Study on the Existence of Sign-Changing Solutions of Case Theory Based a Class of Differential Equations Boundary-Value Problems

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Abstract

By using the fixed point theorem under the case structure, we study the existence of sign-changing solutions of a class of second-order differential equations three-point boundary-value problems, and a positive solution and a negative solution are obtained respectively, so as to popularize and improve some results that have been known.

Keywords
Case Theory, Boundary-Value Problems, Fixed Point Theorem, Sign-Changing Solutions

1. Introduction

The existence of nonlinear three-point boundary-value problems has been studied [1]-[6], and the existence of sign-changing solutions is obtained. In the past, most studies were focused on the cone fixed point index theory [7] [8] [9], just a few took use of case theory to study the topological degree of non-cone mapping and the calculation of fixed point index, and the case theory was combined with the topological degree theory to study the sign-changing solutions. Recent study Ref. [10] [11] have given the method of calculating the topological degree under the case structure, and taken use of the fixed point theorem of non-cone mapping to study the existence of nontrivial solutions for the nonlinear Sturm-Liouville problems. Relevant studies as [12] [13] [14].

Inspired by the Ref. [8]-[13] and by using the new fixed point theorem under the case structure, this paper studies three-point boundary-value problems for A
class of nonlinear second-order equations
\[
\begin{align*}
\left\{ \begin{array}{l}
u''(t) + f(u(t)) = 0, \quad 0 \leq t \leq 1; \\
u'(0) = 0, \ u(1) = \alpha u(\eta),
\end{array} \right.
\end{align*}
\] (1)

Existence of the sign-changing solution, constant \( 0 < \alpha < 1, 0 < \eta < 1 \), 
\( f \in C(R,R) \).

Boundary-value problem (1) is equivalent to Hammerstein nonlinear integral equation hereunder
\[
u(t) = \int_0^1 G(t,s)f(u(s))\,ds, \quad 0 \leq t \leq 1
\] (2)

Of which \( G(t,s) \) is the Green function hereunder
\[
G(t,s) = \begin{cases} 
(1-s) - \alpha(\eta-s), & 0 \leq s \leq \eta, 0 \leq t \leq s; \\
(1-s), & \eta \leq s \leq 1, 0 \leq t \leq s; \\
(1-\alpha\eta) - t(1-\alpha), & 0 \leq s \leq \eta, s \leq t \leq 1; \\
(1-\alpha\eta) - t(1-\alpha), & \eta \leq s \leq 1, s \leq t \leq 1.
\end{cases}
\]

Defining linear operator \( K \) as follow
\[
(Ku)(t) = \int_0^1 G(t,s)u(s)\,ds, \quad u \in C[0,1].
\] (3)

Let \( Fu(t) = f(u(t)), \quad t \in [0,1] \), obviously composition operator \( A = KF \), i.e.
\[
(Au)(t) = \int_0^1 G(t,s)f(u(s))\,ds, \quad 0 \leq t \leq 1
\] (4)

It’s easy to get: \( u \in C^2[0,1] \) is the solution of boundary-value problem (1), and \( u \in C[0,1] \) is the solution of operator equation \( u = Au \).

We note that, in Ref. [9] [10], an abstract result on the existence of sign-changing solutions can be directly applied to problem (1). After the necessary preparation, when the non-linear term \( f \) is under certain assumptions, we get the existence of sign-changing solution of such boundary-value problems. Compared with the Ref. [8], we can see that we generalize and improve the non-linear term \( f \), and remove the conditions of strictly increasing function, and the method is different from Ref. [8].

For convenience, we give the following conditions.
\[\text{(H}_1\text{)} \quad f(u):R \rightarrow R \text{ continues, } f(u)u > 0, \quad \forall u \in R, u \neq 0, \text{ and } f(0) = 0.\]
\[\text{(H}_2\text{)} \quad \lim_{u \rightarrow 0} \frac{f(u)}{u} = \beta, \quad \text{and } \ n_b \in N, \text{ make } \lambda_{2n_b} < \beta < \lambda_{2n_b+1}, \text{ of which } 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots \text{ is the positive sequence of } \cos \sqrt{x} = \alpha \cos \eta \sqrt{x}.\]
\[\text{(H}_3\text{)} \quad \exists \varepsilon > 0, \text{ make } \lim_{u \rightarrow 0} \sup_{x \in [0,1]} \frac{f(u)}{u} \leq \lambda - \varepsilon.\]

2. Knowledge

Provided \( P \) is the cone of \( E \) in Banach space, the semi order in \( E \) is exported by cone \( P \). If the constant \( N > 0 \), and \( \theta \leq x \leq y \Rightarrow \|x\| \leq N\|y\| \), then \( P \) is a normal
cone; if $P$ contains internal point, i.e. $\text{int} P \neq \emptyset$, then $P$ is a solid cone.

