A Class of Constacyclic Codes over $R + vR$ and Its Gray Image

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ABSTRACT
We study $(1 + 2v)$-constacyclic codes over $R + vR$ and their Gray images, where $v^2 + v = 0$ and $R$ is a finite chain ring with maximal ideal $\langle \lambda \rangle$ and nilpotency index $e$. It is proved that the Gray map images of a $(1 + 2v)$-constacyclic codes of length $n$ over $R + vR$ are distance-invariant linear cyclic codes of length $2n$ over $R$. The generator polynomials of this kind of codes for length $n$ are determined, where $n$ is relatively prime to $p$, $p$ is the character of the field $R/\langle \lambda \rangle$. Their dual codes are also discussed.

Keywords: Frobenius Ring; Constacyclic Codes; Generator Polynomial; Dual Codes

1. Introduction
Cyclic codes are a very important class of codes, they were studied for over fifty years. After the discovery that certain good nonlinear binary codes can be constructed from cyclic codes over $Z_4$ via the Gray map, codes over finite rings have received much more attention. In particular, constacyclic codes over finite rings have been a topic of study. For example, Wolfmann [1] studied negacyclic codes over $Z_4$ of odd length and gave some important results about such negacyclic codes. Tapia-recillas and Vega generalized these results to the setting of codes over $Z_d$ in [2]. More generally, the structure of negacyclic codes of length $n$ over a finite chain ring $R$ such that the length $n$ is not divisible by the characteristic $p$ of the residue field $R/\langle \lambda \rangle$ was obtained by Dinh and López-Permouth in [3]. The situation when the code length $n$ is divisible by the characteristic $p$ of residue field of $R$ yields the so-called repeated root codes. Dinh studied the structure of $\lambda$-constacyclic codes of length $2^r$ over $Z_4$ [4] where $\lambda$ is any unit of $Z_4$ with form $4k - 1$, and established the Hamming, homogenous, Lee and Euclidean distances of all such constacyclic codes. Recently, linear codes over the ring $F_2 + uF_2 + vF_2 + uvF_2$ have been considered by Yildiz and Karadeniz in [5], where some good binary codes have been obtained as the images under two Gray maps. Some results about cyclic codes over $F_2 + vF_2$ and $F_p + vF_p$ were given by Zhu et al. in [6] and [7] respectively, where it is shown that cyclic codes over the ring are principally generated. As these two rings are not finite chain rings, some techniques used in the mentioned papers are different from those in the previous papers. It seems to be more difficult to deal with codes over these rings. In this paper, we investigate $(1 + 2v)$-constacyclic codes over $R + vR$ of length $n$ ($n$ is relatively prime to $p$, $p$ is the character of the field $R/\langle \lambda \rangle$, where $R$ is a finite chain ring with maximal ideal $\langle \lambda \rangle$ and nilpotency index $e$, and $v^2 = -v$. We define a Gray map from $R + vR$ to $R^2$ and prove that the Gray map image of $(1 + 2v)$-constacyclic codes over $R + vR$ of length $n$ is a distance invariant linear cyclic codes of length $2n$ over $R$. The generator polynomials of this kind of codes of length $n$ are determined and their dual codes are also discussed. We also prove that this class of constacyclic codes over the ring is principally generated.

