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Bayesian and Frequentist Prediction Using Progressive Type-II Censored with Binomial Removals

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ABSTRACT

In this article, we study the problem of predicting future records and order statistics (two-sample prediction) based on progressive type-II censored with random removals, where the number of units removed at each failure time has a discrete binomial distribution. We use the Bayes procedure to derive both point and interval bounds prediction. Bayesian point prediction under symmetric and symmetric loss functions is discussed. The maximum likelihood (ML) prediction intervals using "plug-in" procedure for future records and order statistics are derived. An example is discussed to illustrate the application of the results under this censoring scheme.

Keywords: Bayesian Prediction; Burr-X Model; Progressive Censoring; Random Removals

1. Introduction

In many practical problems of statistics, one wishes to use the results of previous data (past samples) to predict a future observation (a future sample) from the same population. One way to do this is to construct an interval which will contain the future observation with a specified probability. This interval is called a prediction interval. Prediction has been applied in medicine, engineering, business, and other areas as well. Hahn and Meeker [1] have recently discussed the usefulness of constructing prediction intervals. Bayesian prediction bounds for a future observation based on certain distributions have been discussed by several authors. Bayesian prediction bounds for future observations from the exponential distribution are considered by Dunsmore [2], Lingappaiah [3], Evans and Nigm [4], and Al-Hussaini and Jaheen [5]. Bayesian prediction bounds for future lifetime under the Weibull model have been derived by Evans and Nigm [6,7], and Bayesian prediction bounds for observable having the Burr type-XII distribution were obtained by Nigm [8], Al-Hussaini and Jaheen [9,10], and Ali Mousa and Jaheen [11,12]. Prediction was reviewed by Patel [13], Nagraja [14], Kaminsky and Nelson [15], and Al-Hussaini [16], and for details on the history of statistical prediction, analysis, and applications, see, for example, Aitchison and Dunsmore [17], Geisser [18]. Bayesian prediction bounds for the Burr type-X model based on records have been derived from Ali Mousa [19], and Bayesian prediction bounds from the scaled Burr type X model were obtained by Jaheen and AL-Matrafi [20]. Bayesian prediction with Outliers and random sample size for the Burr-X model was obtained by Soliman [21], Bayesian prediction bounds for order statistics in the one and two-sample cases from the Burr type X model were obtained by Sartawi and Abu-Salih [22]. Recently, Ahmadi and Balakrishnan [23] discussed how one can predict future usual records (order statistics) from an independent Y-sequence based on order statistics (usual records) from an independent X-sequence and developed nonparametric prediction intervals. Ahmadi and Mir Mostafaee [24], Ahmadi *et al.* [25] obtained prediction intervals for order statistics as well as for the mean lifetime from a future sample based on observed usual records from an exponential distribution using the classical and Bayesian approaches, respectively.

The rest of the paper is as follows. In Section 2, we present some preliminaries as the model, priors and the posterior distribution. In Section 3, Bayesian predictive distribution for the future lower records (two-sample prediction) is based on progressive type-II censored with

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random removals. In Section 4, the ML prediction both point and interval prediction using "plug-in" procedure are derived. In Section 5, Bayesian predictive distribution for the future order statistics based on progressive type-II censored with random removals. In Section 6, the ML prediction both point and interval prediction using "plug-in" procedure for the future order statistics are derived. A practical example using generating data set Progressively type-II censored random sample from Burr-X distribution, and a simulation study has been carried out in order to compare the performance of different methods of prediction are presented in Section 8. Finally we conclude the paper in Section 8.

2. The Model, Prior and Posterior Distribution

Let random variable X have an Burr-X distribution with Parameter θ. the probability density function and the cumulative distribution function of X are respectively

$$f(x) = 2\theta x \exp\left(-x^2\right)\left(1-\exp\left(-x^2\right)\right)^{\theta-1}, \; x,\theta>0$$
$$F(x) = \left(1-\exp\left(-x^2\right)\right)^{\theta}, \quad x,\theta>0. \tag{1}$$

