From Braided Infinitesimal Bialgebras to Braided Lie Bialgebras

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Abstract

The present paper is a continuation of [1], where we considered braided infinitesimal Hopf algebras (i.e., infinitesimal Hopf algebras in the Yetter-Drinfeld category \(\mathcal{H}_{YD}^{H}\) for any Hopf algebra \(H\)), and constructed their Drinfeld double as a generalization of Aguiar’s result. In this paper we mainly investigate the necessary and sufficient condition for a braided infinitesimal bialgebra to be a braided Lie bialgebra (i.e., a Lie bialgebra in the category \(\mathcal{H}_{YD}^{H}\)).

Keywords

Braided Infinitesimal Bialgebra, Braided Lie Bialgebra, Yetter-Drinfeld Category, Balanceator

1. Introduction

An infinitesimal bialgebra is a triple \((A,m,\Delta)\), where \((A,m)\) is an associative algebra (possibly without unit), \((A,\Delta)\) is a coassociative coalgebra (possibly without counit) such that

\[
\Delta(xy) = xy_1 \otimes y_2 + x_1 \otimes x_2 y, \; x, y \in A.
\]

Infinitesimal bialgebras were introduced by Joni and Rota in [2], called infinitesimal coalgebra there, in the context of the calculus of divided differences [3]. In combinatorics, they were further studied in [4] [5] [6]. Aguiar established the basic theory of infinitesimal bialgebras in [7] [8] by investigating several examples and the notions of antipode, Drinfeld double and the associative Yang-Baxter equation keeping close to ordinary Hopf algebras. In [9], Yau introduced the notion of Hom-infinitesimal bialgebras and extended Aguiar’s main results in [7] [8] to Hom-infinitesimal bialgebras.

One of the motivations of studying infinitesimal bialgebras is that they are


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closely related to Drinfeld’s Lie bialgebras (see [10]). The cobracket $\Delta$ in a Lie bialgebra is a 1-cocycle in Chevalley-Eilenberg cohomology, which is a 1-cocycle in Hochschild cohomology (i.e., a derivation) in an infinitesimal bialgebra. So the compatible condition in an infinitesimal bialgebra can be seen as an associative analog of the cocycle condition in a Lie bialgebra.

Motivated by [1], in which we considered infinitesimal Hopf algebras in the Yetter-Drinfeld categories, called braided infinitesimal Hopf algebras, the natural idea is whether we can obtain braided Lie bialgebras (called generalized $H$-Lie bialgebras in [11] [12]) from braided infinitesimal Hopf algebras. This becomes our motivation of writing this paper.

To give a positive answer to the question above, we organize this paper as follows.

In Section 1, we recall some basic definitions about Yetter-Drinfeld modules and braided Lie bialgerbas. In Section 2, we introduce the notion of the balanceator of a braided infinitesimal bialgerba and show that a braided infinitesimal bialgebra gives rise to a braided Lie bialgebra if and only if the balanceator is symmetric (see Theorem 2.3).

2. Preliminaries

In this paper, $k$ always denotes a fixed field, often omitted from the notation. We use Sweedler’s ([13]) notation for the comultiplication: $\Delta(h) = h_1 \otimes h_2$, for all $h \in H$. Let $H$ be a Hopf algebra. We denote the category of left $H$-modules by $\mathcal{M}$. Similarly, we have the category $\mathcal{M}^H$ of left $H$-comodules. For a left $H$-comodules $(M, \rho)$, we also use Sweedler’s notation: $\rho(m) = m_{(-1)} \otimes m_0$, for all $m \in M$.

A left-left Yetter-Drinfeld module $M$ is both a left $H$-module and a left $H$-comodule satisfying the compatibility condition

$$h m_{(-1)} \otimes h_2 \cdot m_0 = (h_1 \cdot m)_{(-1)} h_2 \otimes (h \cdot m)_0$$

for all $h \in H$ and $m \in M$. The equation (1.1) is equivalent to

$$\rho(h \cdot m) = h m_{(-1)} S(h_1) \otimes (h_2 \cdot m_0).$$

By [14] [15], the left-left Yetter-Drinfeld category $\mathcal{YD}^H$ is a braided monoidal category whose objects are Yetter-Drinfeld modules, morphisms are both left $H$-linear and $H$-colinear maps, and its braiding $C_{\omega}$ is given by

$$C_{\omega}(m \otimes n) = m_{(-1)} \cdot n \otimes m_0,$$

for all $m \in M \in \mathcal{YD}^H$ and $n \in N \in \mathcal{YD}^H$.