2. Basic Concepts
In this section, we will review some fundamental backgrounds used in this paper. We assume the reader is familiar with standard terms from ring theory, as found in [8]. Let $R$ be a finite commutative ring with identity. A code over $R$ of length $N$ is a nonempty subset of $R^N$, and a code is linear over $R$ of length $N$ if it is an $R$-submodule of $R^N$. For some fixed unit $\mu$ of $R$, the $\mu$-constacyclic shift $\tau_\mu$ on $R^N$ is the shift $\tau_\mu(c_0, c_1, \cdots, c_{N-1}) = (c_{\mu^{-1}} c_0, c_1, \cdots, c_{N-2})$ and a linear code $C$ of length $N$ over $R$ is $\mu$-constacyclic if the code is invariant under the
\(\mu\)-constacyclic shift \(\tau_p\). Note that the \(R\)-module \(R^x\) is isomorphic to the \(R\)-module \(R[x]/(x^\mu - \mu)\). We identify a codeword \((c_0, c_1, \ldots, c_{\nu-1})\) with its polynomial representation \(c(x) = c_0 + c_1 x + \cdots + c_{\nu-1} x^{\nu-1}\). Then \(xc(x)\) corresponds to the \(\mu\)-constacyclic shift of \(c(x)\) in the ring \(R[x]/(x^\mu - \mu)\). Thus \(\mu\)-constacyclic codes of length \(N\) over \(R\) can be identified as ideals in the ring \(R[x]/(x^N - \mu)\). A code \(C\) is said to be cyclic if \(\tau_1(C) = C\), negacyclic if \(\tau_p(C) = C\), \(\mu\)-constacyclic if \(\tau_\mu(C) = C\) respectively. Let \(R\) be a finite chain ring with maximal ideal \(\langle \lambda \rangle\), \(e\) be the nilpotency index of \(\lambda\), where \(p\) is the characteristic of the residue field \(R/\langle \lambda \rangle\).

In this section, we assume \(n\) to be a positive integer which is not divisible by \(p\); that implies \(n\) is not divisible by the characteristic of the residue field \(R/\langle \lambda \rangle\), so that \(x^n - 1\) is square free in \(R[x]\). Therefore, \(x^n - 1\) has a unique decomposition as a product of irreducible pairwise coprime monic polynomials in \(R[x]\). Customarily, for a polynomial \(f\) of degree \(k\), its reciprocal polynomial \(x^k f(1/x)\) will be denoted by \(f^*(x)\). Thus, for example, if \(f(x) = a_0 + a_1 x + \cdots + a_n x^n\), then \(f^*(x) = a_n + a_{n-1} x + \cdots + a_0 x^n\). Moreover, if \(f(x)\) is a factor of \(x^n - 1\), we denote \(\tilde{f}(x) = x^n f(1/x)\), if \(f(x)\) is a factor of \(x^n + 1\), we denote \(\tilde{f}(x) = x^n f(1/x)\), if \(f(x)\) is a factor of \(x^{2n} - 1\), we denote \(\tilde{f}(x) = x^n f(1/x)\). Obviously, we have \((x^n - 1) \tilde{f}(x) = \tilde{f}(x)\), \((x^n + 1) \tilde{f}(x) = \tilde{f}(x)\), \((x^{2n} - 1) \tilde{f}(x) = \tilde{f}(x)\).

The next six lemmas are well known, proof of them can be found in [4].

**Lemma 2.1.** Let \(C\) be a cyclic code of length \(n\) over a finite chain ring \(R\) (\(R\) has maximal ideal \(\langle \lambda \rangle\) and \(e\) is the nilpotency of \(\langle \lambda \rangle)\). Then there exists a unique family of pairwise coprime monic polynomials \(F_0(x), F_1(x), \ldots, F_{n-1}(x)\) in \(R[x]\) such that

\[ F_0(x) F_1(x) \cdots F_{n-1}(x) = x^n - 1 \]

\[ C = \langle F_0(x), F_1(x), \ldots, F_{n-1}(x) \rangle \]

Moreover \(|C| = |R/\langle \lambda \rangle|^{\sum_{i=0}^{n-1} \deg(F_i)}\).

**Lemma 2.2.** Let \(C\) be a cyclic code of length \(n\) with notation as in Lemma 2.1, and \(F = F_0(x) + F_1(x) \lambda + \cdots + F_{n-1}(x) \lambda^{n-1}\). Then \(F\) is a generating polynomial of \(C\), i.e., \(C = \langle F \rangle\).

**Lemma 2.3.** Let \(C\) be a cyclic code over \(R\) with

\[ C = \langle F_0(x), F_1(x), \ldots, F_{n-1}(x) \rangle \]

where \(F_0(x) F_1(x) \cdots F_{n-1}(x) = x^n - 1\) as in Lemma 2.1, and \(F_{n-1}(x) = F_0(x)\), then

\[ C^\perp = \langle \tilde{F}_0(x), \tilde{F}_1(x), \ldots, \tilde{F}_{n-1}(x) \lambda x^{n-1} \rangle \]

and \(|C^\perp| = |R/\langle \lambda \rangle|^{\sum_{i=0}^{n-1} \deg(F_i)}\).

