:

- propositional variables: \( p_0, p_1, \cdots \);
- logical connectives: \( \neg, \land, \lor \).

Formulas are defined as follows:

\[
\phi = p \mid \neg p \mid \phi \land \phi_2 \mid \phi \lor \phi_2.
\]

**Definition 2.1.** Given a consistent set \( \Delta \) of formulas and a finite consistent set \( \Gamma \) of formulas, a consistent set \( \Theta \) of formulas is a pre-revision of \( \Gamma \) by \( \Delta \) if \( \Theta = \Delta \cup \Gamma \) (if \( \Delta \cup \Gamma \) is consistent), or (if \( \Delta \cup \Gamma \) is inconsistent then) \( \Theta \) satisfies the following conditions:

1) \( \Theta \subseteq \Delta \cup \Gamma \),
2) \( \Delta \subseteq \Theta \), and
3) there is a \( \phi \in \Gamma \) such that \( \Theta \cup \{ \phi \} \) is inconsistent.

Each pre-revision \( \Theta \) can be generated by the following procedure: given any consistent set \( \Delta \) and finite consistent set \( \Gamma \), assume that \( \Gamma = \{ \phi_1, \cdots, \phi_n \} \) is ordered by a linear ordering \( \leq \) (without loss of generality, assume that \( \phi_1 \leq \phi_2 \leq \cdots \leq \phi_n \)), define

\[
\Theta_0 = \Gamma \cup \Delta;
\]

\[
\Theta_i = \begin{cases} 
\Theta_{i-1} \setminus \{ \phi_i \} & \text{if } \Theta_{i-1} \vdash \neg \phi_i \\
\Theta_{i-1} & \text{otherwise}
\end{cases}
\]

Let \( \Theta = \Theta_n \). Then, \( \Theta \) is a subset of \( \Delta \cup \Gamma \) such that \( \Theta \supseteq \Delta \), and \( \Theta \) is consistent.

**Lemma 2.2.** \( \Theta \) is a pre-revision of \( \Gamma \) by \( \Delta \).

Moreover, Let \( i_0 \) be the least \( i \) such that

\[
\Theta_{i-1} \setminus \{ \phi_i \} \not\vdash \neg \phi_i.
\]

Then, \( \Theta = \Delta \cup \{ \phi_0, \phi_{i_1}, \cdots, \phi_{i_n} \} \).

**Definition 2.3.** Given a consistent set \( \Delta \) of formulas and a finite consistent set \( \Gamma \) of formulas, a consistent set \( \Xi \) of formulas is a pre-revision of \( \Gamma \) by \( \Delta \) if there is a pre-revision \( \Theta \) of \( \Delta \) by \( \Xi \) such that

1) \( \Theta \vdash \Xi \),
2) \( \Delta \subseteq \Xi \), and
3) no subformula \( \xi \) of \( \Xi \) is contradictory to \( \Delta \).

Each pre-revision \( \Xi \) can be generated by the following procedure: given any consistent set \( \Delta \) and finite consistent set \( \Gamma \), assume that \( \Gamma = \{ \phi_1, \cdots, \phi_n \} \), define

\[
\Xi_i = \begin{cases} 
\Xi_{i-1} \setminus \{ \phi_i \} \text{ \ if } \Xi_{i-1} \vdash \neg \phi_i \\
(\Xi_{i-1} \setminus \{ \phi_i \}) \cup \{ \phi_i \} & \text{otherwise}
\end{cases}
\]

where \( \lambda \) is the empty string.

Let \( \Xi = \Xi_{i_0} \), and \( \Theta \) be the pre-revision of \( \Gamma \) by \( \Delta \) in the same ordering as \( \Xi \). Then, we have the following

**Lemma 2.4.** Let \( i_0 \) be the least \( i \) such that \( \Theta_{i-1} \not\vdash \neg \phi_i \). Then, for any \( j < i_0, \Theta_j = \Xi_{i-1} \). \( \Theta_j \) is a pre-revision of \( \phi_j \), and for any \( j \geq i_0, \phi_j \) is a subformula of \( \Theta_j \).}

**Lemma 2.5.** \( \Xi \) is a pre-revision of \( \Gamma \) by \( \Delta \) such that \( \Xi \vdash \Theta \), and no subformula of \( \Xi \) is contradictory to \( \Delta \).

**Proof.** Let \( i_0 \) be the least \( i \) such that \( \Theta_{i-1} \not\vdash \neg \phi_i \). Then,

\[
\Xi = \Delta \cup \{ \phi_0, \phi_{i_1}, \cdots, \phi_{i_n} \}.
\]

We prove that for any \( i \) with \( i_0 \leq i \leq n, \Xi \vdash \Theta \), and \( \Theta \vdash \Xi_i \) by induction on \( i \).

Let \( \Omega = \Theta_{i-1} \setminus \{ \phi_0 \} \) and \( \psi = \phi_{i_0} \). Then,

\[
\Xi_{i_0} = \Omega \cup \{ \psi \}.
\]

We prove by induction on the structure of \( \psi \) that \( \Omega \vdash \psi' \) and \( \Omega \vdash \psi' \).

If \( \psi = \lambda \) and \( \Omega \vdash \lambda \), then \( \Omega \vdash \psi \), a contradiction to the choice of \( i_0 \).