Suppose that $(x_1,R_1),(x_2,R_2),\cdots,(x_m,R_m)$, denote a progressively type-II censored sample, where $x_1<x_2<\cdots<x_m$ with pre-determined number of removals, say $R_1=r_1,R_2=r_2,\cdots,R_m=r_m$, the conditional likelihood function can be written as,

$$L(\theta;x|R=r) = C\prod_{i=1}^{m} f(x_i)\left[1-F(x_i)\right]^{r_i}, \tag{2}$$

where $C = n(n-r_1-1)(n-r_1-r_2-2),\cdots,(n-r_1-\cdots-r_m-m+1)$ and $0\le r_i\le(n-m-r_1-\cdots-r_{i-1})$ for $i=1,2,3,\cdots,m-1$,

substituting (1), (2) into (3) we get

$$\ell(\theta) \propto \theta^m \sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m} G\exp\left(\theta\sum_{i=1}^{m}(k_i+1)\ln U_i\right), \tag{3}$$

where

$$G = (-1)^{k_1+\cdots+k_m}\binom{r_1}{k_1}\cdots\binom{r_m}{k_m}, \quad U_i = \left(1-\exp\left(-x_i^2\right)\right). \tag{4}$$

If the parameter θ is unknown, from the In-likelihood function given by (3), the MLEs, $\hat{\theta}_{\text{MLE}} = \hat{\theta}$ can be obtained by the following equations,

$$\ln\ell(\theta) = m\ln\theta + \theta\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m} G\sum_{i=1}^{m}(k_i+1)\ln U_i. \tag{5}$$

The first derivative of $\ln\ell(\theta)$ with respect to θ is

$$\frac{\partial\ln\ell(\theta)}{\partial\theta} = \frac{m}{\theta} + \sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m} G\sum_{i=1}^{m}(k_i+1)\ln U_i, \tag{6}$$

setting $\frac{\partial\ln\ell(\theta)}{\partial\theta}=0$ we get the maximum likelihood estimator of θ as the following

$$\hat{\theta}_{\text{MLE}} = \frac{-m}{\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m} G\sum_{i=1}^{m}(k_i+1)\ln U_i}, \tag{7}$$

consider a gamma conjugate prior for θ in the form

$$\pi(\theta|\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{(\alpha-1)}\exp(-\beta\theta), \quad \theta>0, \alpha,\beta>0. \tag{8}$$

From (3) and (8) the conditional posterior (*pdf*) of θ is given by

$$\pi^*(\theta|\underline{x}) = \frac{\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m} G\theta^{(m+\alpha-1)}\exp(-\theta q_k)}{\Gamma(m+\alpha)\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m} Gq_k^{-m-\alpha}} \tag{9}$$

where

$$q_k = \beta - \sum_{i=1}^{m}(k_i+1)\ln U_i.$$

3. Bayesian Prediction for Record Value

Suppose n independent items are put on a test and the lifetime distribution of each item is given by (2). Let $X_1,X_2,X_3,\cdots,X_m$ be the ordered m-failures observed under the type-II progressively censoring plan with binomial removals $(R_1,\cdots,R_m)$, and that $Y_1,Y_2,\cdots,Y_{m_1}$ be a second independent sample (of size m_1) of future lower record observed from the same distribution (future sample). Our aim is to make Bayesian prediction about the s^{th}, then the marginal pdf of V_S is given by see Ahmadi and Mir Mostafaee [24] is

$$f_{V_s}(Y) = \frac{\left[-\log F(Y)\right]^{S-1}}{(S-1)!}f(Y) \tag{10}$$

where

$$F(Y) = \left(1-\exp\left(-Y_s^2\right)\right)^{\theta},$$
$$f(Y) = 2\theta Y_s\exp\left(-Y_s^2\right)\left(1-\exp\left(-Y_s^2\right)\right)^{\theta-1}, \tag{11}$$

$$U_s = \left(1-\exp\left(-Y_s^2\right)\right), \; q_s = -\log U_s,$$
$$\Phi(Y_s) = \left(\frac{2Y_s\exp\left(-Y_s^2\right)}{1-\exp\left(-Y_s^2\right)}\right). \tag{12}$$

Applying (11), (12) in (10) we obtain

$$f_{V_s}(Y_s) = \frac{\theta^S q_s^{(S-1)}}{\Gamma(S)}\exp(-\theta q_s)\Phi(Y_s). \tag{13}$$

Combining the posterior density (9) with (13) and integrating out θ we obtain the Bayes predictive density

$$f\left(Y_s|\underline{\mathbf{X}}\right)=\int_0^\infty f_{V_s}\left(Y_s\right)\Pi^*\left(\theta|\underline{\mathbf{X}}\right)\mathrm{d}\theta$$

$$=\frac{q_s^{(S-1)}\Phi\left(Y_s\right)\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m}G\left(q_k+q_s\right)^{-(m+\alpha+S)}}{B\left(S,m+\alpha\right)\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m}Gq_k^{-(m+\alpha)}}. \quad (14)$$