**Lemma 2.4.** Let \(C\) be a negacyclic code of length \(n\) over a finite chain ring \(R\) (\(R\) has maximal ideal \(\langle \lambda \rangle\) and \(e\) is the nilpotency of \(\lambda\)). Then there exists a unique family of pairwise coprime monic polynomials \(G_0(x), G_1(x), \ldots, G_{n-1}(x)\) in \(R[x]\) such that

\[ G_0(x) G_1(x) \cdots G_{n-1}(x) = x^n + 1 \]

and \(C = \langle \tilde{G}_0(x), \tilde{G}_1(x), \ldots, \tilde{G}_{n-1}(x) \lambda x^{n-1} \rangle\).

Moreover \(|C| = |R/\langle \lambda \rangle|^{\sum_{i=0}^{n-1} \deg(G_i)}\).

**Lemma 2.5.** Let \(C\) be a negacyclic code of length \(n\) with notations as in Lemma 2.6, and

\[ G_0(x) G_1(x) \cdots G_{n-1}(x) = x^n + 1 \]

Then \(G(x)\) is a generating polynomial of \(C\), i.e., \(C = \langle G \rangle\).

**Lemma 2.6.** Let \(C\) be a negacyclic code over \(R\) with

\[ C = \langle \tilde{G}_0(x), \tilde{G}_1(x), \ldots, \tilde{G}_{n-1}(x) \lambda x^{n-1} \rangle\]

where \(G_0(x) G_1(x) \cdots G_{n-1}(x) = x^n + 1\) as in Lemma 2.6 and \(G_{n-1} = G_0\), then

\[ C^\perp = \langle \tilde{G}_0(x), \tilde{G}_1(x), \ldots, \tilde{G}_{n-1}(x) \lambda x^{n-1} \rangle \]

and \(|C^\perp| = |R/\langle \lambda \rangle|^{\sum_{i=0}^{n-1} \deg(G_i)}\).

3. Graymap

Let \(R\) be the commutative ring \(R + vR = \{a + bv | a, b \in R\}\) with \(v^2 = -v\). This ring is a kind of commutative Frobenius ring with two coprime ideals \(\langle v \rangle = \{av | a \in R\}\) and \(\langle 1 + v \rangle = \{a + av | a \in R\}\). Obviously, both \(R/\langle v \rangle\) and \(R/\langle 1 + v \rangle\) is isomorphic to \(R\). By the Chinese Remainder Theorem, we have \(R \cong \langle v \rangle \oplus \langle 1 + v \rangle\).

In the rest of this paper, we denote \(R + vR\) by \(R\), where \(R\) is a finite chain ring with maximal ideal \(\langle \lambda \rangle\), the nilpotency index of \(\langle \lambda \rangle\) is \(e\), the character of the residue field \(R/\langle \lambda \rangle\) is \(p\), a prime odd.

We first give the definition of the Gray map on \(R\). Let \(a + bv\) be an element in \(R\), where \(a, b \in R\). The Gray map \(\phi : R \to R^2\) is given by \(\phi(a + bv) = (ta + sb, (t + 2s)a - sb)\) where \(s, t \in R\).

**Lemma 3.1.** The Gray map is bijection. If \(s(t + s)\) is a unit in \(R\).

**Proof.** Since \(s(t + s)\) is a unit in \(R\), we can define a map \(\phi : R^2 \to R\) by

\[ \phi(x, y) = (2s(t + s))^{-1}(sx + sy + ((t + 2s)x - ty)v) \]
then for any \( c = a + bv \in R \), we have
\[
\phi_{s,t}(c) = \phi_{s,t}(a + bv)
\]
\[
= \phi_{s,t}(a + b(s + 2a)x - s(b + 2b)v)
\]
\[
= (2s + t + s)(a + b(s + 2a)x - s(b + 2b)v)
\]
\[
= a + bv = c
\]

This means that the \( c \) can be recovered from \( \phi_{s,t}(c) \) by the map \( \phi_{s,t} \), hence the Gray map \( \phi_{s,t} \) is bijective.