If \( \psi = \lambda \) and \( \Omega \vdash \lambda \), then \( \psi' = \psi \), and
\[ \Omega, \psi \vdash \Omega, \psi'; \]

If \( \psi = \psi_1 \land \psi_2 \) and \( \Omega \cup \psi \) is consistent then \( \Omega \cup \{ \psi_1 \} \) and \( \Omega \cup \{ \psi_2 \} \) are consistent, and by the induction assumption,

\[ \Omega, \psi_1 \vdash \Omega, \psi_1'; \]
\[ \Omega, \psi_2 \vdash \Omega, \psi_2', \]

and hence,

\[ \Omega, \psi_1 \land \psi_2 \vdash \Omega, \psi_1' \land \psi_2'; \]

If \( \psi = \psi_1 \lor \psi_2 \) and \( \Omega \cup \psi \) is consistent then either \( \Omega \cup \{ \psi_1 \} \) or \( \Omega \cup \{ \psi_2 \} \) is consistent.

If \( \Omega \cup \{ \psi_1 \} \) and \( \Omega \cup \{ \psi_2 \} \) are consistent, then by the induction assumption,

\[ \Omega, \psi_1 \vdash \Omega, \psi_1'; \]
\[ \Omega, \psi_2 \vdash \Omega, \psi_2', \]

and hence, \( \Omega, \psi_1 \lor \psi_2 \vdash \Omega, \psi_1' \lor \psi_2' \);

If \( \Omega \cup \{ \psi_1 \} \) is inconsistent and \( \Omega \cup \{ \psi_2 \} \) is consistent, then \( \Omega, \psi_1 \vdash \Omega, \{ \psi_2 \} \) and by the induction assumption, \( \Omega, \psi_2 \vdash \Omega, \psi_2' \), and hence, \( \Omega, \psi_1 \lor \psi_2 \vdash \Omega, \psi_2' \), because \( \Omega, \psi_2 \vdash \psi_2' \), and \( \Omega, \psi_1 \vdash \psi_1' \) (\( \Omega \cup \{ \psi_1 \} \) is inconsistent, and hence, for any formula \( \theta, \Omega, \psi_1 \vdash \theta \)).

Similar for the case that \( \Omega \cup \{ \psi_1 \} \) is consistent and \( \Omega \cup \{ \psi_2 \} \) is inconsistent.

Similarly we can prove that for any \( i \) with \( i_0 < i \leq n, \exists, t \vdash \Theta_i \).

### 3. The \( R \)-Calculus \( R \)

In this section we give an \( R \)-calculus \( R \) which is sound and complete with respect to the pseudo-revision, where the decision of whether \( \Delta \cup \{ \phi \} \cup \Gamma \) is consistent is needed so that if \( \Delta \cup \{ \phi \} \cup \Gamma \) is consistent then \( \Delta \phi, \Gamma \vdash \Delta \phi, \Gamma \) is provable; otherwise, \( \Delta \phi, \Gamma \vdash \Delta \Gamma \) is provable.

Let \( \Delta, \Gamma \) be any consistent sets of formulas.

**Definition 3.1.** \( t = \Delta \Gamma \) is a term, and \( t \Rightarrow t' \) is a statement, where \( t = \Delta \Gamma \) and \( t' = \Delta \Gamma' \); and \( \frac{S_1, \ldots, S_n}{S} \) is a deduction rule, where \( S_1, \ldots, S_n, S \) are statements.

**\( R \)** has the following deduction rules:

\[
(\text{R}^+) \quad \frac{\Delta \cup \Gamma \vdash \neg \phi}{\Delta \phi, \Gamma \vdash \Delta \phi, \Gamma}
\]

\[
(\text{R}^-) \quad \frac{\Delta \cup \Gamma \vdash \phi}{\Delta \neg \phi, \Gamma \vdash \Delta \Gamma}
\]

\[
(\text{R}^+) \quad \frac{\Delta \cup \Gamma \vdash \neg \phi}{\Delta \neg \phi, \Gamma \vdash \Delta \Gamma}
\]

\[
(\text{R}^-) \quad \frac{\Delta \phi, \Gamma \vdash \Delta \Gamma}{\Delta \phi, \neg \phi, \Gamma \vdash \Delta \Gamma}
\]

**Definition 3.2.** \( \Delta \Gamma \vdash \Theta \) is provable if there is a sequence

\[
\{ \Delta_i | \Gamma_i \vdash \Delta_i' | \Gamma_i' \}, \ldots, \Delta_s | \Gamma_s \vdash \Delta_s' | \Gamma_s' \}
\]

of statements such that

1. \( \Delta_s | \Gamma_s = \Delta | \Gamma \);
2. \( \Delta_i, \Gamma_i = \Theta \) and
3. for each \( i \leq s \), \( \Delta_i \Gamma_i \vdash \Delta_i' \Gamma_i' \) is either an axiom or deduced from the previous statements by the deduction rules.

