International Journal of Astronomy and Astrophysics, 2013, 3, 29-33
http://dx.doi.org/10.4236/ijaa.2013.31004 Published Online March 2013 (http://www.scirp.org/journal/ijaa)

Scientific Research

Approximate Kepler's Elliptic Orbits with the Relativistic Effects

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Received December 10, 2012; revised January 13, 2013; accepted January 21, 2013

ABSTRACT

Beginning with a Lagrangian, we derived an approximate relativistic orbit equation which describes relativistic corrections to Keplerian orbits. The critical angular moment to guarantee the existence of periodic orbits is determined. An approximate relativistic Kepler's elliptic orbit is illustrated by numerical simulation via a second-order perturbation method of averaging.

Keywords: Kepler's Elliptic Orbits; The Relativistic Kepler Problem; Unboundedness; Averaging

1. Introduction

Kepler problem is one of the fundamental problems of orbital mechanics [1,2], which has been studied widely [3-5]. It is regarded as a special case of two-body problems [6], where one body is assumed to be fixed at the origin-say, for example, it is so massive, like the Sun, that to the first approximation it does not move. The Kepler's elliptic orbit is a conic section of the Kepler's equation in polar coordinates with the form

$$r = \frac{r_c}{1 + e\cos f}, (0 < e < 1), \tag{1}$$

where e is the eccentricity and the angle f is often called the true anomaly. Such elliptic orbits are of importance on describing dynamics of orbital mechanics in celestial mechanics and astrophysics.

When dealing with particles moving at speed close to that of light it may be important to take into account the relativistic effects [7-12]. There have been several attempts to obtain the orbit solution for a classical relativistic two-body system interacting electromagnetically, and the concentric circular motion of two classical relativistic point charges interacting electromagnetically had been described [13-16]. In this paper, using a perturbation techniques of averaging we will give the approximate Kepler's ellipse orbits for the Kepler problem with the special relativistic effects. In our results, we will show that once the relativistic contribution to Kepler problem is considered, the Kepler's ellipse orbit may be destroyed. However, they perhaps maintain the original characteristics for a long time.

The paper is organized as follows. Firstly, the Lagrangian equations of motion of the relativistic Kepler problem are deduced, and the elliptic periodic orbits and unbounded orbits of equations are determined. Secondly, by the near-identity transformation, a good approximation of the Kepler's elliptic orbits is obtained via averaging of the angle. An example is given to illustrate the application of the result. Finally, we conclude our results.

2. Periodic and Unbounded Orbits of the Relativistic Kepler Problem

Under relativistic effects, a particle of mass m orbiting a central mass M is commonly described by the Lagrangian in the polar coordinates [17-21]

$$L = -mc\sqrt{c^2 - \dot{r}^2 - r^2\dot{\theta}^2} - \frac{GMm}{r} \tag{2}$$

where G is Newton's universal gravitational constant and c is the speed of light in vacuum. Then the Lagrangian equations of motion are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{r}}{\sqrt{1-\left(\dot{r}^2 - r^2\dot{\theta}^2\right)/c^2}} = \frac{r^2\dot{\theta}}{\sqrt{1-\left(\dot{r}^2 - r^2\dot{\theta}^2\right)/c^2}}, \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\frac{r^2\dot{\theta}}{\sqrt{1-\left(\dot{r}^2 - r^2\dot{\theta}^2\right)/c^2}} = 0. \tag{4}$$

At this moment it is convenient to introduce the relativistic linear momentum p [22]

$$p = \frac{\dot{r}}{\sqrt{1-\left(\dot{r}^2 - r^2\dot{\theta}^2\right)/c^2}} \tag{5}$$

and Equation (4) implies the conservation of the relativistic angle momentum μ, an arbitrary constant of integration,

$$\mu = \frac{r^2\dot{\theta}}{\sqrt{1-\left(\dot{r}^2 - r^2\dot{\theta}^2\right)/c^2}}. \tag{6}$$