The Bayesian prediction bounds for the future Y_S, $s=1,2,\cdots,m_1$ are obtained by evaluating $\Pr\left[Y_S\geq t_1|\underline{\mathrm{x}}\right]$ for some given value of t_1, It follows from (14) that

$$\Pr\left[Y_S\geq t_1|\underline{\mathrm{x}}\right]=\int_{t_1}^\infty f\left(V_S|\underline{\mathrm{x}}\right)\mathrm{d}Y_s$$

$$=\frac{\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m}GI_{t_1}\left(Y_s\right)}{B\left(S,m+\alpha\right)\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m}Gq_k^{-(m+\alpha)}}, \quad (15)$$

where

$$I_{t_1}\left(Y_s\right)=\int_{t_1}^\infty\frac{q_s^{(S-1)}}{\left(q_i+q_s\right)^{(m+\alpha+S)}}\Phi\left(Y_s\right)\mathrm{d}Y_s. \quad (16)$$

The predictive bounds of a two-sided interval with cover τ $\left(\tau 100\%\right)$ for the future lower record Y_S, may thus obtained by solving the following two equation for the lower L_s and upper U_s bounds:

$$\frac{\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m}GI_L\left(Y_S\right)}{B\left(S,m+\alpha\right)\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m}G\ q_k^{-(m+\alpha)}}=\frac{1+\tau}{2}, \quad (17)$$

and

$$\frac{\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m}GI_U\left(Y_S\right)}{B\left(S,m+\alpha\right)\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m}Gq_k^{-(m+\alpha)}}=\frac{1-\tau}{2}, \quad (18)$$

where $I_L\left(Y_S\right)$ and $I_U\left(Y_S\right)$ are given by Equation (16).

Now by using (14) the Bayesian point prediction of the future lower record values Y_S under SE (*BS*) and LINEX loss functions (*BL*) are given, respectively, as

$$\tilde{Y}_{s(BS)}=\int_0^\infty Y_s f\left(V_S|\underline{\mathbf{X}}\right)\mathrm{d}Y_s$$

$$=\frac{\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m}GI_1\left(Y_s\right)}{B\left(S,m+\alpha\right)\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m}Gq_k^{-(m+\alpha)}}, \quad (19)$$

and

$$\tilde{Y}_{s(BL)}=-\frac{1}{c}\mathrm{Log}\left[\int_0^\infty\exp\left(-cY_s\right)f\left(V_S|\underline{\mathbf{X}}\right)\mathrm{d}Y_s\right]$$

$$=-\frac{1}{c}\mathrm{Log}\left[\frac{\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m}GI_2\left(Y_s\right)}{B\left(S,m+\alpha\right)\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m}Gq_k^{-(m+\alpha)}}\right] \quad (20)$$

where

$$I_1\left(Y_s\right)=\int_0^\infty\frac{Y_s q_s^{(S-1)}}{\left(q_i+q_s\right)^{(m+\alpha+S)}}\Phi\left(Y_s\right)\mathrm{d}Y_s, \quad (21)$$

and

$$I_2\left(Y_s\right)=\int_0^\infty\frac{q_s^{(S-1)}\exp\left(-cY_s\right)}{\left(q_i+q_s\right)^{(m+\alpha+S)}}\Phi\left(Y_s\right)\mathrm{d}Y_s. \quad (22)$$

One can use a numerical integration technique to get the above integration, given by (21), (22).

Special case: In special case it is important to predict the first unobserved lower record value Y_1.

When s = 1, in (17) and (18), the lower and upper Bayesian prediction bounds with cover τ of Y_1 are obtained from the numerical solution of the following equations

$$\frac{\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m}GI_L\left(Y_1\right)}{B\left(1,m+\alpha\right)\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m}Gq_k^{-(m+\alpha)}}=\frac{1+\tau}{2}, \quad (23)$$

and

$$\frac{\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m}GI_U\left(Y_1\right)}{B\left(1,m+\alpha\right)\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m}Gq_k^{-(m+\alpha)}}=\frac{1-\tau}{2}, \quad (24)$$

where $I_L\left(Y_1\right)$ and $I_U\left(Y_1\right)$ given by Equation (16), and solving the resulting equations numerically.