Let $A$ be an object in $\mathcal{YD}^H$, the braiding $\tau$ is called symmetric on $A$ if the following condition holds:

$$\left( \left( a_{(-1)} \cdot b \right)_{(-1)} \cdot a_0 \right) \otimes \left( a_{(-1)} \cdot b \right)_0 = a \otimes b,$$

which is equivalent to the following condition:

$$a_{(-1)} \cdot b \otimes a_0 = b_0 \otimes S^{-1}(b_{(-1)}) \cdot a.$$
for any \( a, b \in A \).

In the category \( \mathcal{YD}_H \), we call an (co)algebra simply if it is both a left \( H \)-module (co)algebra and a left \( H \)-comodule (co)algebra. For more details about (co)module-(co)algebras, the reader can refer to [16] [17].

A braided Lie algebra ([11]) in \( \mathcal{YD}_H \), called generalized \( H \)-Lie algebra there, is an object \( L \) in \( \mathcal{YD}_H \) together with a bracket operation \([,]\) : \( L \otimes L \to L \), which is a morphism in \( \mathcal{YD}_H \) satisfying

1. \( H \)-anti-commutativity: \([l, l'] = -[l_{(-1)}, l'_{(-1)}], l, l' \in L\),
2. \( H \)-Jacobi identity:

\[
[l \otimes l' \otimes l''] + \left( (\tau \otimes 1)(l \otimes l' \otimes l'') \right) + \left( (1 \otimes \tau)(l \otimes l' \otimes l'') \right) = 0,
\]

for all \( l, l', l'' \in L \), where \( \{l \otimes l' \otimes l''\} \) denotes \([l, [l', l'']]\) and \( \tau \) the braiding for \( L \).

Let \( A \) be an associative algebra in \( \mathcal{YD}_H \). Assume that the braiding is symmetric on \( A \). Define

\[
[a, b] = ab - (a_{(-1)} \cdot b)a_0, a, b \in A.
\]

Then \( (A,[,]) \) is a braided Lie algebra (see [11]).

A braided Lie coalgebra ([12]) \( \Gamma \) is an object in \( \mathcal{YD}_H \) together with a linear map \( \delta: \Gamma \to \Gamma \otimes \Gamma \) (called the cobracket), which is also a morphism in \( \mathcal{YD}_H \) subject to the following conditions:

1. \( H \)-anti-cocommutativity: \( \delta = -\tau \delta \),
2. \( H \)-coJacobi identity:

\[
(id + (id \otimes \tau)(\tau \otimes id))(id \otimes \delta)\delta = 0,
\]

where \( \tau \) denotes the braiding for \( L \).

Dually, let \( (C, \Delta) \) be a coassociative coalgebra in \( \mathcal{YD}_H \). Assume that the braiding on \( C \) is symmetric. Define \( \delta: C \to C \otimes C \) by

\[
c \mapsto c_1 \otimes c_2 - c_{(-1)} \cdot c_{(-1)} \cdot c_0, c \in C.
\]

Then \( (C, \delta) \) is a braided Lie coalgebras in \( \mathcal{YD}_H \) (see [12]).

A braided Lie bialgebra ([18]) \( (L,[,],\delta) \) in \( \mathcal{YD}_H \), where \( (L,[,]) \) is a braided Lie algebra, and \( (L,\delta) \) is a braided Lie coalgebra, such that the compatibility condition holds:

\[
\delta(x, y) = \left( ([,] \otimes id)(id \otimes \delta) + (id \otimes [,])(\tau \otimes id)(id \otimes \delta)\right)(id \otimes id - \tau)(x \otimes y), x, y \in L,
\]

where \( \tau \) denotes the braiding for \( L \).

3. Main Results

In this section, we will study the relation between braided infinitesimal bialgebras and braided Lie bialgebras as a generalization of Aguiar’s result in [8].