For example, the following

\[
(1) \quad \neg p \lor \neg q, p \vdash \neg q
\]

\[
(2) \quad \neg p \lor \neg q, p \lor q \vdash \neg p \lor \neg q, p \lor q \vdash \neg q \quad (11) \quad (R_1^+)
\]

\[
(3) \quad \neg p \lor \neg q, p \lor q \vdash \neg p \lor \neg q \quad (12, 4) \quad (R_1^-)
\]

is a proof and so \( \neg p \lor \neg q, p \lor q \vdash \neg p \lor \neg q \) is provable.

Also, the following

\[
(1) \quad \neg p \land \neg q \vdash \neg p
\]

\[
(2) \quad \neg p \land \neg q \vdash \neg p \land \neg q \quad (R_1^+)
\]

\[
(3) \quad \neg p \land \neg q \vdash \neg q
\]

\[
(4) \quad \neg p \land \neg q \vdash \neg p \land \neg q \quad (R_1^-)
\]

\[
(5) \quad \neg p \land \neg q \vdash \neg p \land \neg q \quad (2, 4) \quad (R_1^-)
\]

is a proof and so \( \neg p \land \neg q, p \lor q \vdash \neg p \land \neg q \) is provable.

**Theorem 3.3.** For any consistent sets \( \Gamma, \Delta \) of formulas and \( \phi \), if \( \Delta \phi, \Gamma \vdash \Delta \Gamma \) is provable then \( \Delta \cup \{ \phi \} \cup \Gamma \) is inconsistent; and if \( \Delta \phi, \Gamma \vdash \Delta \phi, \Gamma \) is provable then \( \Delta \cup \{ \phi \} \cup \Gamma \) is consistent.

**Proof.** If \( \Delta \phi, \Gamma \vdash \Delta \phi, \Gamma \) is provable then \( R^{\text{con}} \) is used and \( \Delta \cup \{ \phi \} \cup \Gamma \) is consistent.

If \( \Delta \phi, \Gamma \vdash \Delta \phi, \Gamma \) is provable then we prove that \( \Delta \cup \{ \phi \} \cup \Gamma \vdash \neg \phi \), i.e., \( \Delta \cup \{ \phi \} \cup \Gamma \) is inconsistent, by the induction on the length of a proof of \( \Delta \phi, \Gamma \vdash \Delta \Gamma \) and the cases that the last inference rule is used.

If the last rule used is \( R^+ \) then \( \phi = \neg p \), and \( \Delta \cup \{ \phi \} \vdash \neg p \), i.e., \( \Delta \cup \{ \phi \} \vdash \neg p \); if the last rule used is \( R^- \) then \( \phi \), i.e., \( \Delta \cup \{ \phi \} \vdash \neg \phi \); if the last rule used is \( R^+ \) then \( \phi = \phi_1 \land \phi_2 \), and \( \Delta \phi_1, \Gamma \vdash \Delta \Gamma \). By the induction assumption, \( \Delta \cup \{ \phi \} \vdash \neg \phi_1 \), and hence, \( \Delta \cup \{ \phi \} \vdash \neg \phi_1 \lor \neg \phi_2 \), i.e., \( \Delta \cup \{ \phi \} \vdash \neg \phi \); if the last rule used is \( R^- \) then \( \phi = \phi_1 \land \phi_2 \), and
Δ, φ₁ | φ₂, Γ ⇒ Δ, φ₁ | Γ. By the induction assumption, 
\[ Δ \cup \{ φ₁ \} \vdash \neg φ₂, \] and hence, \[ Δ \cup Γ \vdash \neg φ₁ \land \neg φ₂, \] i.e., \[ Δ \cup Γ \vdash \neg φ. \]

If the last rule used is \( R_1^c \) then \( φ = φ₁ \lor φ₂. \)

\[ Δ | φ₁, Γ ⇒ Δ | Γ, \]
\[ Δ | φ₂, Γ ⇒ Δ | Γ. \]

By the induction assumption, \( Δ \cup Γ \vdash \neg φ₁, \)
\[ Δ \cup Γ \vdash \neg φ₂, \] and hence, \[ Δ \cup Γ \vdash \neg φ₁ \land \neg φ₂, \] i.e., \[ Δ \cup Γ \vdash \neg φ. \]

**Theorem 3.4.** For any consistent sets \( Γ, Δ \) of formulas and formula \( φ \), if \( Δ \cup \{ φ \} \cup Γ \) is inconsistent then \( Δ | φ, Γ ⇒ Δ | Γ \) is provable; and if \( Δ \cup \{ φ \} \cup Γ \) is consistent then \( Δ | φ, Γ ⇒ Δ, φ | Γ \) is provable.

**Proof.** If \( φ \) is consistent with \( Δ \cup Γ \) then by \( (R^{inc}) \),
\[ Δ | φ, Γ ⇒ Δ | Γ \] is provable;

Assume that \( φ \) is inconsistent with \( Δ \). We prove by the induction on the structure of \( φ \) that \( Δ | φ, Γ ⇒ Δ | Γ \) is provable.

If \( φ = p \) then \( Δ \cup Γ \vdash \neg p \) and by \( (R_1^c) \), \( Δ | p, Γ ⇒ Δ | Γ \) is provable.

If \( φ = \neg p \) then \( Δ \cup Γ \vdash p \) and by \( (R_2) \), \( Δ | \neg p, Γ ⇒ Δ | Γ \) is provable.