4. ML Prediction for Record Value

The commonly used frequentist approaches such as the maximum likelihood estimate and the "plug-in" procedure, which is to substitute a point estimate of the unknown parameters into the predictive distribution are reviewed and discussed. In this section the ML prediction both point and interval using "plug-in" procedure for future lower record based on progressive type-II censored sample defined by (2). By replacing θ in the marginal pdf of V_S (13) by $\hat{\theta}$ which we can find it from the numerical solution of the Equation (7), then

$$f_{1V_s}\left(Y_s\right)=\frac{\hat{\theta}^S q_s^{(S-1)}}{\Gamma(S)}\Phi\left(Y_s\right)\exp\left(-\hat{\theta}q_s\right). \quad (25)$$

The ML prediction bounds for the future Y_S are obtained by evaluating $\Pr\left[Y_S \ge t_2 \mid \underline{x}\right]$ for some given value of t_2, It follows from (25) that

$$\Pr\left[Y_S \ge t_2 \mid \underline{x}\right]=\int_{t_2}^{\infty} f_{1V_s}\left(Y_s\right)\mathrm{d}Y_s=\frac{1}{\Gamma(S)}\text{Ingamma}\left(t^*;S,1\right) \quad (26)$$

where Ingamma $\left(\varphi;t_1,t_2\right)$ is the incomplete gamma function defined by

$$\text{Ingamma}\left(\varphi;t_1,t_2\right)=\int_0^{\varphi} z^{t_1-1}\exp\left[-t_2 z\right]\mathrm{d}z., \text{ and } t^*=-\hat{\theta}\log\left(1-\exp\left[-t_2^2\right]\right). \quad (27)$$

The predictive bounds of a two-sided interval with cover τ for the future lower record Y_S, may thus obtained by solving the following two equation for the lower L_s and upper U_s bounds:

$$\frac{1}{\Gamma(S)}\text{Ingamma}\left(L_s^*;S,1\right)=\frac{1+\tau}{2}, \quad (28)$$

and

$$\frac{1}{\Gamma(S)}\text{Ingamma}\left(U_s^*;S,1\right)=\frac{1-\tau}{2}. \quad (29)$$

Special case

When S = 1, in (28) and (29), the lower and upper ML prediction bounds with cover τ of Y_1 are obtained from the numerical solution of the following equations:

$$\text{Ingamma}\left(L_1^*;1,1\right)=\frac{1+\tau}{2}, \quad (30)$$

and

$$\text{Ingamma}\left(U_1^*;1,1\right)=\frac{1-\tau}{2}. \quad (31)$$

The ML point prediction of the lower record value Y_S is given, from (25), as

$$\tilde{Y}_{s(ML)}=\int_0^{\infty} Y_s f_{1V_s}\left(Y_s\right)\mathrm{d}Y_s=\frac{\hat{\theta}^S}{\Gamma(S)}\int_0^{\infty} q_s^{(S-1)}\exp\left(-\hat{\theta}q_s\right)Y_s\Phi\left(Y_s\right)\mathrm{d}Y_s. \quad (32)$$

One can use a numerical integration technique to get the above integration, given by (32).

5. Bayesian Prediction for Order Statistics

Order statistics arise in many practical situations as well as the reliability of systems. It is well-known that a system is called a *k-out-of-m* system if it consists of *m* components functioning satisfactorily provided that at least $k(\le m)$ components function. If the lifetimes of the components are independently distributed, then the lifetime of the system coincides with that of the $(m-k+1)$th order statistic from the underlying distribution. Therefore, order statistics play a key role in studying the lifetimes of such systems. See Arnold *et al*. [26] and David and Nagaraja [27] for more details concerning the applications of order statistics.

Suppose *n* independent items are put on a test and the lifetime distribution of each item is given by (2). Let $X_1, X_2, X_3, \cdots, X_m$ be the ordered m-failures observed under the type-II progressively censoring plan with binomial removals $\left(R_1,\cdots,R_m\right)$, and that $Y_1,Y_2,\cdots,Y_{m_2}$ be a second independent random sample (of size m_2) of future order statistics observed from the same distribution. Our aim is to obtain Bayesian prediction about some functions of $Y_1,Y_2,\cdots,Y_{m_2}$.

