On Some $I$-Convergent Double Sequence Spaces Defined by a Modulus Function

Vakeel A. Khan, Nazneen Khan

Department of Mathematics, Aligarh Muslim University, Aligarh, India
Email: vakhanmaths@gmail.com, nazneen4maths@gmail.com

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ABSTRACT

In 2000, Kostyrko, Salat, and Wilczynski introduced and studied the concept of $I$-convergence of sequences in metric spaces where $I$ is an ideal. The concept of $I$-convergence has a wide application in the field of Number Theory, trigonometric series, summability theory, probability theory, optimization and approximation theory. In this article we introduce the double sequence spaces $\ell^p(I)$, $\ell^p(f)$ and $\ell^p(\omega)$ for a modulus function $f$ and study some of the properties of these spaces.

Keywords: Ideal; Filter; Modulus Function; Lipschitz Function; $I$-Convergent Field; $I$-Convergent; Monotone and Solid Double Sequence Spaces

1. Introduction

The notion of $I$-Convergence is a generalization of the concept statistical convergence which was first introduced by H. Fast [1] and later on studied by J. A. Fridy [2,3] from the sequence space point of view and linked it with the summability theory. At the initial stage $I$-Convergence was studied by Kostyrko, Salat and Wilczynski [4]. Further it was studied by Salat, Tripathy, Ziman [5] and Demirci [6]. Throughout the article $I$, $J$, $K$ denote the double sequence respectively.

A non-empty family of sets $\mathcal{I}$ is said to be an ideal if $\mathcal{I}$ is additive i.e. $A,B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$ and hereditary i.e. $A \in \mathcal{I}, B \subseteq A \Rightarrow B \in \mathcal{I}$.

An ideal $I \subseteq 2^X$ is called non-trivial if $I \neq 2^X$. A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\}: x \in X\} \subseteq I$.

A non-trivial ideal $I$ is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing $I$ as a subset.

For each ideal $I$, there is a filter $\mathcal{F}(I)$ corresponding to $I$. i.e. $\mathcal{F}(I) = \{K \subseteq N: K^c \in I\}$, where $K^c = N - K$.

The idea of modulus was structured in 1953 by Nakano (See [9]).
A function $f: [0, \infty) \to [0, \infty)$ is called a modulus if

1. $f(t) = 0$ if and only if $t = 0$,
2. $f(t + u) \leq f(t) + f(u)$ for all $t, u \geq 0$,
3. $f$ is nondecreasing, and
4. $f$ is continuous from the right at zero.

Ruckle [10] used the idea of a modulus function $f$ to construct the sequence space

$X(f) = \left\{ x = (x_i): \sum_{i=1}^{\infty} f(|x_i|) < \infty \right\}$.

This space is an FK space, and Ruckle [10] proved that the intersection of all such $X(f)$ spaces is $\phi$, the space of all finite sequences.

The space $X(f)$ is closely related to the space $l_1$, which is an $\ell(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle [11] proved that, for any modulus $f$. 

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\[ X(f) \subset l, \text{ and } X(f)^a = l^\infty \]

where
\[
X(f)^a = \left\{ y = (y_i) \in \omega : \sum_{k=1}^{\infty} f\left( \|y_k\| \right) < \infty \right\}
\]

The space \( X(f) \) is a Banach space with respect to the norm
\[
\|x\| = \sum_{k=1}^{\infty} f\left( \|x_k\| \right) < \infty \quad \text{(See [10])}.
\]

Spaces of the type \( X(f) \) are a special case of the spaces structured by B. Gramsch in [12]. From the point of view of local convexity, spaces of the type \( X(f) \) are quite pathological. Therefore symmetric sequence spaces, which are locally convex have been frequently studied by D. J. H. Garling [13,14], G. Kothe [15] and W. H. Ruckle [10,16].

**Definition 2.1.** A sequence space \( E \) is said to be solid or normal if \( (x_j) \in E \) implies \( (\alpha_j x_j) \in E \) for all sequence of scalars \( (\alpha_j) \) with \( |\alpha_j| < 1 \) for all \( i, j \in \mathbb{N} \) (see [17]).