If \( φ = φ₁ \lor φ₂ \) then there are two subcases: \( φ₁ \) is inconsistent with \( Δ \cup Γ \), or \( φ₂ \) is consistent with \( Δ \cup \{ φ₁ \} \cup Γ \).

In the first subcase, by the induction assumption, \( Δ | φ₁, Γ ⇒ Δ | Γ \) is provable, and by \( (R_1^c) \),
\[ Δ | φ₁ \lor φ₂, Γ ⇒ Δ | Γ \Rightarrow Δ | Γ \] is provable; in the second subcase, \( Δ \cup \{ φ₁ \} \cup Γ \) is consistent and \( Δ \cup \{ φ₁, φ₂ \} \cup Γ \) is inconsistent. By the induction assumption, \( Δ | φ₁ \lor φ₂, Γ ⇒ Δ, φ₁ | Γ \) is provable, and by \( (R_2) \), \( Δ | \neg φ₁ \lor \neg φ₂, Γ ⇒ Δ | Γ \Rightarrow Δ, Γ \)

If \( φ = φ₁ \land φ₂ \) then both \( Δ \cup \{ φ₁ \} \cup Γ \) and \( Δ \cup \{ φ₂ \} \cup Γ \) are inconsistent. By the induction assumption, both \( Δ | φ₁, Γ ⇒ Δ | Γ \) and \( Δ | φ₂, Γ ⇒ Δ | Γ \) are provable, and by \( (R_1^c) \), \( Δ | φ₁ \land φ₂, Γ ⇒ Δ | Γ \) is provable.

**Theorem 3.5.** For any consistent sets \( Γ, Δ \) of formulas, if \( Γ \) is finite then there is a set \( Θ \subseteq Γ \) of formulas such that \( Δ | Γ ⇒ Θ \) is provable.

**Proof.** Let \( Γ = \{ φ₁, \ldots, φₙ \} \).

We prove the theorem by the induction on \( n \).

If \( n = 1 \) then by theorem 3.3, let
\[ Θ = \{ Δ \} \text{ if } Δ \cup \{ φ₁ \} \text{ is inconsistent } \]
\[ otherwise. \]

and \( Θ \) satisfies the theorem.

Assume that the theorem holds for \( n = k \), that is, there is a set \( Θ \) such that \( Δ | Γ ⇒ Θ \) is provable. Let \( n = k + 1 \).

If \( φ₁ \vdash Θ \) then \( Δ | Γ ⇒ Θ' \) is provable, where \( Θ' = Θ \cup \{ φ₁ \} \).

If \( φ₁ \vdash Θ \) is inconsistent with \( Θ \) then
\[ Δ | Γ ⇒ Θ, \]

**Theorem 3.6.** (The soundness theorem for \( Γ \)). If \( Δ | Γ ⇒ Θ \) is provable then \( Θ \) is a pseudo-revision of \( Γ \) by \( Δ \).

**Proof.** Firstly we prove that if \( Δ | φ ⇒ Θ \) is provable then \( Θ \) is a pseudo-revision of \( φ \) by \( Δ \).

Assume that \( Δ | φ ⇒ Θ \) is provable.

If \( Θ = Δ \cup \{ φ \} \) then \( Δ \) is consistent with \( φ \), and \( Θ \) is a pseudo-revision of \( φ \) by \( Δ \).

Similarly, by the induction on the number of formulas in \( Γ \), we can prove that if \( Δ | Γ ⇒ Θ \) then \( Θ \) is a pseudo-revision of \( Γ \) by \( Δ \).

**Theorem 3.7.** (The completeness theorem for \( Γ \)). If \( Θ \) is a pseudo-revision of \( Γ \) by \( Δ \) then \( Δ | Γ ⇒ Θ \) is provable.

**Proof.** Let \( Θ \) be a pseudo-revision of \( Γ \) by \( Δ \) under the ordering \( φ₁, \ldots, φₙ \) of \( Γ \).

We prove by induction on \( i < n \) that there is a formula set \( Θ' \) such that \( Θ' | φ_i, Γ_{i+1} ⇒ Θ_{i+1} | Γ_{i+1} \) is provable, where \( Θ_0 = Δ \) and \( Γ_{i+1} = \{ φ₁, \ldots, φᵢ \} \).

If \( Θ_j | Γ_{i+1} ⇒ Θ_{i+1} | Γ_{i+1} \) is consistent then let \( Θ_{i+1} = Δ \cup \{ φᵢ \} \), and \( Θ_j | φ_i, Γ_{i+1} ⇒ Θ_{i+1} | Γ_{i+1} ⇒ Θ_{i+1} | Γ_{i+1} \) is provable, where \( Θ' = Θ_{i+1} | Γ_{i+1} \).

Assume that \( Θ_j | Γ_{i+1} ⇒ Θ' | Γ_{i+1} \) is inconsistent. Then, \( Θ_j | Γ_{i+1} ⇒ Θ_{i+1} \), and let \( Θ_{i+1} = Θ_j \), by theorem 3.4,
\[ Θ | φ_i, Γ_{i+1} ⇒ Θ_{i+1} | Γ_{i+1} \] is provable.

