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I-Pre-Cauchy Double Sequences and Orlicz Functions

Vakeel A. Khan[1], Nazneen Khan[1], Ayhan Esi[2], Sabiha Tabassum[3]
[1]Department of Mathematics, Aligarh Muslim University, Aligarh, India
[2]Department of Mathematics, Science and Art Faculty, Adiyaman University, Adiyaman, Turkey
[3]Department of Applied Mathematics, Zakir Hussain College of Engineering and Technology, Aligarh Muslim University, Aligarh, India
Email: vakhanmaths@gmail.com, nazneen4maths@gmail.com, aesi23@hotmail.com

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ABSTRACT

Let $x=(x_{ij})$ be a double sequence and let M be a bounded Orlicz function. We prove that x is I-pre-Cauchy if and only if $I-\lim_{mn}\frac{1}{m^2n^2}\sum_{i,p\le m}\sum_{j,q\le n}M\left(\frac{|x_{ij}-x_{pq}|}{\rho}\right)=0.$ This implies a theorem due to Connor, Fridy and Klin [1], and Vakeel A. Khan and Q. M. Danish Lohani [2].

Keywords: Ideal; Filter; Paranorm; I-Convergent; Invariant Mean; Monotone and Solid Space

1. Introduction

The concept of statistical convergence was first defined by Steinhaus [3] at a conference held at Wroclaw University, Poland in 1949 and also independently by Fast [4], Buck [5] and Schoenberg [6] for real and complex sequences. Further this concept was studied by Salat [7], Fridy [8], Connor [9] and many others. Statistical convergence is a generalization of the usual notation of convergence that parallels the usual theory of convergence.

A sequence $x=(x_i)$ is said to be statistically convergent to L if for a given $\varepsilon>0$

$$\lim_k\frac{1}{k}\left|\left\{i:|x_i-L|\ge\varepsilon, i\le k\right\}\right|=0.$$

A sequence $x=(x_i)$ is said to be statistically pre-cauchy if

$$\lim_k\frac{1}{k^2}\left|\left\{(j,i):|x_i-x_j|\ge\varepsilon,\ j,i\le k\right\}\right|=0.$$

Connor, Fridy and Klin [1] proved that statistically convergent sequences are statistically pre-cauchy and any bounded statistically pre-cauchy sequence with a nowhere dense set of limit points is statistically convergent. They also gave an example showing statistically pre-cauchy sequences are not necessarily statistically convergent (see [10]).

Throughout a double sequence is denoted by $x=(x_{ij})$. A double sequence is a double infinite array of elements $x_{ij}\in\mathbb{R}$ for all $i,j\in\mathbb{N}$.

The initial works on double sequences is found in Bromwich [11], Tripathy [12], Basarir and Solancan [13] and many others.

Definition 1.1. A double sequence (x_{ij}) is called statistically convergent to L if

$$\lim_{m,n\to\infty}\frac{1}{mn}\left|(i,j):|x_{ij}-L|\ge\varepsilon, i\le m,\ j\le n\right|=0,$$

where the vertical bars indicate the number of elements in the set.

Definition 1.2. A double sequence (x_{ij}) is called statistically pre-cauchy if for every $\varepsilon>0$ there exist $p=p(\varepsilon)$ and $q=q(\varepsilon)$ such that

$$\lim_{m,n\to\infty}\frac{1}{m^2n^2}\left|(i,j):|x_{ij}-x_{pq}|\ge\varepsilon, i\le m,\ j\le n\right|=0.$$

Definition 1.3. An *Orlicz Function* is a function $M:[0,\infty)\to[0,\infty)$ which is continuous, nondecreasing and convex with $M(0)=0, M(x)>0$ for $x>0$ and $M(x)\to\infty$, as $x\to\infty$.

If convexity of M is replaced by $M(x+y)\le M(x)+M(y)$, then it is called a *Modulus function* (see Maddox [14]). An Orlicz function may be bounded or unbounded. For example, $M(x)=x^p\ (0<p\le1)$ is unbounded and $M(x)=\frac{x}{x+1}$

is bounded (see Maddox [14]).