By a simply algebraic computation, we have

$$\begin{cases} \dot{r} = \dfrac{crp}{\sqrt{c^2r^2 + p^2r^2 + \mu^2}}, \\ \dot{\theta} = \dfrac{c\mu}{r\sqrt{c^2r^2 + p^2r^2 + \mu^2}}. \end{cases} \tag{7}$$

Substituting (7) into (3) and together with (5), it yields that

$$\begin{cases} \dot{r} = \dfrac{crp}{\sqrt{c^2r^2 + p^2r^2 + \mu^2}}, \\ \dot{p} = \dfrac{c\mu^2}{r^2\sqrt{c^2r^2 + p^2r^2 + \mu^2}} - \dfrac{GM}{r^2}. \end{cases} \tag{8}$$

Note that Equation (8) have periodic orbits if and only if the relativistic angle momentum μ is large enough, precisely, $\mu > \mu_c := GM/c$. In fact, the derivation of the relativistic linear momentum p always be negative when the opposite direction of inequality holds, since

$$\dot{p} = \frac{1}{r^2}\left[\frac{c\mu}{\sqrt{1+\left(c^2r^2 + p^2r^2\right)/\mu^2}} - GM\right] < \frac{1}{r^2}\left(\mu - GM\right).$$

For example, the mass of Sun, the Newton's universal gravitational constant and the speed of the light are taken to be $M = 1.989\times10^{30}\,\mathrm{kg}, G = 6.670\times10^{-11}\,\mathrm{m}^3\cdot\mathrm{kg}^{-1}\cdot\mathrm{s}^{-2}$ and $c = 2.998\times10^{8}\,\mathrm{m}\cdot\mathrm{s}^{-1}$, respectively, then the periodic orbits exist only for $\mu > 4.425\times10^{11}\,\mathrm{s}^{-1}$.

Since the change of the polar coordinates preserve the symplectic form, Equation (8) retains the Hamilton structure with the Hamiltonian

$$H\left(r,p\right) = \frac{c}{r}\sqrt{c^2r^2 + p^2r^2 + \mu^2} - \frac{GM}{r}. \tag{9}$$

The curve of level set with the Hamiltonian

$$H\left(r,p\right) = h, h \in \left[c^2\sqrt{1 - G^2M^2/c^2\mu^2}, +\infty\right)$$

defines the "energy" of the Kepler system (8). When

$$h = c^2\sqrt{1 - G^2M^2/c^2\mu^2},$$

the curve of level set reduces to an elliptic equilibrium point

$$\left(r^*, p^*\right) = \left(\frac{\mu}{c}\sqrt{1 - \frac{G^2M^2}{c^2\mu^2}}, 0\right)$$

of Equations (8), which is corresponding to the circle of Keplerian orbits of the form (1) with $e = 0$. At the same time, every curve of level set with

$$h \in \left[c^2\sqrt{1 - G^2M^2/c^2\mu^2}, c^2\right)$$

is corresponding to a periodic orbit of Equation (8). In case of $h \in \left[c^2, +\infty\right)$, the orbits become unbounded and insect the r-axis only one time. The orbits in the phase plane $\left(r,p\right)$ for Equation (8) are depicted in **Figure 1** using the parameters mentioned above.

3. Approximate Kepler's Elliptic Orbits

In previous section, we find that the large relativistic angle momentum μ is necessary and sufficient to guarantee the existence of the periodic orbits. At the same time, μ also as a constant of integration can be taken arbitrarily large. Consequently, in this section we will assume that μ is so large that

$$\varepsilon = \frac{G^2M^2}{c^2\mu^2} \ll 1.$$

In the following, with this assumption by the method of averaging, we will show that for a long time the orbit on the $\left(r,\theta\right)$ plane is an approximate Kepler elliptic orbits. The averaged method has been used widely [23-26].