**Definition 2.2.** Let
\[
K = \left\{ (n, k) : i, j \in \mathbb{N}; n_1 < n_2 < \cdots \text{ and } k_1 < k_2 < k_3 \cdots \right\} \subset \mathbb{N} \times \mathbb{N}
\]

and \( E \) be a double sequence space. A \( K \)-step space of \( E \) is a sequence space
\[
\mathcal{L}^K = \left\{ (\alpha_j x_j) : (x_j) \in E \right\}.
\]

**Definition 2.3.** A canonical preimage of a sequence \( (x_n, k_n) \in E \) is a sequence \( (b_{n,k}) \in E \) defined as follows:
\[
b_{n,k} = \begin{cases} 
\alpha_n, & \text{for } n, k \in K, \\
0, & \text{otherwise}. 
\end{cases}
\]

(See [18].)

**Definition 2.4.** A sequence space \( E \) is said to be monotone if it contains the canonical preimages of all its stepspaces (see [19]).

**Definition 2.5.** A sequence space \( E \) is said to be convergent free if \( (y_j) \in E \), whenever \( (x_j) \in E \) and \( x_j = 0 \) implies \( y_j = 0 \).

**Definition 2.6.** A sequence space \( E \) is said to be a sequence algebra if \( (x_j, y_j) \in E \) whenever \( (x_j) \in E \) and \( (y_j) \in E \).

**Definition 2.7.** A sequence space \( E \) is said to be symmetric if \( (x_{\pi(i)}, y_{\pi(i)}) \in E \) whenever \( (x_j, y_j) \in E \) where \( \pi(i) \) and \( \pi(j) \) is a permutation on \( N \).

**Definition 2.8.** A sequence \( (x_j) \in \omega \) is said to be \( l \)-convergent to a number \( L \) if for every \( \epsilon > 0 \), \( \{i, j\} \in I \cap I \) \( \implies \|x_j - L\| \geq \epsilon \) \( i \in I \). In this case we write \( \lim_{I} x_j = L \).

The space \( c' \) of all \( l \)-convergent sequences to \( L \) is given by
\[
c' = \left\{ (x_j) \in \omega : \{i, j\} \in I \cap I \implies \|x_j - L\| \geq \epsilon \right\} \in I,
\]

for some \( L \in \mathbb{C} \).

**Definition 2.9.** A sequence \( (x_j) \in \omega \) is said to be \( I \)-null if \( L = 0 \). In this case we write \( \lim_{I} x_j = 0 \).

**Definition 2.10.** A sequence \( (x_j) \in \omega \) is said to be \( I \)-cauchy if for every \( \epsilon > 0 \) there exists a number \( m = m(\epsilon) \) and \( n = n(\epsilon) \) such that
\[
\{i, j\} \in I \cap I \implies \|x_j - x_m\| \geq \epsilon \in I.
\]

**Definition 2.11.** A sequence \( (x) \in \omega \) is said to be \( I \)-bounded if there exists \( M > 0 \) such that
\[
\{i, j\} \in I \implies \|x_j\| > M \right\} \in I
\]

**Definition 2.12.** A modulus function \( f \) is said to satisfy \( \Delta_2 \) condition if for all values of \( \epsilon \) there exists a constant \( K > 0 \) such that \( f(Lu) \leq Kf(u) \) for all values of \( L > 1 \).

**Definition 2.13.** Take for the class \( I_f \) of all finite subsets of \( I \). Then \( I_f \) is a non-trivial admissible ideal and \( I_f \)-convergence coincides with the usual convergence with respect to the metric in \( X \) (see [4]).

**Definition 2.14.** For \( I = I_g \) and \( A \subset \mathbb{N} \) with \( \delta(A) = 0 \) respectively. \( I_g \) is a non-trivial admissible ideal, \( I_g \)-convergence is said to be logarithmic statistical convergence (see [4]).