Let \( Θ = Θ_j \). Then, \( Δ | Γ ⇒ Θ \) is provable.

4. The \( R \)-Calculus \( R₂ \)

In this section we give an \( R \)-calculus \( R₂ \) which is sound and complete with respect to the pre-revision, where the decision of whether \( Δ \cup \{ φ \} \cup Γ \) is consistent is deduced by a set of \( λ \)-rules.

\( R₁ \) is used to reduce \( Δ | φ, Γ \) to \( Δ | Γ \) when \( Δ \cup \{ φ \} \cup Γ \) is inconsistent. When \( Δ \cup \{ φ \} \cup Γ \) is consistent, there are subformulas in \( φ \) which is inconsistent with \( Δ \), we hope to reduce those subformulas into the empty string. For example, let
\[ Δ = \{ \neg p, q \}, \]
\[ Γ = \{ p, (q \land r) \lor s \}. \]

Then, by \( R₁ \) we have the following reduction:
\[ Δ | Γ ⇒ \neg p, q \lor (q \land r) \lor s \]
\[ ⇒ \neg p, q \lor (q \land r) \lor s; \]
and by \( R₂ \) we shall have the following one:
\[ Δ | Γ ⇒ \neg p, q \lor (q \land r) \lor s \]
\[ \Rightarrow \neg p, \neg q, r \lor s. \]

For the two reductions, we have
\[ \neg p, \neg q, r \lor s \models \neg p, \neg q, (q \land r) \lor s. \]

Let \( \Delta \) be a consistent set of formulas and \( \Gamma \) a finite consistent set of formulas.

\( R_2 \) consists of two parts: \( R_1 \), which we use to decompose formula \( \phi \) in \( \Gamma \) if \( \Delta \cup \{ \phi \} \cup \Gamma \) is inconsistent; and \( \lambda \) - deduction rules, which we use to decompose \( \phi \) if \( \Delta \cup \{ \phi \} \cup \Gamma \) is consistent.

\( R_2 \) has the following \( \lambda \) - deduction rules to reduce \( \Delta \models \phi, \Gamma \) when \( \Delta \cup \{ \phi \} \cup \Gamma \) is consistent:
\[
\begin{align*}
\left( \lambda^1_{\text{con}} \right) & \quad \frac{\Delta \cup \Gamma \not \vdash \neg p}{\Delta \mid p, \Gamma \models \Delta, \{ p \}}, \\
\left( \lambda^2_{\text{con}} \right) & \quad \frac{\Delta \cup \Gamma \not \vdash p}{\Delta \mid \neg p, \Gamma \models \Delta, \{ \neg p \}}, \\
\left( \lambda_1 \right) & \quad \frac{\Delta \models \phi, \Gamma \models \Delta, \{ \phi \} \cup \Gamma}{\Delta \mid \phi, \Gamma \models \Delta, \{ \phi \} \cup \Gamma}, \\
\left( \lambda^2 \right) & \quad \frac{\Delta \models \phi, \Gamma \models \Delta, \{ \phi \} \cup \Gamma}{\Delta \mid \phi, \Gamma \models \Delta, \{ \phi \} \cup \Gamma}, \\
\left( \lambda^3 \right) & \quad \frac{\Delta \mid \phi, \Gamma \models \Delta, \{ \phi \} \cup \Gamma}{\Delta \models \phi, \Gamma \models \Delta, \{ \phi \} \cup \Gamma}, \\
\left( \lambda^4 \right) & \quad \frac{\Delta \models \phi, \Gamma \models \Delta, \{ \phi \} \cup \Gamma}{\Delta \mid \phi, \Gamma \models \Delta, \{ \phi \} \cup \Gamma},
\end{align*}
\]

where if \( \theta \) is consistent then
\[
\lambda \lor \theta = \theta \lor \lambda = \theta, \lambda \land \theta = \theta \land \lambda = \theta, \lambda, \lambda = \Delta;
\]
and if \( \theta \) is inconsistent then
\[
\lambda \lor \theta = \theta \lor \lambda = \lambda, \\
\lambda \land \theta = \theta \land \lambda = \lambda.
\]

The deductions for the inconsistent \( \Delta \cup \{ \phi \} \cup \Gamma \) are the same as in \( R_1 \) minus \( \{ R^\text{con} \} \).

**Definition 4.1.** \( \Delta \mid \Gamma \models \Sigma \) is provable if there is a sequence
\[
\{ \Delta_1 \mid \Gamma_1 \Rightarrow \Delta'_1, \Gamma'_1, \cdots, \Delta_n \mid \Gamma_n \Rightarrow \Delta'_n, \Gamma'_n \}
\]

of statements such that
1) \( \Delta_1 \mid \Gamma_1 = \Delta \mid \Gamma \);
2) \( \Delta'_n \mid \Gamma'_n = \Sigma \), and
3) for each \( i \leq n, \Delta_1 \mid \Gamma_1 \Rightarrow \Delta'_1 \mid \Gamma'_1 \) is either an axiom or deduced from the previous statements by the deduction rules.

We call the sequence a proof of statement \( \Delta \mid \Gamma \models \Sigma \).