Let \( (A, m, \Delta) \) be a braided \( \varepsilon \)-bialgebra in \( \mathcal{YD}_H \). For any \( x, y, z \in A \), define an action of \( A \) on \( A \otimes A \) by

\[
x \mapsto (y \otimes z) = xy \odot z - x_{(-1)} \cdot y \odot (x_0 \cdot z) \cdot x_{00}.
\]
Then the action $\to$ is a morphism in $H \mathcal{YD}$. In fact, for any $x, y, z \in A$ and $h \in H$, we have

$$h_1 \cdot x \to h_2 \cdot (y \otimes z) = h_1 \cdot x \to (h_2 \cdot y \otimes h_3 \cdot z)$$

$$= (h_1 \cdot x)(h_2 \cdot y) \otimes (h_3 \cdot z) - (h_1 \cdot x)_{(1)} \cdot h_2 \cdot y \otimes ((h_1 \cdot x)_{(2)} \cdot h_3 \cdot z)(h_1 \cdot x)_{(0)}$$

$$= (h_1 \cdot x)(h_2 \cdot y) \otimes (h_3 \cdot z) - h_1 x_{(-1)} \cdot S(h_3) \cdot h_2 \cdot y \otimes ((h_2 \cdot x_{(0)})_{(-1)} \cdot h_3 \cdot z)(h_2 \cdot x_{(0)})$$

$$= (h_1 \cdot x)(h_2 \cdot y) \otimes (h_3 \cdot z) - h_1 x_{(-1)} \cdot y \otimes ((h_2 \cdot x_{(0)})_{(-1)} \cdot h_3 \cdot z)(h_2 \cdot x_{(0)})$$

$$= h_1 (xy) \otimes (h_2 \cdot z) - h_1 x_{(-1)} \cdot y \otimes h_2 \cdot ((x_{(0)} \cdot z) x_{(0)})$$

$$= h_1 (xy) \otimes (h_2 \cdot z) - h_1 x_{(-1)} \cdot y \otimes h_2 \cdot ((x_{(0)} \cdot z) x_{(0)})$$

So $\to$ is left $H$-linear. To show the left $H$-colinearity of the action $\to$, we compute

$$\rho(x \to (y \otimes z)) = \rho(xy \otimes z - x_{(-1)} \cdot y \otimes (x_{(0)} \cdot z) x_{(0)})$$

$$= x_{(-1)}y_{(-1)}z_{(-1)} \otimes (x_{(-1)} \cdot y)_{(-1)} \cdot (x_{(0)} \cdot z)_{(-1)} \otimes (x_{(-1)} \cdot y)_{(0)} \otimes (x_{(0)} \cdot z)_{(0)} x_{(0)}$$

$$= x_{(-1)}y_{(-1)}z_{(-1)} \otimes (x_{(-1)} \cdot y)_{(-1)} \cdot (x_{(0)} \cdot z)_{(-1)} \otimes (x_{(-1)} \cdot y)_{(0)} \otimes (x_{(0)} \cdot z)_{(0)} x_{(0)}$$

and

$$(id \otimes \to) \rho(x \otimes y \otimes z) = (id \otimes \to)(x_{(-1)}y_{(-1)}z_{(-1)} \otimes x_0 \otimes y_0 \otimes z_0)$$

$$= x_{(-1)}y_{(-1)}z_{(-1)} \otimes (x_{(-1)} \cdot y)_{(-1)} \cdot (x_{(0)} \cdot z)_{(-1)} \otimes (x_{(-1)} \cdot y)_{(0)} \otimes (x_{(0)} \cdot z)_{(0)} x_{(0)}$$

$$= x_{(-1)}y_{(-1)}z_{(-1)} \otimes (x_{(-1)} \cdot y)_{(-1)} \cdot (x_{(0)} \cdot z)_{(-1)} \otimes (x_{(-1)} \cdot y)_{(0)} \otimes (x_{(0)} \cdot z)_{(0)} x_{(0)}$$

as desired.

**Definition 2.1.** Let $(A, m, \Delta)$ be a braided infinitesimal bialgebra and $\tau$ the braiding of $A$. The map $B : A \otimes A \to A \otimes A$ defined by

$$B(x, y) = x \to \tau \Delta(y) + \tau(y \to \tau \Delta(x)), x, y \in A,$$

(3.1)

is called the balanceator of $A$. The balanceator $B$ is called symmetric if $B = B \circ \tau$. The braided infinitesimal bialgebra $A$ is called balanced if $B \equiv 0$ on $A$.