**Definition 2.15.** A map \( h \) defined on a domain \( D \subset X \) i.e. \( h : D \subset X \implies IR \) is said to satisfy Lipschitz condition if \( h(x) - h(y) \leq K|x - y| \) where \( K \) is known as the Lipschitz constant. The class of \( K \)-Lipschitz functions defined on \( D \) is denoted by \( h \in (D, K) \) (see [20]).

**Definition 2.16.** A convergence field of \( I \)-convergence is a set
\[
F(I) = \{ x = (x_i) \in l : \exists \text{ there exists } I - \lim x \in IR \}
\]

The convergence field \( F(I) \) is a closed linear subspace of \( l \) with respect to the supremum norm,
\[
F(I) = l \cap c'.
\]

(See [5]).

Define a function \( h : F(I) \implies IR \) such that
\[
h(x) = \lim_{I} x, \quad \forall x \in F(I), \quad \text{then the function}
\]

\[
h : F(I) \implies IR \quad \text{is a Lipschitz function (see [20]).}
\]

(See [18,20-30])

Throughout the article \( l, c', c', m_i \) and \( m_i' \) represent the bounded, \( l \)-convergent, \( l \)-null, bounded \( l \)-convergent and bounded \( l \)-null sequence spaces respectively.

In this article we introduce the following classes of sequence spaces.

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Theorem 3.1. For any modulus function $f$, the classes of sequences $z c^l (f)$, $z c^0 (f)$, $z m^l (f)$ and $z m^0 (f)$ are linear spaces.

Proof: We shall prove the result for the space $z c^l (f)$.

The proof for the other spaces will follow similarly.

Let $(x_y), (y_y) \in z c^l (f)$ and let $\alpha, \beta$ be scalars. Then

$$I - \lim f \left( \| x_y - L_1 \| \right) = 0, \text{ for some } L_1 \in A;
$$

$$I - \lim f \left( \| y_y - L_2 \| \right) = 0, \text{ for some } L_2 \in A;$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \left\{ (i, j) \in IN \times IN : f \left( \| x_y - L_1 \| \right) > \frac{\epsilon}{2} \right\} \subseteq I,$n

$$A_2 = \left\{ (i, j) \in IN \times IN : f \left( \| y_y - L_2 \| \right) > \frac{\epsilon}{2} \right\} \subseteq I.$$

Since $f$ is a modulus function, we have

$$f \left( \| \alpha x_y + \beta y_y \| - (\alpha L_1 + \beta L_2) \right)$$

$$\leq f \left( \| x_y - L_1 \| \right) + f \left( \| y_y - L_2 \| \right)$$

Now, by (1) and (2),

$$\left\{ i, j \in N : f \left( \| \alpha x_y + \beta y_y \| - (\alpha L_1 + \beta L_2) \right) > \epsilon \right\} \subseteq A_1 \cup A_2.$$

Therefore $(\alpha x_y + \beta y_y) \in z c^l (f)$

Hence $z c^l (f)$ is a linear space.

Theorem 3.2. A sequence $x = (x_y) \in z m^l (f)$ is $l$-convergent if and only if for every $\epsilon > 0$ there exists $I_i, J_i, \in IN$ such that

$$\left\{ (i, j) \in IN \times IN : f \left( \| x_y - x_{i, j} \| \right) < \epsilon \right\} \subseteq z m^l (f)$$

Proof: Suppose that $L = I - \lim x$. Then

$$B_i = \left\{ (i, j) \in IN \times IN : |x_y - L| < \frac{\epsilon}{2} \right\} \subseteq z m^l (f)$$

For all $\epsilon > 0$.

Fix an $I_i, J_i \in B_i$. Then we have

$$|x_{i, j} - x_y| \leq |x_{i, j} - L| + |L - x_y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $i, j \in B_i$.