The Gray map can be extended to \( R^n \) in a natural way: \( \phi_{s,t} : R^n \to R^{2n} \)
\[
(c_0, c_1, \ldots, c_{n-1}) \mapsto (a_0 + b(s + 2a)x_0, \ldots, a_{n-1} + b(s + 2a)x_{n-1},
\]
\[
(t + 2s)a_0 - sb_0, \ldots, (t + 2s)a_{n-1} - sb_{n-1})
\]

It is obvious that for any \( k, l \in R, x, y \in R^n \), we have
\[
\phi_{s,t}(kx + ly) = \phi_{s,t}(k, l)(\phi_{s,t}(x) + \phi_{s,t}(y))
\]
which means the Gray map \( \phi_{s,t} \) is R-linear.

**Lemma 3.2.** Let \( \tau \) denote the \((1 + 2v)\)-constacyclic shift of \( R^n \) and \( \sigma \) denote the cyclic shift of \( R^{2n} \). Let \( \phi_{s,t} \) be the Gray map of \( R^n \to R^{2n} \), then \( \phi_{s,t} \tau = \sigma \phi_{s,t} \).

**Proof.** Let \( c = (c_0, c_1, \ldots, c_{n-1}) \in R^n \), where \( c_i = a_i + b_i \) with \( a_i, b_i \in R \) for \( 0 \leq i \leq n-1 \). From the definition of the Gray map, we have
\[
\phi_{s,t}(c) = (a_0 + b(s + 2a)x_0, \ldots, a_{n-1} + b(s + 2a)x_{n-1},
\]
\[
(t + 2s)a_0 - sb_0, \ldots, (t + 2s)a_{n-1} - sb_{n-1})
\]

hence,
\[
\sigma(\phi_{s,t}(c)) = ((t + 2s)a_0 - sb_0, a_0, \ldots, a_{n-1} + b(s + 2a)x_{n-1},
\]
\[
(t + 2s)a_{n-1} - sb_{n-1})
\]

On the other hand,
\[
\tau(c) = ((1 + 2v)c_0, c_1, \ldots, c_{n-2})
\]
\[
= (a_0 + (2a_{n-1} - b_1)v, a_0, a_1 + b_2v, \ldots, a_{n-2} + b_{n-2}v)
\]

We can deduce that
\[
\phi_{s,t}(\tau(c)) = ((t + 2s)a_{n-1} - sb_{n-1}, a_0 + b(s + 2a)x_0, \ldots,
\]
\[
(t + 2s)a_{n-1} - sb_{n-1})
\]

Therefore, \( \phi_{s,t} \tau = \sigma \phi_{s,t} \).

**Theorem 3.1.** A linear code \( C \) of length \( n \) over \( R \) is a \((1 + 2v)\)-constacyclic code if and only if \( \phi_{s,t}(C) \) is a cyclic code of length \( 2n \) over \( R \).

**Proof.** It is an immediately consequence of Lemma 3.2.

Now we define a Gray weight for codes over \( R \) as follows.

**Definition 3.1.** The gray weight on \( R \) is a weight function on \( R \) defined as
\[
W_{s,t} : R \to N
\]
\[
W_{s,t}(a + bv) =
\]
\[
0: \text{ if } a = 0, b = 0
\]
\[
1: \text{ if } ta + sb = 0, (t + 2s)a - sb \neq 0
\]
\[
2: \text{ if } ta + sb \neq 0, (t + 2s)a - sb = 0
\]