Lindenstrauss and Tzafriri [15] used the idea of Orlicz functions to construct the sequence space,

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{for some } \rho > 0 \right\}.$$

The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}$$

The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = x^p$ for $1 \le p < \infty$.

An Orlicz function *M* is said to satisfy Δ_2 condition for all values of x if there exists a constant $K > 0$ such that $M(Lx) \le KLM(x)$ for all values of $L > 1$.

The study of Orlicz sequence spaces have been made recently by various authors [1,2,16-20]).

In [1], Connor,Fridy and Klin proved that a bounded sequence $x = (x_k)$ is statistically pre-cauchy if and only if

$$\lim_k \frac{1}{k^2} \sum_{i,j \le k} \left(|x_i - x_j|\right) = 0.$$

The notion of I-convergence is a generalization of statistical convergence. At the initial stage it was studied by Kostyrko, Salat, Wilezynski [21]. Later on it was studied by Salat, Tripathy, Ziman [22] and Demirci [23], Tripathy and Hazarika [24-26]. Here we give some pre-liminaries about the notion of I-convergence.

Definition 1.4. [20,27] Let *X* be a non empty set. Then a family of sets $I \subseteq 2^X$ (2^X denoting the power set of *X*) is said to be an ideal in *X* if

(i) $\varnothing \in I$

(ii) *I* is additive *i.e* $A, B \in I \Rightarrow A \bigcap B \in I$.

(iii) *I* is hereditary *i.e* $A \in I, B \subseteq A \Rightarrow B \in I$.

An Ideal $I \subseteq 2^X$ is called non-trivial if $I \ne 2^X$. A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$.

A non-trivial ideal *I* is maximal if there cannot exist any non-trivial ideal $J \ne I$ containing *I* as a subset.

For each ideal *I*, there is a filter $£(I)$ corresponding to *I*. *i.e.*

$$£(I) = \{K \subseteq N : K^c \in I\},$$

where $K^c = N - K$.

Definition 1.5. [10,21,28] A double sequence $(x_{ij}) \in \omega$ is said to be I-convergent to a number *L* if for every $\epsilon > 0$,

$$\{i, j \in \mathbb{N} : |x_{ij} - L| \ge \epsilon\} \in I.$$

In this case we write $I - \lim x_{ij} = L.$

Definition 1.6. [21] A non-empty family of sets $£(I) \subseteq 2^X$ is said to be filter on *X* if and only if

(i) $\Phi \notin £(I)$,

(ii) For $A, B \in £(I)$ we have $A \bigcap B \in £(I)$

(iii) For each $A \in £(I)$ and $A \subseteq B$ implies $B \in £(I)$.

2. Main Results

In this article we establish the criterion for any arbitrary double sequence to be I-pre-cauchy.

Theorem 2.1. Let $x = (x_{ij})$ be a double sequence and let *M* be a bounded Orlicz function then x is I-pre-Cauchy if and only if

$$I - \lim_{mn} \frac{1}{m^2 n^2} \sum_{i,p \le m} \sum_{j,q \le n} M\left(\frac{|x_{ij} - x_{pq}|}{\rho}\right) = 0, \text{ for some } \rho > 0.$$

Proof: Suppose that

$$I - \lim_{mn} \frac{1}{m^2 n^2} \sum_{i,p \le m} \sum_{j,q \le n} M\left(\frac{|x_{ij} - x_{pq}|}{\rho}\right) = 0, \text{ for some } \rho > 0.$$