Hence

$$\left\{ (i, j) \in IN \times IN : f \left( \| x_y - x_{i, j} \| \right) < \epsilon \right\} \subseteq z m^l (f).$$

Conversely, suppose that

$$\left\{ (i, j) \in IN \times IN : f \left( \| x_y - x_{i, j} \| \right) < \epsilon \right\} \subseteq z m^l (f).$$

That is

$$\left\{ (i, j) \in IN \times IN : \left| x_y - x_{i, j} \right| < \epsilon \right\} \subseteq z m^l (f)$$

for all $\epsilon > 0$. Then the set

$$z C_i = \left\{ (i, j) \in IN \times IN : x_y \in [x_{i, j} - \epsilon, x_{i, j} + \epsilon] \right\}$$

$$\subseteq z m^l (f)$$

for all $\epsilon > 0$.

Let $N_i = [x_{i, j} - \epsilon, x_{i, j} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $z C_i \subseteq z m^l (f)$ as well as $z C_i \subseteq z m^l (f)$.

Hence $z C_i \subseteq z m^l (f)$. This implies that

$$\text{diam } N \leq \text{diam } N_i$$

where the diam of $N$ denotes the length of interval $N$.

In this way, by induction we get the sequence of closed intervals

$$N_i = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

with the property that

$$\text{diam } I_i \leq \frac{1}{2^i} \text{diam } I_{(i-1)(i-1)}$$

for

$$i = 2, 3, 4, \ldots$$

and

$$\left\{ (i, j) \in IN \times IN : x_y \in I_i \right\} \subseteq z m^l (f)$$

for

$$i = 1, 2, 3, 4, \ldots.$$
Then there exists a $\zeta \in \bigcap_{i} I_{i}$ where $i, j \in \mathbb{N}$ such that $\zeta = I - \lim x$. So that $f(\zeta) = I - \lim f(x)$, that is $L = I - \lim f(x)$.

**Theorem 3.3.** Let $f$ and $g$ be modulus functions that satisfy the $\Delta_{2}$-condition. If $X$ is any of the spaces $z^{c}, z^{c_{0}}, z^{m}$ and $z^{m_{0}}$, etc. then the following assertions hold.

(i) $X(g) \subseteq X(f \cdot g)$.

(ii) $X(f) \nabla X(g) \subseteq X(f + g)$.

**Proof:** (i) Let $\{x_{i}\} \in \cdot z^{c_{0}}(g)$. Then

$$I - \lim g\left(x_{i}\right) = 0$$

(4)

Let $\epsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 < t < \delta$.

Write $y_{ij} \equiv g\left(x_{i}\right)$ and consider

$$\lim_{i} f\left(y_{ij}\right) = \lim_{i} f\left(y_{ij}\right)_{x_{i} < \delta} + \lim_{i} f\left(y_{ij}\right)_{x_{i} > \delta}$$

We have

$$\lim_{i} f\left(y_{ij}\right) \leq f\left(2\right) \lim_{i} f\left(y_{ij}\right)$$

(5)

For $y_{ij} > \delta$, we have $y_{ij} < y_{ij}^{\delta} < 1 + \frac{y_{ij}^{\delta}}{\delta}$. Since $f$ is non-decreasing, it follows that

$$f\left(y_{ij}\right) < f\left(1 + \frac{y_{ij}^{\delta}}{\delta}\right) < f\left(2\right) + \frac{1}{2} f\left(\frac{2y_{ij}^{\delta}}{\delta}\right)$$

Since $f$ satisfies the $\Delta_{2}$-condition, we have

$$f\left(y_{ij}\right) < \frac{1}{2} K \frac{y_{ij}^{\delta}}{\delta} f\left(2\right) + \frac{1}{2} K \frac{y_{ij}^{\delta}}{\delta} f\left(2\right) = K \frac{y_{ij}^{\delta}}{\delta} f\left(2\right)$$

Hence

$$\lim_{i} f\left(y_{ij}\right) \leq \max \left(1, K\right) \delta^{-1} f\left(2\right) \lim_{i} f\left(y_{ij}\right).$$

(6)

From (4), (5) and (6), we have $\{x_{i}\} \in z^{c_{0}}(f \cdot g)$. Thus $z^{c_{0}}(g) \subseteq z^{c_{0}}(f \cdot g)$. The other cases can be proved similarly.