For example, the following
\[
\begin{align*}
(1) & \quad \neg p \land \neg q \models \neg p, \\
(2) & \quad \neg p \land \neg q \mid p \Rightarrow \neg p \lor \neg q, \lambda \quad \left(1 \left( \lambda^1_{\text{con}} \right) \right), \\
(3) & \quad \neg p \land \neg q \not \models \neg r, \\
(4) & \quad \neg p \land \neg q \mid r \Rightarrow \neg p \lor \neg q, r \quad \left(3 \left( \lambda^2_{\text{con}} \right) \right), \\
(5) & \quad \neg p \land \neg q \mid p \lor r \Rightarrow \neg p \lor \neg q, \lambda \lor r \quad \left(4 \left( \lambda^3 \right) \right),
\end{align*}
\]
is a proof and \( \neg p \land \neg q \mid p \lor r \Rightarrow \neg p \land \neg q, r \) is provable.

**Theorem 4.2.** For any consistent sets \( \Gamma, \Delta \) of formulas and formula \( \phi \), if \( \Delta \models \phi, \Gamma \models \Delta, \Gamma \) is provable then \( \Delta \cup \{ \phi \} \cup \Gamma \) is inconsistent; and if there is a formula \( \theta \neq \lambda \) such that \( \Delta \models \phi, \Gamma \models \Delta, \{ \theta \} \mid \Delta \) is provable then \( \Delta \cup \{ \phi \} \cup \Gamma \) is consistent.

**Proof.** If \( \Delta \mid \phi, \Gamma \models \Delta, \Gamma \) is provable then similar to the proof of theorem 3.3, \( \Delta \cup \{ \phi \} \cup \Gamma \) is inconsistent.

Assume that there is a formula \( \theta \neq \lambda \) such that \( \Delta \models \phi, \Gamma \models \Delta, \{ \theta \} \mid \Delta \) is provable. We prove by induction on the length of a proof of \( \Delta \models \phi, \Gamma \models \Delta, \{ \theta \} \mid \Delta \) and the cases that the last inference rule is used that \( \Delta \cup \{ \phi \} \cup \Gamma \) is consistent.

If the last rule used is \( \lambda^1_{\text{con}} \) then \( \phi = p, \Delta \cup \Gamma \not \vdash \neg p \), and \( \Delta \mid p, \Gamma \models \Delta, \{ \neg p \} \mid \Delta \) is provable, where \( \theta = p \neq \lambda \).

Hence, \( \Delta \cup \Gamma \cup \{ p \} \mid \Delta \) is consistent.

If the last rule used is \( \lambda^2_{\text{con}} \) then \( \phi = p, \Delta \cup \Gamma \not \vdash p \), and \( \Delta \mid \neg p, \Gamma \models \Delta, \{ p \} \mid \Delta \) is provable, where \( \theta = \neg p = \phi \). Hence, \( \Delta \cup \Gamma \cup \{ \neg p \} \mid \Delta \) is consistent.

If the last rule used is \( \lambda^3 \) then \( \phi = \phi_1 \lor \phi_2 \), and there are formulas \( \theta_1 \lor \theta_2 \) such that
\[
\Delta \mid \phi_1, \Gamma \models \Delta, \{ \phi_1 \} \mid \Delta \}
\]
and
\[
\Delta \mid \phi_2, \Gamma \models \Delta, \{ \phi_2 \} \mid \Delta \}
\]
By the induction assumption, if \( \theta_1 \neq \lambda \) and \( \theta_2 \neq \lambda \) then \( \Delta \cup \{ \phi_1 \} \mid \Delta \cup \{ \phi_2 \} \cup \Gamma \) is consistent and \( \Delta \cup \{ \phi_1 \} \cup \{ \phi_2 \} \cup \Gamma \) is consistent, and therefore, \( \Delta \cup \{ \phi_1 \} \cup \{ \phi_2 \} \cup \Gamma \) is consistent.

If the last rule used is \( \lambda^4 \) then \( \phi = \phi_1 \lor \phi_2 \), and
\[
\Delta \mid \phi_1, \Gamma \models \Delta, \{ \phi_1 \} \mid \Delta \}
\]
and
\[
\Delta \mid \phi_2, \Gamma \models \Delta, \{ \phi_2 \} \mid \Delta \}
\]
where either \( \theta_1 \neq \lambda \) or \( \theta_2 \neq \lambda \).

If \( \theta_1 \neq \lambda \) and \( \theta_2 \neq \lambda \) then by the induction assumption, \( \Delta \cup \{ \phi_1 \} \cup \Gamma \) and \( \Delta \cup \{ \phi_2 \} \cup \Gamma \) are consistent, and so is \( \Delta \cup \{ \phi_1 \} \cup \{ \phi_2 \} \cup \Gamma \).