For each $\varepsilon > 0, \rho > 0$ and $m, n \in IN$ we have that

$$A_1 = \left\{ m, n \in IN : M\left(\frac{|x_{ij} - x_{pq}|}{\rho}\right) \ge \frac{\varepsilon}{2mn}, i, p \le m, \ j, q \le n \right\} \in I, \tag{1}$$

$$A_1^c = \left\{ m, n \in IN : M\left(\frac{|x_{ij} - x_{pq}|}{\rho}\right) < \frac{\varepsilon}{2mn}, i, p \le m, \ j, q \le n \right\} \in I. \tag{2}$$

$$\begin{aligned} &\lim_{mn} \frac{1}{m^2 n^2} \sum_{i,p \le m} \sum_{j,q \le n} M\left(\frac{|x_{ij} - x_{pq}|}{\rho}\right) \\ &= \lim_{mn} \frac{1}{m^2 n^2} \sum_{|x_{ij} - x_{pq}| < \frac{\varepsilon}{2mn}} M\left(\frac{|x_{ij} - x_{pq}|}{\rho}\right) \\ &+ \lim_{mn} \frac{1}{m^2 n^2} \sum_{|x_{ij} - x_{pq}| \ge \frac{\varepsilon}{2mn}} M\left(\frac{|x_{ij} - x_{pq}|}{\rho}\right) \\ &\ge \lim_{mn} \frac{1}{m^2 n^2} \sum_{|x_{ij} - x_{pq}| \ge \frac{\varepsilon}{2mn}} M\left(\frac{|x_{ij} - x_{pq}|}{\rho}\right) \end{aligned}$$

Now by (1) and (2) we have

$$\left\{m,n\in IN:\lim_{mn}\frac{1}{m^2n^2}\sum_{i,p\le m}\sum_{j,q\le n}M\left(\frac{\left|x_{ij}-x_{pq}\right|}{\rho}\right)\ge\varepsilon,i,p\le m\ j,q\le n\right\}\subset A_1\cup A_1^c\in I.$$

thus x is I-pre-Cauchy.

Now conversely suppose that x is I-pre-Cauchy, and that ε has been given.

Then we have

$$\left\{m,n\in IN:\lim_{mn}\frac{1}{m^2n^2}\sum_{i,p\le m}\sum_{j,q\le n}M\left(\frac{\left|x_{ij}-x_{pq}\right|}{\rho}\right)\ge\varepsilon,i,p\le m\ \ j,q\le n\right\}\subset A_1\cup A_1^c\in I.$$

where,

$$A_1=\left\{m,n\in IN:M\left(\frac{\left|x_{ij}-x_{pq}\right|}{\rho}\right)\ge\frac{\varepsilon}{2mn},i,p\le m\ j,q\le n\right\}\in I,$$

$$A_1^c=\left\{m,n\in IN:M\left(\frac{\left|x_{ij}-x_{pq}\right|}{\rho}\right)<\frac{\varepsilon}{2mn},i,p\le m\ \ j,q\le n\right\}\in I.$$

Let $\delta>0$ be such that $M(\delta)<\frac{\varepsilon}{2}$. Since M is a bounded Orlicz function there exists an integer B such that $M(x)<\frac{B}{2}$ for all $x\ge 0$. Therefore, for each $m,n\in IN$,

$$\begin{aligned}
&\lim_{mn}\frac{1}{m^2n^2}\sum_{i,p\le m}\sum_{j,q\le n}M\left(\frac{\left|x_{ij}-x_{pq}\right|}{\rho}\right)\\
&=\lim_{mn}\frac{1}{m^2n^2}\sum_{\left|x_{ij}-x_{pq}\right|<\frac{\varepsilon}{2mn}}M\left(\frac{\left|x_{ij}-x_{pq}\right|}{\rho}\right)\\
&+\lim_{mn}\frac{1}{m^2n^2}\sum_{\left|x_{ij}-x_{pq}\right|\ge\frac{\varepsilon}{2mn}}M\left(\frac{\left|x_{ij}-x_{pq}\right|}{\rho}\right)\\
&\le M(\delta)+\lim_{m,n}\frac{1}{m^2n^2}\sum_{i,p\le m}\sum_{j,q\le n}M\left(\frac{\left|x_{ij}-x_{pq}\right|}{\rho}\right)\\
&\le\frac{\varepsilon}{2}+\frac{B}{2}\left(\frac{1}{m^2n^2}\left|\left\{(i,j):\left|x_{ij}-x_{pq}\right|\ge\varepsilon,i,p\le m,j,q\le n\right\}\right|\right)\\
&\le\varepsilon+B\left(\frac{1}{m^2n^2}\left|\left\{(i,j):\left|x_{ij}-x_{pq}\right|\ge\varepsilon,i,p\le m,j,q\le n\right\}\right|\right)
\end{aligned}\tag{3}$$