(ii) Let $\{x_{i}\} \in z^{c_{0}}(f) \sqcap z^{c_{0}}(g)$. Then

$$I - \lim_{i} f\left(x_{i}\right) = 0 \quad \text{and} \quad I - \lim_{i} g\left(x_{i}\right) = 0$$

$$\lim_{i} f\left(g\right)\left(x_{i}\right) = \lim_{i} f\left(x_{i}\right) + g\left(x_{i}\right)$$

$$= \lim_{i} f\left(x_{i}\right) + \lim_{i} g\left(x_{i}\right) = 0$$

Therefore

$$\lim_{i} f\left(g\right)\left(x_{i}\right) = 0$$

which implies $\{x_{i}\} \in X(f + g)$, that is $X(f + g) \subseteq X(f + g)$.

**Corollary 3.4.** $X \subseteq X(f)$ for $X = z^{c}, z^{c_{0}}, z^{m}$ and $z^{m_{0}}$.

**Proof:** The result can be easily proved using $f(x) = x$ for $x = \{x_{i}\} \in X$.

**Theorem 3.5.** The spaces $z^{c_{0}}(f)$ and $z^{m_{0}}(f)$ are solid and monotone.

**Proof:** We shall prove the result for $z^{c_{0}}(f)$. Let $x_{ij} \in z^{c_{0}}(f)$. Then

$$I - \lim_{i} f\left(x_{ij}\right) = 0$$

(7)

Let $\{\alpha_{ij}\}$ be a sequence of scalars with $|\alpha_{ij}| \leq 1$ for all $i, j \in \mathbb{N}$. Then we have

$$I - \lim_{i} f\left(\alpha_{ij}x_{ij}\right) = \lim_{i} f\left(\alpha_{ij}\right)|x_{ij}|$$

$$= |\alpha_{ij}| I - \lim_{i} f\left(x_{ij}\right) = 0$$

$$I - \lim_{i} f\left(\alpha_{ij}x_{ij}\right) = 0 \quad \text{for all } i, j \in \mathbb{N}.$$
Let \( I = I_f \) and \( f(x) = x^3 \) for all \( x \in [0, \infty) \). Consider the sequence \( (x_y) \) and \((y_y)\) defined by
\[
x_y = \frac{1}{i+j} \quad \text{and} \quad y_y = i + j \quad \text{for all} \quad i, j \in \mathbb{N}.
\]
Then \((x_y) \in c^f(f)\) and \(c_y^f(f)\), but \((y_y) \notin c^f(f)\) and \(c^f_y(f)\).

Hence the spaces \( c^f(f) \) and \( c^f_y(f) \) are not convergence free.

Theorem 3.9. If \( I \) is not maximal and \( I \neq I_f \), then the spaces \( c^f(f) \) and \( c^f_y(f) \) are not symmetric.

Proof: Let \( A \in I \) be infinite and \( f(x) = x \) for all \( x \in [0, \infty) \).
\[
x_y = \begin{cases} 1, & \text{for } i, j \in A, \\ 0, & \text{otherwise.} \end{cases}
\]
Then by Lemma (3) we have \( x_y \in c^f_y(f) \subset c^f(f) \).
Let \( K \subset \mathbb{N} \) be such that \( K \notin I \) and \( \mathbb{N} - K \notin I \).
Let \( \phi : K \to A \) and \( \psi : \mathbb{N} - K \to \mathbb{N} - A \) be bijections, then the map \( \pi : \mathbb{N} \to \mathbb{N} \) defined by
\[
\pi(ij) = \begin{cases} \phi(ij), & \text{for } i, j \in K, \\ \psi(ij), & \text{otherwise.} \end{cases}
\]
is a permutation on \( \mathbb{N} \), but \( x_{\pi(ij)} \notin c^f(f) \) and \( x_{\pi(ij)} \notin c^f_y(f) \).
Hence \( c^f_y(f) \) and \( c^f(f) \) are not symmetric.

Theorem 3.10. Let \( f \) be a modulus function. Then \( c^f_y(f) \subset c^f(f) \subset c^f_y(f) \) and the inclusions are proper.