Together with (7) and (8), by successive applications of the chain rule, we get

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}t}\Big/\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{pr^2}{\mu} \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\mu}{r} = -p, \tag{10}$$

$$\frac{\mathrm{d}p}{\mathrm{d}\theta} = \frac{\mathrm{d}p}{\mathrm{d}t}\Big/\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{r}{\mu} - \frac{GM}{\mu}\sqrt{1 + \frac{p^2}{c^2} + \frac{1}{c^2}\left(\frac{\mu}{r}\right)^2}. \tag{11}$$

So it follows that

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\frac{r_c}{r} + \frac{r_c}{r} = \sqrt{1 + \varepsilon\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{r_c}{r}\right)^2 + \left(\frac{r_c}{r}\right)^2}, \tag{12}$$

where $r_c = \mu^2/GM$.

Let

$$\rho = r_c/r - 1, \rho \in \left(-\infty, +\infty\right),$$

then we obtain that

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}\theta^2} + \rho = \sqrt{1 + \varepsilon\left[\dot{\rho}^2 + \left(\rho + 1\right)^2\right]} - 1. \tag{13}$$

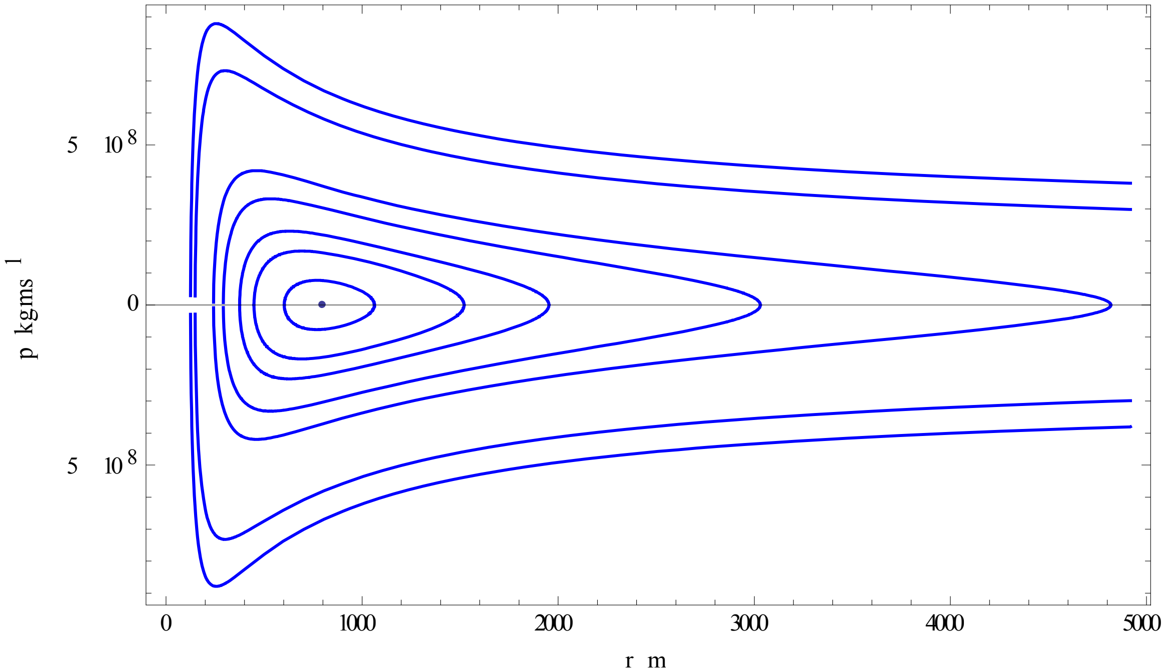

Figure 1. Periodic and unbounded orbits in the plane for the relativistic Kepler problem.