Since x is I-pre-Cauchy, there is an IN such that the right hand side of (3) is less than ε for all $m,n\in IN$. Hence

$$I-\lim_{m,n}\frac{1}{m^2n^2}\sum_{i,p\le m}\sum_{j,q\le n}M\left(\frac{\left|x_{ij}-x_{pq}\right|}{\rho}\right)=0.$$

Theorem 2.2. Let $x=(x_{ij})$ be a double sequence and let M be a bounded Orlicz function then x is I-convergent to L if and only if

$$I-\lim_{m,n}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}M\left(\frac{\left|x_{ij}-L\right|}{\rho}\right)=0,\text{for some }\rho>0.$$

Proof: Suppose that

$$I-\lim_{m,n}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}M\left(\frac{\left|x_{ij}-L\right|}{\rho}\right)=0,\text{for some }\rho>0.$$

with an Orlicz function M, then x is I-convergent to L (See [1])

Conversely suppose that x is I-convergent to L. We can prove this in similar manner as in Theorem 2.1 assuming that

$$I-\lim_{m,n}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}M\left(\frac{\left|x_{ij}-L\right|}{\rho}\right)=0,\text{for some }\rho>0.$$

and M being a bounded Orlicz function.

Corollary 2.3. A sequence $x=(x_{ij})$ is I-convergent if and only if

$$I-\lim_{m,n}\frac{1}{m^2n^2}\sum_{i,p\le m}\sum_{j,q\le n}\left|x_{ij}-x_{pq}\right|=0.$$

Proof: Let $M(x)=x.$ Then

$$M\left(\frac{\left|x_{ij}-x_{pq}\right|}{\rho}\right)\le\left|x_{ij}-x_{pq}\right|$$

for all $i,p\le m,\ j,q\le n$ and for $m,n\in IN$

Let

$$B_1=\left\{m,n\in IN:M\left(\frac{\left|x_{ij}-x_{pq}\right|}{\rho}\right)<\varepsilon,i,p\le m,j,q\le n\right\}\in I.\tag{4}$$

and

$$B_1^c = \left\{ m,n \in IN : M\left(\frac{\left|x_{ij} - x_{pq}\right|}{\rho}\right) \geq \varepsilon, i, p \leq m, j, q \leq n \right\} \in I. \quad (5)$$

Therefore from (4) and (5) we have,

$$\left\{ m,n \in IN : M\left(\frac{\left|x_{ij} - x_{pq}\right|}{\rho}\right) \geq \varepsilon, i, p \leq m, j, q \leq n \right\} \subset B_1 \cup B_1^c \in I.$$

Hence

$$I - \lim_{m,n} \frac{1}{m^2 n^2} \sum_{i,p\leq m} \sum_{j,q\leq n} \left|x_{ij} - x_{pq}\right| = 0.$$

if and only if

$$I - \lim_{m,n} \frac{1}{m^2 n^2} \sum_{i,p\leq m} \sum_{j,q\leq n} M\left(\frac{\left|x_{ij} - x_{pq}\right|}{\rho}\right) = 0.$$

By an immediate application of Theorem 2.1 we get the desired result.

Corollary 2.4. A sequence $x = (x_{ij})$ is I-convergent to L if and only if

$$I - \lim_{m,n} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \left|x_{ij} - L\right| = 0$$

Proof: Let $M(x) = x.$

We can prove this in the similar manner as in the proof of Corollary 2.3.

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