Proof: The inclusion \( c^f_y(f) \subset c^f(f) \) is obvious.
Let \( x = x_y \in c^f_y(f) \). Then there exists \( L \in \mathbb{C} \) such that
\[
I - \lim f(|x_y - L|) = 0.
\]
We have \( f(|x_y|) \leq \frac{1}{2} f(|x_y - L|) + \frac{1}{2} f(|L|) \).

Taking the supremum over \( i \) and \( j \) on both sides we get \( x_y \in c^f_y(f) \).
Next we show that the inclusion is proper.
(i) \( c^f_y(f) \subset c^f(f) \)
Let \( x = (x_y) \in c^f(f) \) then \( I - \lim f(|x_y|) = L \) for some \( L \neq 0 \) \( \in \mathbb{C} \), which implies \( x \notin c^f_y(f) \). Hence the inclusion is proper.
(ii) \( c^f(f) \subset c^f_y(f) \).
Let \( x = (x_y) \in c^f_y(f) \) then
\[
I - \lim f(|x_y|) < \infty \\
I - \lim f(|x_y - L + L|) < \infty \\
I - \lim f(|x_y - L|) + I - \lim f(|L|) < \infty \\
I - \lim f(|x_y - L|) < \infty \\
I - \lim f(|x_y - L|) \neq 0
\]
Therefore \( x \notin c^f_y(f) \), and hence the inclusion is proper.

Theorem 3.11. The function \( h : \mathbb{R}[m(f) \to \mathbb{R} \) is the Lipschitz function, where
\( \mathbb{R}[m(f) = c^f(f) \cap \mathbb{R}[\infty] \), and hence uniformly continuous.

Proof: Let \( x, y \in \mathbb{R}[m(f), x \neq y \). Then the sets
\[
A_x = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_y - h(x)| \geq \|x - y\| \} \in I, \\
A_y = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |y_y - h(y)| \geq \|x - y\| \} \in I. 
\]
Thus the sets,
\[
B_x = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_y - h(x)| < \|x - y\| \in \mathbb{R}[m(f), \\
B_y = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |y_y - h(y)| < \|x - y\| \in \mathbb{R}[m(f).
\]
Hence also \( B_x \cap B_y \in \mathbb{R}[m(f) \), so that \( B \neq \emptyset \).
Now taking \( i, j \) in \( B \),
\[
|h(x) - h(y)| \\
\leq |h(x) - x_y| + |x_y - y_y| + |y_y - h(y)| \\
\leq 3|x - y|.
\]
Thus \( h \) is a Lipschitz function. For \( \mathbb{R}[m(f) \) the result can be proved similarly.

Theorem 3.12. If \( x, y \in \mathbb{R}[m(f) \), then \( (x - y) \in \mathbb{R}[m(f) \) and \( h(x - y) = h(x) - h(y) \).

Proof: For \( \epsilon > 0 \)
\[
B_x = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_y - h(x)| < \epsilon \} \in \mathbb{R}[m(f), \\
B_y = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |y_y - h(y)| < \epsilon \} \in \mathbb{R}[m(f).
\]
Now,
\[
|x_y - h(x)| \\
= |x_y - x_y - h(x) + x_y - h(x) + h(x) - h(x)| \\
\leq |x_y - y_y| + |h(x) - h(y)| + |y_y - h(x)| \\
\leq |x_y - y_y| + |h(x) - h(y)| + |y_y - h(x)| \\
\leq 3|x - y|.
\]
As \( \mathbb{R}[m(f) \subset \mathbb{R}[\infty] \), there exists an \( \mathbb{M} \in \mathbb{R} \) such that \( |x| < \mathbb{M} \) and \( |h(x)| < \mathbb{M} \).
Using Equation (8) we get \( |x - y| - h(x)| \leq \mathbb{M} \epsilon + \mathbb{M} \epsilon = 2\mathbb{M} \epsilon \)
For all \( i, j \in B_x \cap B_y \in \mathbb{R}[m(f) \). Hence \( (x - y) \in \mathbb{R}[m(f) \) and \( h(x) - h(y) \).
For \( \mathbb{R}[m(f) \) the result can be proved similarly.

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