Define the gray weight of a codeword \( c = (c_0, c_1, \ldots, c_{n-1}) \in R^n \) to be the rational sum of the Gray weights of its components, i.e.
\[
W_{s,t}(c) = \sum_{i=0}^{n-1} W_{s,t}(c_i).
\]
The Gray distance \( d_{s,t} \) is given by \( d_{s,t}(c_1, c_2) = W_{s,t}(c_1 - c_2) \). The minimum Gray distance of \( C \) is the smallest nonzero Gray distance between all pairs of distinct codeword of \( C \). The minimum Gray weight of \( C \) is the smallest nonzero Gray weight among all codeword of \( C \). If \( C \) is linear, the minimum Gray distance of \( C \) is the same as the minimum Gray weight of \( C \). The Hamming weight \( W(c) \) of a codeword \( c \) is the number of nonzero components in \( c \). The Hamming distance \( d(C) \) of two codewords \( c_1 \) and \( c_2 \) is the Hamming weight of the codeword \( c_1 - c_2 \). The minimum Hamming distance \( \lambda(C) \) of \( C \) is defined as min \( \{d(c_1, c_2) \mid c_1, c_2 \in C, c_1 \neq c_2\} \) (cf [7]). It is obviously that for any codeword \( c \) of \( C \), we have \( W_{s,t}(c) = W(\phi_{s,t}(c)) \).

**Lemma 3.3.** The Gray map \( \phi_{s,t} \) is a distance-preserving map from \( (R^n, \text{ Gray distance}) \) to \( (R^{2n}, \text{ Hamming distance}) \).

**Proof.** Let \( x = a_0 + b_1v, y = a_1 + b_2v \). From the definition of \( \phi_{s,t} \), we have
\[
\phi_{s,t}(x - y) = \phi_{s,t}((a_0 - a_1) + (b_1 - b_2)v)
\]
\[
= (a_0 - a_1 + (2a_0 - b_1)v, a_0, a_1 + b_2v, \ldots, a_{n-1} + b_{n-1}v)
\]
\[
= ((t + 2s)a_{n-1} - sb_{n-1}, a_0 + b(s + 2a)x_0, \ldots,
\]
\[
(t + 2s)a_{n-1} - sb_{n-1})
\]

for any \( x, y \in R^n \). Then
\[
d_{s,t}(x, y) = W_{s,t}(x - y) = W(\phi_{s,t}(x - y))
\]
\[
= W(\phi_{s,t}(x) - \phi_{s,t}(y)) = d(\phi_{s,t}(x), \phi_{s,t}(y)).
\]

**Corollary 3.1.** The Gray images of a \((1 + 2v)\)-constacyclic code of length \( n \) over \( R \) under the Gray map is a distance invariant linear cyclic code of length \( 2n \) over \( R \).

4. \((1 + 2v)\)-Constacyclic Codes of Length \( n \) over \( R \) and Their Gray Images

In this section, we study \((1 + 2v)\)-constacyclic codes of length \( n \) over \( R \) and their Gray images, where \( n \) is a
Lemma 4.1. ([8], Theorem 1.3). Let $m_1,m_2,...,m_n$ be ideals of a ring $R$. The following are equivalent:

1) For $i \neq j$, $m_i$ and $m_j$ are relatively prime;
2) The canonical homomorphism $R \rightarrow \oplus_{i=1}^n R/m_i$ is surjective.

Let $m = \cap_{i=1}^n m_i$, then the canonical homomorphism $R/m \rightarrow \oplus_{i=1}^n R/m_i$ is bijective.

A finite family $(a_i)_{i=1}^n$ of ideals of a commutative ring, such that the canonical homomorphism of $R$ to $\oplus_{i=1}^n R/a_i$ is an isomorphism is called a direct decomposition of $R$. The next lemma is well-known.

Lemma 4.2. Let $R$ be a commutative ring, $(a_i)_{i=1}^k$ a direct decomposition of $R$ and $M$ an $R$-module. With the notation we have:

1) There exists a family $(e_i)_{i=1}^k$ of idempotents of $R$ such that $e_ie_j = 0$ for $i \neq j$, $\sum e_i = 1$ and $a = R(1-e_i)$ for $i = 1, 2, \ldots, k$.