$E$ becomes a case when semi order $\leq$, i.e. any $x, y \in E$, $\sup \{x, y\}$ and $\inf \{x, y\}$ is existed, for $x \in E$, $x^+ = \sup \{x, \theta\}$, $x^- = \sup \{-x, \theta\}$, we call positive and negative of $x$ respectively, call $|x| = x^+ + x^-$ as the modulus of $x$. Obviously, $x^+ \in P$, $x^- \in (-P)$, $|x| \in P$, $x = x^+-x^-$. For convenience, we use the following signs: $x_+ = x^+$, $x_- = -x^-$. Such that

$$x = x_+ + x_-,$$

Provided Banach space $E = C[0,1]$, and $E$'s norm as $\|\cdot\|$, i.e.

$$\|u\| = \max_{t \in [0,1]} |u(t)|$$. Let $P = \{u \in E | u(t) \geq 0, t \in [0,1]\}$, then $P$ is the normal cone of $E$, and $E$ becomes a case under semi order $\leq$.

Now we give the definitions and theorems

**Def 1** [10] provided $D \subseteq E, A : D \to E$ is an operator (generally a nonlinear). If $Ax = Ax_+ + Ax_-, \forall x \in E$, then $A$ is an additive operator under case structure; if $v^* \in E$, and $Ax = Ax_+ + Ax_+ + v^*, \forall x \in E$, then $A$ is a quasi additive operator.

**Def 2** provided $x$ is a fixed point of $A$, if $x \in (P \setminus \{\theta\})$, then $x$ is a positive fixed point; if $x \in (-P \setminus \{\theta\})$, then $x$ is a negative fixed point; if $x \not\in (P \cup (-P))$, then $x$ is a sign-changing fixed point.

**Lemma 1** [6] $G(t,s)$ is a nonnegative continuous function of $[0,1] \times [0,1]$, and when $t, s \in [0,1]$, $G(t,s) \geq \gamma G(0,s)$, of which $\gamma = \alpha (1-\eta) \over 1 - \alpha \eta$.

**Lemma 2** $K : P \to P$ is completely continuous operator, and $A : E \to E$ is completely continuous operator.

**Lemma 3** $A$ is a quasi additive operator under case structure.

**Proof:** Similar to the proofs in Lemma 4.3.1 in Ref. [10], get Lemma 3 works.

**Lemma 4** [6] the eigenvalues of the linear operator $K$ are

$$\lambda_1 = \lambda_2 = \cdots, \lambda_1 = \lambda_2 = \cdots$$. And the sum of algebraic multiplicity of all eigenvalues is $1$, of which $\lambda_1$ is defined by (H$_2$).

The lemmas hereunder are the main study bases.

**Lemma 5** [10] provided $E$ is Banach space, $P$ is the normal cone in $E$, $A : E \to E$ is completely continuous operator, and quasi additive operator under case structure. Provided that

1) There exists positive bounded linear operator $B_1$, and $B_1$’s $r(B_1) < 1$, and $u^* \in P, u^*_1 \in P$, get

$$-u^* \leq Ax \leq B_1 x + u_1, \forall x \in P;$$

2) There exists positive bounded linear operator $B_2$, $B_2$’s $r(B_2) < 1$, and $u_2 \in P$, get

$$Ax \geq B_2 x - u_2, \forall x \in (-P);$$

3) $A\theta = \theta$, there exists Frechet derivative $A_\theta'$ of $A$ at $\theta$, $1$ is not the eigenvalue of $A_\theta'$, and the sum $\mu$ of algebraic multiplicity of $A_\theta'$s all eigenvalues in the range $(1, \infty)$ is a nonzero even number,
Then $A$ exists three nonzero fixed points at least: one positive fixed point, one negative fixed point and a sign-changing fixed point.

### 3. Results

**Theorem** provided (H\(_1\)) (H\(_2\)) (H\(_3\)) works, boundary-value problem (1) exists a sign-changing solution at least, and also a positive solution and a negative solution.

**Proof** provided linear operator $B = \left( \frac{\lambda_i - \varepsilon}{2} \right) K$, Lemma 2 knows $B : C[0,1] \rightarrow C[0,1]$ is a positive bounded linear operator. Lemma 4 gets $K$'s $r(K) = \frac{1}{\lambda_i}$, so $r(B) = \left( \frac{\lambda_i - \varepsilon}{2} \right) r(K) = 1 - \frac{\varepsilon}{2\lambda_i} < 1$.