Let

$$\rho = J\sin(\theta-\phi), \dot{\rho} = J\cos(\theta-\phi). \tag{14}$$

The perturbed Equation (13) becomes

$$\begin{aligned}\begin{bmatrix}\dot{J}\\ \dot{\phi}\end{bmatrix} &= \begin{bmatrix}\cos(\theta-\phi)\\ \frac{1}{J}\sin(\theta-\phi)\end{bmatrix}\times\left(\sqrt{1+\varepsilon\left[\left(J\cos(\theta-\phi)\right)^2+\left(J\sin(\theta-\phi)+1\right)^2\right]}-1\right)\\ &= \varepsilon\begin{bmatrix}\frac{1}{2}\cos(\theta-\phi)\left[J^2\cos^2(\theta-\phi)+1+J^2\sin^2(\theta-\phi)+2J\sin(\theta-\phi)\right]\\ \frac{1}{2J}\sin(\theta-\phi)\left[J^2\cos^2(\theta-\phi)+1+J^2\sin^2(\theta-\phi)+2J\sin(\theta-\phi)\right]\end{bmatrix}+O\left(\varepsilon^2\right).\end{aligned} \tag{15}$$

By the near-identity transformation

$$\begin{bmatrix}J\\ \phi\end{bmatrix} = \begin{bmatrix}\bar{J}\\ \bar{\phi}\end{bmatrix}+\varepsilon\begin{bmatrix}U^{[1]}\left(\bar{J},\bar{\phi},\theta\right)\\ U^{[2]}\left(\bar{J},\bar{\phi},\theta\right)\end{bmatrix} \tag{16}$$

the system (15) transforms into

$$\dot{\bar{J}} = 0+O\left(\varepsilon^2\right), \dot{\bar{\phi}} = \frac{1}{2}\varepsilon+O\left(\varepsilon^2\right), \tag{17}$$

where

$$\begin{bmatrix}U^{[1]}\left(\bar{J},\bar{\phi},\theta\right)\\ U^{[2]}\left(\bar{J},\bar{\phi},\theta\right)\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\sin\left(\theta-\bar{\phi}\right)\left(1+J^2+J\sin\left(\theta-\bar{\phi}\right)\right)-\frac{\left(1+J^2\right)\left(\cos\left(\theta-\bar{\phi}\right)-1\right)}{2J}-\frac{1}{4}\sin 2\left(\theta-\bar{\phi}\right)\end{bmatrix}.$$

Similarly, for the equation, averaged to second-order, we obtain

$$\dot{\bar{J}} = 0+O\left(\varepsilon^3\right), \dot{\bar{\phi}} = \frac{1}{2}\varepsilon+\frac{1}{8}\varepsilon^2+O\left(\varepsilon^3\right). \tag{18}$$

The Equation (18) with the initial value

$$\bar{J}(0) = e, \bar{\phi}(0) = -\pi/2$$

is easily solved by

$$\begin{cases}\bar{J} = e+O\left(\varepsilon^3\right),\\ \bar{\phi} = -\frac{\pi}{2}+\frac{1}{2}\varepsilon\theta+\frac{1}{8}\varepsilon^2\theta+O\left(\varepsilon^3\right).\end{cases} \tag{19}$$

Combining with the transformation (14) and (16), we have

$$\frac{r_c}{r}=1+J\sin\left(\theta-\phi\right)=1+\left(J+\varepsilon U^{[1]}\left(J,\bar{\phi},\theta\right)\right)\sin\left(\theta-\bar{\phi}-\varepsilon U^{[2]}\left(J,\bar{\phi},\theta\right)\right)$$

$$=1+\left\{e+\frac{\varepsilon}{2}\cos\left[\left(1-\frac{\varepsilon}{2}-\frac{\varepsilon^2}{8}\right)\theta\right]\times\left(1+e^2+e\cos\left[\left(1-\frac{\varepsilon}{2}-\frac{\varepsilon^2}{8}\right)\theta\right]\right)\right\}$$

$$\times\cos\left\{\theta-\varepsilon\frac{\theta}{2}-\varepsilon^2\frac{\theta}{8}-\varepsilon\frac{\left(1+\varepsilon^2\right)}{2e}\sin\left[\theta-\varepsilon\frac{\theta}{2}-\varepsilon^2\frac{\theta}{8}\right]+\frac{\varepsilon}{4}\cos\left(2\theta-\varepsilon\theta-\varepsilon^2\theta/4\right)\right\}+O\left(\varepsilon^2\right).$$

As an example, we illustrate our results for Mercury of our solar system which is described by the near-circular orbit. Mercury has the eccentricity $\left(e=0.2056\right)$ by the classical Newton mechanics. The other parameters are taken as follows:

Newton's universal gravitational constant G = 6.670 × 10^{-11} $m^3 \cdot kg^{-1} \cdot s^{-2}$; the mass of the Sun M = 1.989 × 10^{30} kg; the speed of light G = 2.998 × 10^8 $m \cdot s^{-1}$; the relativistic angular moment $\mu = 10\mu_c$ = 10 GM/c.