2) For $i = 1, 2, \ldots, k$, the submodule $M_i = e_iM$ is a complement in $M$ of the submodule $a_iM = (1-e_i)M$ so the $R/a_i$-modules $M_i$ and $M/a_iM$ are isomorphic via the map

$$\psi_i : M_i \rightarrow M/a_iM, x \mapsto x + a_iM.$$ 

3) Every submodule $N$ of $M$ is an internal direct sum $M_i \oplus N_i$ of submodules of $N_i = e_iN \subseteq M_i$, which are isomorphic to $N_i = e_iN$ by the map $\psi_i$.

Let $N = \{v_1, v_2, \ldots, v_n\}$ be a $(1 + 2v)$-constacyclic codes of length $n$ over $\mathbb{F}$, then there are polynomials $F_i, G_i, \ldots, G_i$ over $\mathbb{F}$, $G_i = G_i + \lambda G_{i+1} + \cdots + \lambda^{i-1} G_i$, where $F_i, G_i, \ldots, G_i$ are pairwise coprime monic polynomials over $R$, such that $F_iF_{i+1} \cdots F_n = F_i^rG_i + \cdots + \lambda^{i-1} F_iG_i$.

Proof. Since $C$ is a $(1 + 2v)$-constacyclic codes of length $n$ over $\mathbb{F}$, then by Theorem 4.1, $C_i$ and $C_{i+1}$ are negacyclic and cyclic codes of length $n$ over $\mathbb{F}$, respectively, then by Lemma 2.2 and Lemma 2.5, there are polynomials $F_iF_{i+1} \cdots F_n = x^r + 1$, $G_iG_{i+1} \cdots G_n = x^r + 1$. For any $c \in C_i$, then $c \in C_i, c_1, c_{i+1}$, there are $k_1(x), k_2(x) \in R[x]$ such that $c_i(x) = k_1(x)G_i(x) \mod (x^r + 1)$, $c_{i+1}(x) = k_2(x)F_i(x) \mod (x^r + 1)$, that means, there are $r_1(x), r_2(x) \in R[x]$ such that

$$c_i(x) = k_1(x)G_i(x) + r_1(x)(x^r + 1)$$

and

$$c_{i+1}(x) = k_2(x)F_i(x) + r_2(x)(x^r + 1).$$
hence \( c(x) = vk_i(x)G(x)+(1+v)k_2F(x) \mod (x^n-(1+2v)) \). So \( c(x) \in \langle vG,(1+v)F \rangle \subseteq R[x]/(x^n-(1+2v)) \).

On the other hand, For any \( d(x) \in \langle vG,(1+v)F \rangle \subseteq R[x]/(x^n-(1+2v)) \), then there are polynomials \( k_1(x),k_2(x) \in R[x] \) such that \( d(x) = k_1(x)G(x)+k_2(x)F(x) \) mod \( (x^n-(1+2v)) \) then there are \( r_1(x),r_2(x) \in R[x] \) such that \( vk_1(x) = vr_1(x) \), \( (1+v)k_2(x) = (1+v)r_2(x) \), and there is \( r(x) = vr_1(x)+(1+v)r_2(x) \) such that

\[
d(x) = vd_r(x)+(1+v)dr_v(x)
\]

\[
= G(x)r_1(x)+(1+v)F(x)r_2(x)
+ r(x)(x^n-(1+2v)),
\]

then

\[
v = G(x)r_1(x)+(1+v)F(x)r_2(x) + r(x)(x^n-(1+2v))
\]

This means \( d_r(x) \in G \subseteq R[x]/(x^n+1) \), and \( d_{r_v}(x) \in F \subseteq R[x]/(x^n+1) \), hence

\[
d_r(x) \in G, d_{r_v}(x) \in G
\]

then \( d \in G \), so \( \langle vG,(1+v)F \rangle \subseteq G \). This gives that \( C = \langle vG,(1+v)F \rangle \).

From Lemma 2.1, 2.4, and the proof of Theorem 4.2, we immediately obtain the following result.