Let Y_s be the s^{th} ordered lifetime in the future sample of m_2 lifetimes. The density function of Y_s for given θ is of the form

$$h^*\left(Y_s \mid \theta\right)=D^*(s)\left[1-F\left(Y_s \mid \theta\right)\right]^{m_2-s}\left[F\left(Y_s \mid \theta\right)\right]^{s-1} f\left(Y_s \mid \theta\right), \quad (33)$$

where $D^*(s)=s\binom{m_2}{s}$. For the Burr-X model, substituting (11) and (12) in (22), we obtain

$$h^*\left(Y_s \mid \theta\right)=D^*(s)\sum_{l=0}^{m_2-s}(-1)^l\binom{m_2-s}{l}\left(2\theta Y_s\exp\left(-Y_s^2\right)\right)U_s^{\theta(l+s)-1} \quad (34)$$

Therefore, from (9) and (34), the Bayes predictive density function of Y_s will be (see Equation (35)).

where $\Psi(l)=D^*(s)\sum_{l=0}^{m_2-s}(-1)^l\binom{m_2-s}{l}$ and $\Phi\left(Y_s\right)$, U_s

$$f^{**}\left(Y_s \mid \underline{x}\right)=\int_0^{\infty} h^*\left(Y_s \mid \theta\right)\pi^*\left(\theta \mid \underline{x}\right)\mathrm{d}\theta=\Psi(l)\Phi\left(Y_s\right)(m+\alpha)\frac{\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m} G\left(q_k-(s+l)\log U_s\right)^{-(m+\alpha+1)}}{\sum_{k_1=0}^{r_1}\cdots\sum_{k_m=0}^{r_m} G q_k^{-(m+\alpha)}}, \quad (35)$$

are given by Equation (12).

The Bayesian prediction bounds for the future Y_s, $s = 1, 2, \cdots, m_2$ are obtained by evaluating $\Pr\left[Y_S \geq \varepsilon_1 | \underline{x}\right]$ for some given value of ε_1, It follows from (35) that

$$\Pr\left[Y_S \geq \varepsilon_1 | \underline{x}\right] = \int_{\varepsilon_1}^{\infty} f^{**}\left(Y_s | \underline{x}\right) \mathrm{d}Y_s = \frac{\Psi(l) \sum_{k_1=0}^{r_1} \cdots \sum_{k_m=0}^{r_m} G\left(q_k^{-(m+\alpha)} - \left[q_k - (s+l)\log\left(1-\exp\left(-\varepsilon_1^2\right)\right)\right]^{-(m+\alpha)}\right)}{(s+l) \sum_{k_1=0}^{r_1} \cdots \sum_{k_m=0}^{r_m} G q_k^{-(m+\alpha)}}. \quad (36)$$

The predictive bounds of a two-sided interval with cover τ $(\tau 100\%)$ for the future order statistics Y_S, may thus obtained by solving the following two equation for the lower LL_s and upper UU_s bounds:

$$\frac{\Psi(l) \sum_{k_1=0}^{r_1} \ldots \sum_{k_m=0}^{r_m} G\left(q_k^{-(m+\alpha)} - \left[q_k - (s+l)\log\left(1-\exp\left(-LL_s^2\right)\right)\right]^{-(m+\alpha)}\right)}{(s+l) \sum_{k_1=0}^{r_1} \cdots \sum_{k_m=0}^{r_m} G q_k^{-(m+\alpha)}} = \frac{1+\tau}{2}, \quad (37)$$

$$\frac{\Psi(l) \sum_{k_1=0}^{r_1} \cdots \sum_{k_m=0}^{r_m} G\left(q_k^{-(m+\alpha)} - \left[q_k - (s+l)\log\left(1-\exp\left(-UU_s^2\right)\right)\right]^{-(m+\alpha)}\right)}{(s+l) \sum_{k_1=0}^{r_1} \cdots \sum_{k_m=0}^{r_m} G q_k^{-(m+\alpha)}} = \frac{1-\tau}{2}. \quad (38)$$

Now by using (35) the Bayesian point prediction of the future order statistics Y_s under SE (*BS*) and LINEX loss functions (*BL*) are given, respectively, as

$$\tilde{Y}_{s(BS)} = \int_0^{\infty} Y_s f^{**}\left(Y_s | \underline{x}\right) \mathrm{d}Y_s = \Psi(l)(m+\alpha) \frac{\sum_{k_1=0}^{r_1} \cdots \sum_{k_m=0}^{r_m} G I_1^*(Y_s)}{\sum_{k_1=0}^{r_1} \cdots \sum_{k_m=0}^{r_m} G q_k^{-(m+\alpha)}} \quad (39)$$

and

$$\tilde{Y}_{s(BL)} = -\frac{1}{c}\mathrm{Log}\left[\int_0^{\infty} \exp(-cY_s) f^{**}\left(Y_s | \underline{x}\right) \mathrm{d}Y_s\right] = \Psi(l)(m+\alpha) \frac{\sum_{k_1=0}^{r_1} \cdots \sum_{k_m=0}^{r_m} G I_2^*(Y_s)}{\sum_{k_1=0}^{r_1} \cdots \sum_{k_m=0}^{r_m} G q_k^{-(m+\alpha)}} \quad (40)$$

where

$$I_1^*(Y_s) = \int_0^{\infty} Y_s \left(q_k - (s+l)\log U_s\right)^{-(m+\alpha+1)} \Phi(Y_s) \mathrm{d}Y_s, \quad (41)$$

and

$$I_2^*(Y_s) = \int_0^{\infty} \exp(-cY_s)\left(q_k - (s+l)\log U_s\right)^{-(m+\alpha+1)} \Phi(Y_s) \mathrm{d}Y_s. \quad (42)$$