An approximate Kepler elliptic orbit due to special relativity is illustrated in **Figure 2**.

4. Conclusion

The relativistic angle momentum μ determines the existence of periodic orbits. When μ is smaller than the critical angle momentum μ_c, the Kepler system (8) has no periodic orbits. For $\mu > \mu_c$, if the energy defined by (9) lies in a proper interval $h\in\left(c^2\sqrt{1-G^2M^2/c^2\mu^2},c^2\right)$, then every orbit is closed and periodic; otherwise, it leads to the unbounded orbits. The approximate relativistic Kepler elliptic orbit is illustrated by numerical simulation via a second-order perturbation method of averaging, and it is valid only for timescale of the order of $1/\varepsilon^2$.

5. Acknowledgements

This work is supported by the National Natural Science

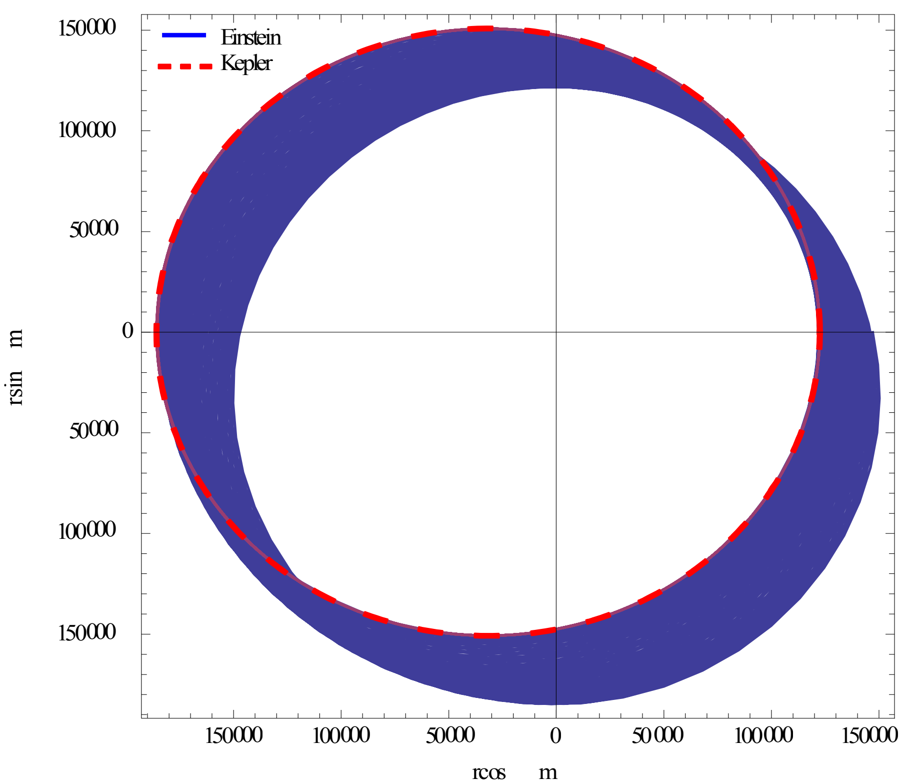

Figure 2. Relativistic orbit in a Keplerian limit (blue solid line), as described by Equation (13), compared to a corresponding Keplerian orbit (red dashed line) with $\varepsilon = 0$. The approximate Kepler elliptic orbit due to special relativity is illustrated here for $0 \le \theta \le 100\pi$.

Foundation of China (Grant No. 11226130 and No. 11261013).

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