**Corollary 4.1.** Let \( C = \langle vG,(1+v)F \rangle \) be a \((1+2v)\)-constacyclic codes of length \( n \) over \( R \), then

\[
|C| = |R/(\lambda^{(e-v)(deg G_{r_v}+deg F_{r_v}))}|
\]

**Theorem 4.3.** Let \( C \) be a \((1+2v)\)-constacyclic code of length \( n \) over \( R \), then there is a polynomial \( f(x) \) over \( R \) such that \( C = \langle f(x) \rangle \).

**Proof.** By Theorem 4.2, there are polynomials

\[
F = F_1 + \lambda F_2 + \cdots + \lambda^{e-1}F_e
\]

and \( G = G_1 + \lambda G_2 + \cdots + \lambda^{e-1}G_e \) over \( R \) such that \( C = \langle vG,(1+v)F \rangle \), where \( F_0,F_1,\cdots,F_e, G_0,G_1,\cdots,G_e \) are pairwise coprime monic polynomials over \( R \), such that \( F_0F_1\cdots F_e = x^n-1 \), \( G_0G_1\cdots G_e = x^n+1 \).

Let \( f(x) = vG(x)+(1+v)F(x) \), obviously, \( \langle f(x) \rangle \subseteq C \).

Note that \( vF(x) = -vG(x) \), \( (1+v)f(x) = (1+v)F(x) \), then hence \( C = \langle f(x) \rangle \).

We now give the definition of polynomial Gray map over \( R \). For any polynomial \( c(x) \in R[x] \) with degree less then \( n \) can be represented as \( c(x) + vb(x) \), where \( a(x),b(x) \in R[x] \) and their degrees are less than \( n \). Define the polynomial Gray map as follows:

\[
\varphi_{r,s,j}(x) : R[x]/(x^n-(1+2v)) \rightarrow R[x]/(x^{2n}-1)
\]

\[
\varphi_{r,s,j}(c(x)) = ta(x) + sb(x) + x^n((1+2s)a(x) - sb(x))
\]

It is obviously that \( \varphi_{r,s,j}(c(x)) \) is the polynomial representation of \( \varphi_{r,s,j}(c) \).

**Theorem 4.4.** Let \( C = \langle vG,(1+v)F \rangle \) be a \((1+2v)\)-constacyclic code of length \( n \) over \( R \), where

\[
F = F_1 + \lambda F_2 + \cdots + \lambda^{e-1}F_e
\]

and

\[
G = G_1 + \lambda G_2 + \cdots + \lambda^{e-1}G_e
\]

are polynomials over \( R \), \( F_0,F_1,\cdots,F_e,G_0,G_1,\cdots,G_e \) are pairwise coprime monic polynomials over \( R \), such that \( F_0F_1\cdots F_e = x^n-1 \), \( G_0G_1\cdots G_e = x^n+1 \).

If \( s(t+s) \neq 0 \), then \( \varphi_{r,s,j}(C) = \langle g(x) \rangle \), where

\[
g(x) = G_1 + \lambda G_2 + \cdots + \lambda^{e-1}G_e
\]

**Proof.** By Lemma 4.3, we know that \( C = \langle f(x) \rangle \), where \( f(x) = vG(x)+(1+v)F(x) \). Let \( c(x) = f(x)r(x) \) be any element in \( C \), where \( r(x) \) can be written as \( r(x) = vr_1(x)+(1+v)r_2(x) \). \( r_1(x),r_2(x) \in R[x] \), it is obviously that \( c(x) = vG(x)r_1(x)+(1+v)F(x)r_2(x) \). Then we have

\[
\varphi_{r,s,j}(c(x)) =

\[
tl(x)r_2(x) + s(G(x)r_1(x) + F(x)r_2(x))
\]

\[
= t(x)r_2(x) \left( F_1 + \lambda F_2 + \cdots + \lambda^{e-1}F_e \right) (x^n+1)
\]

\[
- sG(x)(G_1 + \lambda G_2 + \cdots + \lambda^{e-1}G_e)(x^n-1)
\]

\[
= tl(x) \left( G_1 + \lambda G_2 + \cdots + \lambda^{e-1}G_e \right)
\]

\[
\left( G_1 + \lambda G_2 + \cdots + \lambda^{e-1}G_e \right) = \langle g(x) \rangle.
\]