We can use a numerical integration technique to get the above integration, given by (41), (42).

6. ML Prediction for Order Statistics

In this section the ML prediction both point and interval using "plug-in" procedure for the future order statistics based on progressive type-II censored sample defined by (2). From (34) by replacing θ in the density function of Y_s for given θ by $\hat{\theta}$ which we can find it from the numerical solution of the Equation (7), then the density function of Y_s for given $\hat{\theta}$

$$h_1^*\left(Y_s | \hat{\theta}\right) = D^*(s) \sum_{l=0}^{m_2-s} (-1)^l \binom{m_2-s}{l} \left(2\hat{\theta} Y_s \exp\left(-Y_s^2\right)\right) U_s^{\hat{\theta}(l+s)-1}. \quad (43)$$

The ML prediction bounds for the future Y_s are obtained by evaluating $\Pr\left[Y_S \geq \varepsilon_2 | \underline{x}\right]$ for some given value of ε_2, It follows from (43) that

$$\Pr\left[Y_S \geq \varepsilon_2 \middle| \underline{x}\right] = \int_{\varepsilon_2}^{\infty} h_1^*\left(Y_s \middle| \hat{\theta}\right) \mathrm{d}Y_s = \frac{\Psi(l)}{(l+s)}\left[1-\left(1-\exp\left(-\varepsilon_2^2\right)\right)^{\hat{\theta}(l+s)}\right]. \quad (44)$$

The predictive bounds of a two-sided interval with cover τ for the future order statistics Y_s, may thus obtained by solving the following two equation for the lower LL_s^* and upper UU_s^* bounds:

$$\frac{\Psi(l)}{(l+s)}\left[1-\left(1-\exp\left(-LL_s^{*2}\right)\right)^{\hat{\theta}(l+s)}\right] = \frac{1+\tau}{2}, \quad (45)$$

and

$$\frac{\Psi(l)}{(l+s)}\left[1-\left(1-\exp\left(-UU_s^{*2}\right)\right)^{\hat{\theta}(l+s)}\right] = \frac{1-\tau}{2}. \quad (45)$$

From (43) the ML point prediction of the order statistics Y_S is given by

$$\tilde{Y}_{s(\mathrm{ML})} = \int_0^{\infty} Y_s h_1^*\left(Y_s \middle| \hat{\theta}\right) \mathrm{d}Y_s = \Psi(l) \int_0^{\infty} Y_s \left(2\hat{\theta} Y_s \exp\left(-Y_s^2\right)\right) U_s^{\hat{\theta}(l+s)-1} \mathrm{d}Y_s. \quad (46)$$

One can use a numerical integration technique to get the above integration, given by (46)

7. Illustrative Example and Simulation Study

Example 1: In this example, a progressive type-II censored sample with random removals from the Burr-X distribution have been generated using the following algorithm.

Algorithm 1.

1) Specify the value of n.
2) Specify the value of m.
3) Specify the value of parameters θ and p
4) Generate a random sample with size m from Burr-X and sort it.
5) Generate a random number r_1 from $bio(n-m, p)$.
6) Generate a random number r_i from $bio\left(n-m-\sum_{l=1}^{i-1} r_i, p\right)$, for each $i, i = 2,3,\cdots,m-1$.
7) Set r_m according to the following relation,

$$r_m = \begin{cases} n-m-\sum_{l=1}^{i-1} r_l & \text{if } n-m-\sum_{l=1}^{i-1} r_l > 0 \\ 0 & o.w \end{cases}$$

In these sample, we assumed that the exact value of θ and P are respectively 1.6374, 0.4 and $n = 10$ and $m = 7$, the sample obtained is given as follows (X_i, R_i): (0.4406,1), (0.5449,1), (0.6728,0), (0.9914,0), (1.2655,1), (1.3680,0), (1.4271,0).