By the proof of the theorem, we have
\[
\begin{align*}
\lambda & \quad \text{if } \phi = \lambda \text{ and } \Delta, \Gamma \not \vdash \lambda, \\
l & \quad \text{if } \phi = \lambda \text{ and } \Delta, \Gamma \not \vdash l, \\
\theta_1 \land \theta_2 & \quad \text{if } \phi = \theta_1 \lor \phi_2 \text{ and } \text{con} \left( \Delta, \phi_1 \right), \\
\theta_1 \lor \theta_2 & \quad \text{if } \phi = \phi_1 \lor \phi_2 \text{ and } \text{con} \left( \Delta, \phi_2 \right), \\
\theta_1 & \quad \text{if } \phi = \phi_1 \lor \phi_2 \text{ and } \text{con} \left( \Delta, \phi_2 \right), \\
\theta_2 & \quad \text{if } \phi = \phi_1 \lor \phi_2 \text{ and } \text{con} \left( \Delta, \phi_1 \right), \\
t & \quad \text{incon} \left( \Delta, \phi_1 \right), \\
r & \quad \text{incon} \left( \Delta, \phi_2 \right).
\end{align*}
\]
Theorem 4.3. For any formula sets $\Gamma, \Delta$ and formula $\varphi$, if $\Gamma \cup \Delta \cup \{\varphi\}$ is consistent then $\Delta, \varphi, \Gamma \vdash \Delta, \theta, \Gamma$.

Proof. We prove the theorem by the induction on the structure of $\varphi$. Assume that $\Gamma \cup \Delta \not\vdash \varphi$.

If $\varphi = \bot$ then $\Gamma \cup \Delta \not\vdash l$, and $\theta = l$. Hence, $\Delta, \varphi, \Gamma \vdash \Delta, \theta, \Gamma$.

If $\varphi = \varphi_1 \land \varphi_2$ then $\Gamma \cup \Delta \not\vdash \varphi_1$, and $\Gamma \cup \Delta \not\vdash \varphi_2$. By the induction assumption,

$$\Delta, \varphi_1, \Gamma \vdash \Delta, \theta_1, \Gamma,$$

$$\Delta, \varphi_2, \Gamma \vdash \Delta, \theta_2, \Gamma.$$

Hence, we have

$$\Delta, \varphi_1 \land \varphi_2, \Gamma \vdash \Delta, \theta_1 \land \theta_2, \Gamma.$$

If $\varphi = \varphi_1 \lor \varphi_2$ then $\Gamma \cup \Delta \cup \{\varphi_1\}$ is consistent or $\Gamma \cup \Delta \cup \{\varphi_2\}$ is consistent. If $\Gamma \cup \Delta \cup \{\varphi_1\}$ and $\Gamma \cup \Delta \cup \{\varphi_2\}$ are consistent then $\Gamma \cup \Delta \not\vdash \varphi_1$, and $\Gamma \cup \Delta \not\vdash \varphi_2$. By the induction assumption,

$$\Delta, \varphi_1, \Gamma \vdash \Delta, \theta_1, \Gamma,$$

$$\Delta, \varphi_2, \Gamma \vdash \Delta, \theta_1, \Gamma.$$

Hence, we have

$$\Delta, \varphi_1 \lor \varphi_2, \Gamma \vdash \Delta, \theta_1 \lor \theta_2, \Gamma.$$

If $\Gamma \cup \Delta \cup \{\varphi_1\}$ is inconsistent and $\Gamma \cup \Delta \cup \{\varphi_2\}$ is consistent then $\Gamma \cup \Delta \not\vdash \varphi_2$. By the induction assumption, $\Delta, \varphi_2, \Gamma \vdash \Delta, \theta_1, \Gamma$. Hence, by Lemma 2.5, we have

$$\Delta, \varphi_1 \land \varphi_2, \Gamma \vdash \Delta, \lambda \lor \theta_2, \Gamma$$

$$\vdash \Delta, \theta_2, \Gamma$$

$$\vdash \Delta, \theta, \Gamma.$$ 

If $\Gamma \cup \Delta \cup \{\varphi_2\}$ is consistent and $\Gamma \cup \Delta \cup \{\varphi_1\}$ is inconsistent then $\Gamma \cup \Delta \not\vdash \varphi_1$. By the induction assumption, $\Delta, \varphi_1, \Gamma \vdash \Delta, \theta_1, \Gamma$. Hence, by Lemma 2.5, we have

$$\Delta, \varphi_1 \lor \varphi_2, \Gamma \vdash \Delta, \lambda \lor \theta_2, \Gamma$$

$$\vdash \Delta, \theta_1, \Gamma$$

$$\vdash \Delta, \theta, \Gamma.$$ 

Theorem 4.4. For any consistent sets $\Gamma, \Delta$ of formulas and formula $\varphi$, if $\Delta \cup \{\varphi\} \cup \Gamma$ is inconsistent then $\Delta \vdash \Delta, \theta, \Gamma$ is provable; and if $\Delta \cup \varphi \cup \Gamma$ is consistent then there is a formula $\theta \neq \lambda$ such that $\Delta, \varphi, \Gamma \vdash \Delta, \theta, \Gamma$ is provable.

Proof. If $\varphi$ is inconsistent with $\Delta \cup \Gamma$ then similar to theorem 3.5, $\Delta, \varphi, \Gamma \vdash \Delta, \theta, \Gamma$ is provable.

Assume that $\varphi$ is consistent with $\Delta \cup \Gamma$. We prove the theorem by the induction on the structure of $\varphi$.

If $\varphi = p$ then $\Delta \cup \Gamma \not\vdash p$ and by $(\lambda^x), \Delta | p, \Gamma \vdash \Delta [p], \Gamma$ is provable, where $\theta = p$.