(H\(_3\)) knows $m > 0$ and gets

$$f(u) \leq \left( \frac{\lambda_i - \varepsilon}{2} \right) u + m, \forall \in [0,1], u \geq 0$$  \hfill (5)

$$f(u) \geq \left( \frac{\lambda_i - \varepsilon}{2} \right) u - m, \forall \in [0,1], u \leq 0$$  \hfill (6)

Let $u_0(t) = \int_0^t G(t,s) ds$, obviously, $u_0 \in P$. Such that, for any $u(t) \in P$, there

$$\left( Au \right)(t) = \int_0^t G(t,s) f(u(s)) ds$$

$$\leq \int_0^t G(t,s) \left( \frac{\lambda_i - \varepsilon}{2} u + m \right) ds$$

$$\leq \left( \frac{\lambda_i - \varepsilon}{2} \right) \int_0^t G(t,s) u(s) ds + m \int_0^t G(t,s) ds$$

$$= \left( \frac{\lambda_i - \varepsilon}{2} \right) Ku(t) + u_0(t)$$

$$= Bu(t) + u_0(t)$$

And for any $u^* \in P$, from (H\(_1\)), obviously gets $\left( Au \right)(t) \geq -u^*(t)$.

For any $u(t) \in -P$, there

$$\left( Au \right)(t) = \int_0^t G(t,s) f(u(s)) ds$$

$$\geq \int_0^t G(t,s) \left( \frac{\lambda_i - \varepsilon}{2} u - m \right) ds$$

$$\geq \left( \frac{\lambda_i - \varepsilon}{2} \right) \int_0^t G(t,s) u(s) ds - m \int_0^t G(t,s) ds$$

$$= \left( \frac{\lambda_i - \varepsilon}{2} \right) Ku(t) - u_0(t)$$

$$= Bu(t) - u_0(t)$$

Consequently (1) (2) in lemma 5 works.
We note that \( f(0) = 0 \) can get \( A\theta = \theta \), from (H2), we know \( \forall \epsilon > 0 \), and \( \exists r > 0 \) gets

\[
|f(u) - \beta u| \leq \epsilon u, |u| \leq r
\]

Then

\[
( Fu)(t) - \lambda u(t) = |f(u(t)) - \beta u(t)| \leq \epsilon \|u\|, \forall \|u\| \leq r
\]

\[
\|Au - A\theta - \beta Ku\| = \|K(Fu) - \beta Ku\| \leq \epsilon \|K\|\|u\|, \forall \|u\| \leq r
\]

Such that

\[
\lim_{t \to \infty} \frac{\|Au - A\theta - \beta Ku\|}{\|u\|} = 0
\]

i.e. \( A' = \beta K \), from lemma 4 we get linear operator K's eigenvalue is \( \lambda_n \), then \( A' \)'s eigenvalue is \( \frac{\beta}{\lambda_n} \). Because \( \lambda_{2n_0} < \beta < \lambda_{2n_0+1} \), let \( \mu \) be the sum of algebraic multiplicity of \( A' \)'s all eigenvalues in the range \((1, \infty)\), then \( \mu = 2n_0 \) is an even number.

From (H1) \( f(u)u > 0 \), \( u \in R \setminus \{0\} \), there

\[
f(u(t)) > 0, \forall t \in [0,1], u(t) > 0,
\]

\[
f(u(t)) < 0, \forall t \in [0,1], u(t) < 0.
\]

Easy to get

\[
F(P \setminus \{\theta\}) \subset P \setminus \{\theta\}, F((-P) \setminus \{\theta\}) \subset (-P) \setminus \{\theta\},
\]

Lemma (1) for any \( u(t) \in P \),

\[
(Ku)(t) = \int_0^t G(t, s)u(s)ds \geq \gamma \int_0^t G(0, s)u(s)ds,
\]

consequently \( K(P \setminus \{\theta\}) \subset P \). Such that

\[
A(P \setminus \{\theta\}) \subset P, A((-P) \setminus \{\theta\}) \subset -P,
\]

Such that (3) in lemma 5 works. According to lemma 5, \( A \) exists three nonzero fixed points at least: one positive fixed point, one negative fixed point and one sign-changing fixed point. Which states that boundary-value problem (1) has three nonzero solutions at least: one positive solution, one negative solution and one sign-changing solution.

4. Conclusion

Provided that all conditions of the theorem are satisfied, and \( f(u) \) is an odd function, then boundary-value problem (1) has four nonzero solutions at least: one positive solution, one negative solution and two sign-changing solutions.

Note

By using case theory to study the topological degree of non-cone mapping and
the calculation of fixed point index, it’s an attempt to combine case theory and topological degree theory, the author thinks it’s an up-and-coming topic and expects to have further progress on that.

References