We used the above sample to compute:

1) Bayesian point prediction, under SE and LINEX loss function;
2) The 95% Bayesian prediction intervals of the s^{th} unobserved lower records (order statistics);
3) The maximum likelihood prediction ML;
4) The 95% maximum likelihood prediction intervals of the s^{th} unobserved lower records (order statistics);
5) The results obtained are given in **Tables 1-4**.

Example 2: Simulation Study In this example, we discuss results of a simulation study comparing the performance of the prediction results obtained in this paper. Firstly we generate ($m_1 = 5$) lower record values (order statistics) from the Burr-X distribution $\theta = 1.6374$. By using the generating data we predict the 90% and 95% Bayesian prediction intervals for the future observations S^{th} lower record (order statistics) from the the Burr-X distribution and by repeated the generations 1000 time we can find the Percentage (C.P) and we use prior (α, β) equal (2,3). **Tables 5-8** show the 90% and 95% Bayesian (B) and maximum likelihood (ML) prediction intervals for the future S^{th} lower record (order statistics). The sample obtained is given as follows, (0.4640,1), (0.5124,1), (0.7323,0), (0.8293,1), (0.8665,0), (0.8713,1), (1.1322,0), (1.4969,0).

8. Conclusions

In this paper, we consider the two-sample prediction wherein the observed progressive Type-II censored samples with random removals from the Burr-X distribution form the informative samples and discussed how point prediction and prediction intervals can be constructed for future lower records (order statistics). Bayesian and ML predictions both the point prediction and the prediction intervals are presented and discussed in this paper.

The commonly used frequentist approaches such as the maximum likelihood estimate and the "plug-in" procedure, which is to substitute a point estimate of the unknown parameters into the predictive distribution are reviewed and discussed. Numerical example using simulated data were used to illustrate the procedures developed here. Finally, simulation studies are presented to compare the performance of different methods of prediction. A study of 1000 randomly generated future samples from the same distribution shows that the actual prediction levels are satisfactory. From the results we note the following:

1) The results in **Tables 1-4** show that the lengths of the prediction intervals using "plug-in" procedure (MLPI) are shorter than that of prediction intervals using Bayes procedure.
2) The simulation results show that, for all cases (lower records and order statistics), the proposed prediction levels are satisfactory compared with the actual prediction levels 90% and 95%.

Table 1. Point and interval BP for the future lower record Y_S.

Y_s	SE	LINEX			95% **BPI** for Y_s	
		$c_1 = -1$	$c_2 = 0.0001$	$c_3 = 1$	[Lower, Upper]	Length
Y_1	0.9681	1.0863	0.9735	0.8672	[0.1892,1.9866]	1.7973
Y_2	0.5931	0.6535	0.5933	0.5407	[0.0727,1.3417]	1.2690
Y_3	0.3998	0.4366	0.4010	0.3679	[0.0305,1.0073]	0.9768
Y_4	0.2799	0.3033	0.2849	0.2596	[0.0133,0.7844]	0.7711
Y_5	0.2000	0.2150	0.2335	0.1869	[0.0059,0.6211]	0.6152

Table 2. Point and interval 95% MLPI for Y_S.

Y_s	ML	[Lower, Upper]	Length
Y_1	1.0686	[0.3295,2.0415]	1.7120
Y_2	0.6966	[0.1810,1.4055]	1.2245
Y_3	0.4958	[0.1082,1.0717]	0.9635
Y_4	0.3644	[0.0671,0.8455]	0.7784
Y_5	0.2721	[0.0426,0.6769]	0.6343

Table 3. Point and interval BP for the future order statistics Y_S.

Y_s	SE	LINEX			95% **BPI** for Y_s	
		$c_1 = -1$	$c_2 = 0.0001$	$c_3 = 1$	[Lower, Upper]	Length
Y_1	0.4847	0.5150	0.4848	0.4563	[0.0777,0.9992]	0.9215
Y_2	0.7202	0.7540	0.7201	0.6875	[0.2425,1.2453]	1.0028
Y_3	0.9334	0.9716	0.9334	0.8963	[0.4145,1.4887]	1.0742
Y_4	1.1730	1.2201	1.1730	1.1278	[0.6040,1.7968]	1.1928
Y_5	1.5292	1.6070	1.5292	1.4592	[0.8470,2.3554]	1.5085

Table 4. Point and interval 95% MLPI for Y_S.