If $\varphi = \neg p$ then $\Delta \cup \Gamma \not\vdash \neg p$ and by $(\lambda^x), \Delta | \neg p, \Gamma \vdash \Delta [p], \Gamma$ is provable, where $\theta = \neg p$.

If $\varphi = \varphi_1 \land \varphi_2$ then $\varphi_1$ is consistent with $\Delta \cup \Gamma$, and $\varphi_2$ is consistent with $\Delta \cup \{\varphi_1\} \cup \Gamma$. By the induction assumption, there are formulas $\theta_1, \theta_2$ such that $\Delta, \varphi_1, \Gamma \vdash \Delta, \theta_1, \Gamma$, and $\Delta, \varphi_2, \Gamma \vdash \Delta, \theta_2, \Gamma$ are provable. By $(\lambda^x)$, we have

$$\Delta | \varphi_1 \land \varphi_2, \Gamma \vdash \Delta, \theta_1 \land \theta_2, \Gamma$$

is provable, where $\theta = \theta_1 \land \theta_2$.

If $\varphi = \varphi_1 \lor \varphi_2$ then either $\Delta \cup \{\varphi_1\} \cup \Gamma$ or $\Delta \cup \{\varphi_2\} \cup \Gamma$ is consistent. By the induction assumption, if $\Delta \cup \{\varphi_1\} \cup \Gamma$ is consistent then there is a formula $\theta \neq \lambda$ such that $\Delta, \varphi_1, \Gamma \vdash \Delta, \theta, \Gamma$; and if $\Delta \cup \{\varphi_2\} \cup \Gamma$ is consistent then there is a formula $\theta \neq \lambda$ such that $\Delta, \varphi_2, \Gamma \vdash \Delta, \theta, \Gamma$. Then, by $(\lambda^x), \Delta | \varphi_1 \lor \varphi_2, \Gamma \vdash \Delta, \theta \lor \theta, \Gamma$ is provable, where

$$\theta \lor \theta =$$

$$\Delta, \theta_1 \lor \theta_2$$

if both $\varphi_1, \varphi_2$ are consistent with $\Delta \cup \Gamma$

$$\Delta, \theta \lor \lambda$$

if only $\varphi_1$ is consistent with $\Delta \cup \Gamma$

$$\lambda \lor \theta_2$$

if only $\varphi_2$ is consistent with $\Delta \cup \Gamma$

Remark. In fact, in theorem 4.3, if $\Delta \cup \{\varphi\} \cup \Gamma$ is consistent then there is a formula $\theta \neq \lambda$ such that $\Delta, \varphi, \Gamma \vdash \Delta, \theta, \Gamma$ is provable.

By Theorem 4.3, we have the following

Theorem 4.5. (The soundness theorem for $\Gamma$). If $\Delta | \Gamma \vdash \Xi$ is provable then $\Xi$ is a pre-revision of $\Gamma$ by $\Delta$.

Proof. We only prove that no subformula $\xi$ of $\Xi$ is contradictory to $\Delta$.

Assume that there is a subformula $\xi$ of some formula $\theta$ in $\Xi$ such that $\Delta \vdash \neg \xi$. Let $\Gamma' = \{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma$ such that $\varphi = \varphi$.

If $\Delta \cup \Gamma' \cup \{\varphi\}$ is inconsistent then $\theta = \lambda$, a contradiction.

If $\Delta \cup \Gamma' \cup \{\varphi\}$ is consistent then by Lemma 3.5,

$$\Delta, \varphi, \Gamma' \vdash \Delta, \theta, \Gamma'$$

and for any subformula $\xi$ of $\theta$, if $\Delta, \varphi, \Gamma' \not\vdash \neg \xi$ then, by the definition of $\theta$, $\xi$ is replaced by $\lambda$ in $\theta$, a contradiction to the assumption that $\xi$ is a subformula of $\theta$.

Theorem 4.6. (The completeness theorem for $\Gamma$). If $\Xi$ is a pre-revision of $\Gamma$ by $\Delta$ then $\Delta | \Gamma \vdash \Xi$ is provable.

Proof. The proof is similar to theorem 3.7 and omitted.

5. Conclusion

This paper gave two $R$-calculi which are sound and

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complete with respect to the pseudo-revision and pre-revision, respectively. The calculi are of Gentzen-type, in which each statement is of form \( \Delta | \varphi, \Gamma \Rightarrow \Delta | \Gamma' \). Different orderings of \( \Gamma \) give different results of revision \( \Delta | \Gamma \). Correspondingly, if \( \Delta | \Gamma' \) is irreducible, that is, no deduction rule can be used to reduce \( \Delta | \Gamma' \), then \( \Gamma' \) may be a minimal change of \( \Gamma \) by \( \Delta \). A further work is to give an \( R \) -calculus such that if \( \Delta | \Gamma \Rightarrow \Delta | \Gamma' \) is irreducible then \( \Delta \cup \Gamma'' \) is consistent and \( \Gamma' \) is a minimal change of \( \Gamma \) by \( \Delta \), that is, for any \( \Gamma'' \) with \( \Gamma'' \subset \Gamma'' \subseteq \Gamma, \Delta \cup \Gamma'' \) is inconsistent.

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