On the other hand, by Lemma 2.1, Lemma 2.5, Lemma 3.1 and Corollary 4.1, we know that

\[
|\varphi_{r,s,j}(C)| = |C| = |R/(\lambda)^{(e-v)(deg G_{r_v}+deg F_{r_v})}|
\]
\[ \|g(x)\| = |R/\langle \lambda \rangle|^{\sum_{i=0}^{t}(\deg G_{i} + \deg F_{i})} \]

Hence,
\[ \varphi_{F_{i},2}(C) = \{g(x)\} \]

We now study the dual codes of a \((1 + 2v)\)-constacyclic code of length \(n\) over \(R\).

Since \((1 + 2v)^{2} = 1\), then the dual of a \((1 + 2v)\)-constacyclic code is also a \((1 + 2v)\)-constacyclic code. We have following result similar to Theorem 3.2 in [7].

**Theorem 4.5.** Assume the notation as Theorem 4.1. Let \(C\) be a \((1 + 2v)\)-constacyclic code of length \(n\) over \(R\), then \(C^{\perp} = vC^{\perp}_{e} + (1 + v)C^{\perp}_{e} \).

By Theorem 4.5, Lemma 2.3 and Lemma 2.6, It is obviously that the above results of \((1 + 2v)\)-constacyclic code can be carried over respectively to their dual codes. We list them here for the sake of completeness.

**Corollary 4.2.** Let \(C = \langle vF(x), (1 + v)F(x) \rangle\) be a \((1 + 2v)\)-constacyclic codes of length \(n\) over \(R\), and \(F(x), G(x)\) are generator polynomials of \(C_{e}\) and \(C_{e}^{\perp}\) respectively. Where \(F = F_{0} + \lambda F_{1} + \cdots + \lambda^{e-1}F_{e}\) and \(G = G_{0} + \lambda G_{1} + \cdots + \lambda^{e-1}G_{e}\) are polynomials over \(R\), \(F_{0}, F_{1}, \ldots, F_{e}, G_{0}, G_{1}, \ldots, G_{e}\) are pairwise coprime monic polynomials over \(R\), such that \(F_{0}F_{1}\cdots F_{e} = x^{e} - 1\), \(G_{0}G_{1}\cdots G_{e} = x^{e} + 1\).

Let
\[ h_{0}^{*} = F_{0}^{*} + \lambda F_{1}^{*} + \cdots + \lambda^{e-1}F_{e}^{*}, \]
\[ h_{e}^{*} = G_{0}^{*} + \lambda G_{1}^{*} + \cdots + \lambda^{e-1}G_{e}^{*}, \]
\[ g^{*} = F_{0}^{*}G_{0}^{*} + \lambda F_{1}^{*}G_{1}^{*} + \cdots + \lambda^{e-1}F_{e}^{*}G_{e}^{*}. \]

Then
1) \(C^{\perp} = \langle vh_{0}^{*}(x),(1 + v)h_{e}^{*}(x) \rangle \).
2) \(C^{\perp} = \langle h(x) \rangle\) where \(h(x) = vh_{0}^{*}(x) + (1 + v)h_{e}^{*}(x)\).
3) \(\varphi(C^{\perp}) = \{g^{*}(x)\} \).
4) \(\varphi(C^{\perp}) = (\varphi(C))^{\perp} \).

### 5. Conclusion

In this paper, we establish the structure of \((1 + 2v)\)-constacyclic codes of length \(n\) over \(R\) and classified Gray maps from \((1 + 2v)\)-constacyclic codes of length \(n\) over \(R\) to \(R^{2}\), prove that the image of a \((1 + 2v)\)-constacyclic codes of length \(n\) over \(R + vR\) under the Gray map is a distance-invariant linear cyclic code of length \(2n\) over \(R\), where \(R\) is a finite chain ring. The generator polynomial of this kind of codes of length \(n\) are determined and their dual codes are also discussed.

### REFERENCES