Y_s	ML	[Lower, Upper]	Length
Y_1	0.5895	[0.1978,1.0525]	0.8546
Y_2	0.8301	[0.4219,1.2973]	0.8754
Y_3	1.0394	[0.6046,1.5407]	0.9361
Y_4	1.2707	[0.7848,1.8482]	1.0634
Y_5	1.6134	[1.0033,1.4023]	1.3990

Table 5. Two sample prediction for the future lower record.-90% and 95% BPI for Y_S, $S = 1,2,\cdots,5$ and their actual prediction with $\alpha = 2$, $\beta = 3$, $\theta = 1.6374$, $p = 0.4$, $n = 12$, $m = 8$.

	90% BPI for Y_S			95% BPI for Y_S		
Y_s	[Lower, Upper]	Length	C.P	[Lowe, Upper]	Length	C.P
Y_1	[0.2723,1.7970]	1.5247	0.911	[0.1902,1.9814]	1.7912	0.960
Y_2	[0.1174,1.1967]	1.0793	0.921	[0.0743,1.3332]	1.2589	0.964
Y_3	[0.0550,0.8810]	0.8261	0.911	[0.0318,0.9962]	0.9644	0.959
Y_4	[0.0266,0.6709]	0.6443	0.903	[0.0141,0.7715]	0.7573	0.949
Y_5	[0.0131,0.5184]	0.5053	0.909	[0.0064,0.6070]	0.6006	0.954

Table 6. Two sample prediction for the future lower record-90% and 95% MLPI for Y_S, $S = 1,2,\cdots,5$ and their actual prediction with $\alpha = 2$, $\beta = 3$, $\theta = 1.6374$, $p = 0.4$, $n = 12$, $m = 8$.

	90% MLPI for Y_S			95% MLPI for Y_S		
Y_s	[Lowe, Upper]	Length	C.P	[Lower, Upper]	Length	C.P
Y_1	[0.3850,1.8439]	1.4589	0.896	[0.3018,2.0242]	1.7224	0.949
Y_2	[0.2104,1.2501]	1.0397	0.913	[0.1593,1.3820]	1.2227	0.960
Y_3	[0.1250,0.9341]	0.8091	0.908	[0.0918,1.0444]	0.9526	0.955
Y_4	[0.0770,0.7204]	0.6434	0.899	[0.0550,0.8160]	0.7610	0.945
Y_5	[0.0484,0.5628]	0.5144	0.910	[0.0338,0.6464]	0.6126	0.953

Table 7. Two sample prediction for the future order statistics.-90% and 95% BPI for Y_S, $S = 1,2,\cdots,5$ and their actual prediction with $\alpha = 2$, $\beta = 3$, $\theta = 1.6374$, $p = 0.4$, $n = 12$, $m = 8$ $m_2 = 5$.

	90% BPI for Y_S			95% BPI for Y_S		
Y_s	[Lower, Upper]	Length	C.P	[Lowe, Upper]	Length	C.P
Y_1	[0.1201,0.8999]	0.7796	0.912	[0.0801,0.9867]	0.9065	0.952
Y_2	[0.3104,1.1472]	0.8368	0.897	[0.2470,1.2344]	0.9874	0.943
Y_3	[0.4938,1.3862]	0.8924	0.916	[0.4191,1.4797]	1.0606	0.963
Y_4	[0.6910,1.6811]	0.9901	0.924	[0.6075,1.7900]	1.1825	0.963
Y_5	[0.9442,2.1960]	1.2518	0.920	[0.8483,2.3511]	1.5028	0.971

Table 8. Two sample prediction for the future order statistics-90% and 95% MLPI for Y_S, $S = 1,2,\cdots,5$ and their actual prediction with $\alpha = 2$, $\beta = 3$, $\theta = 1.6374$, $p = 0.4$, $n = 12$, $m = 8$ $m_2 = 5$.

	90% MLPI for Y_S			95% MLPI for Y_S		
Y_s	[Lowe, Upper]	Length	C.P	[Lower, Upper]	Length	C.P
Y_1	[0.2221,0.9442]	0.7224	0.906	[0.1752,1.0249]	0.8498	0.952
Y_2	[0.4488,1.1905]	0.7417	0.897	[0.3924,1.2725]	0.8801	0.950
Y_3	[0.6356,1.4294]	0.7938	0.914	[0.5740,1.5187]	0.9447	0.961
Y_4	[0.8232,1.7236]	0.9004	0.914	[0.7548,1.8292]	1.0744	0.955
Y_5	[1.0568,2.2351]	1.1784	0.901	[0.9753,2.3875]	1.4122	0.955

3) In general, the simulation results show that the "plug-in" procedure (MLPI) performs better than the Bayes method (BPI), in the sense of shorted interval lengths.

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