Condition (2.1) can be written as follows:

$$B(x, y) = x (y_{(0)} \cdot y_{(2)}) \otimes y_{(0)} - x_{(-1)}y_{(-1)}y_{(-1)} \cdot y_{(0)} \otimes (x_{(0)} \cdot y_{(0)}) x_{(0)}$$

$$+ (x_{(-1)} \cdot y)_{(-1)} \cdot x_{(0)} \otimes (x_{(0)} \cdot y)_{(0)} x_{(0)} - ((x_{(0)} \cdot y)_{(-1)} \cdot x_{(0)}) (x_{(0)} \cdot y)_{(0)} \otimes x_{(0)}$$

Obviously,

$$B(x_{(-1)} \cdot y, x_{(0)}) = (x_{(-1)} \cdot y) (x_{(0)} \cdot x_{(0)}) \otimes x_{(0)} + (x_{(-1)} \cdot y) x_{(0)} \otimes x_{(0)} x_{(0)}$$

$$- (x_{(-1)} \cdot y) x_{(0)} \otimes (x_{(0)} \cdot x_{(0)}) x_{(0)} - (x_{(-1)} \cdot y) x_{(0)} \otimes (x_{(0)} \cdot x_{(0)}) x_{(0)}$$

$$= (x_{(-1)} \cdot y) x_{(0)} \otimes x_{(0)} x_{(0)} - (x_{(-1)} \cdot y) x_{(0)} \otimes (x_{(0)} \cdot x_{(0)}) x_{(0)}.$$
Lemma 2.2. Let \((A, m, \Delta)\) be a braided infinitesimal bialgebra and \(x, y \in A\). Assume that the braiding \(\tau\) on \(A\) is symmetric. Then the following equations hold:

1. \(\left( (x_{(-1)} \cdot y)_{(-1)} \right) \cdot x_{01} \otimes \left( (x_{(-1)} \cdot y)_{q(-1)} \right) \cdot x_{02} \left( (x_{(-1)} \cdot y)_{0} \right) = x_{1} \otimes x_{2} y,\)

2. \(\left( x_{(-1)} \cdot y \right)_{(-1)} \cdot x_{00(-1)} \cdot x_{02} \left( (x_{(-1)} \cdot y)_{0} \right) = x_{(-1)} \cdot (x_{2} y) \otimes x_{10}, \)

3. \(\left( x_{(-1)} \cdot y \right)_{(-1)} \left( (x_{(-1)} \cdot y)_{2} \right)_{0} \otimes \left( x_{(-1)} \cdot y \right)_{1} = \left( x_{(-1)} \cdot y \right)_{(-1)} \cdot y_{2} \right) x_{0} \otimes y_{10}.\)

Proof. (1) Since the braiding \(\tau\) on \(A\) is symmetric, for all \(x, y \in A\), we have \(\left( (x_{(-1)} \cdot y)_{(-1)} \right) \cdot x_{0} \otimes \left( (x_{(-1)} \cdot y)_{(-1)} \right) \cdot x_{0} \left( (x_{(-1)} \cdot y)_{0} \right) = x \otimes y,\)

so (1) holds.

(2) To show the Equation (2.2), we need the following computation:

The last equality holds since \(\tau\) is symmetric on \(A\). Hence (2) holds.

(3) Finally, we check the Equation (2.3) as follows:

The last equality holds since \(\tau\) is symmetric on \(A\). Hence (3) holds.
The last equality holds since \( \tau \) is symmetric on \( A \). Hence (3) holds as required.

**Theorem 2.3.** Let \((A, m, \Delta)\) be a braided infinitesimal bialgebra. Assume that the braiding \( \tau \) on \( A \) is symmetric. Then \((A, [\cdot] = m - m\tau, \delta = \Delta - \tau\Delta)\) is a braided Lie bialgebra if and only if \( B = B \circ \tau \).

**Proof.** Since \((A, m)\) is an associative algebra and \((A, \Delta)\) is a coassociative coalgebra in \( \mathcal{O} \), \((A, [\cdot] = m - m\tau)\) is a braided Lie algebra and \((A, \delta = \Delta - \tau\Delta)\) is a braided Lie coalgebra. Therefore it remains to check the compatible condition:

\[
\delta[x, y] = \left( ([\cdot] \otimes id)(id \otimes \delta) + (id \otimes [\cdot])(\tau \otimes id)(id \otimes \delta) \right)(id \otimes id \tau)(x \otimes y),
\]

for all \( x, y \in A \). In fact, on the one hand, we have

\[
\delta[x, y] = \delta(xy - (x_{[-1]} \cdot y)x_0)
= (1 - \tau)\Delta(xy) - (1 - \tau)\Delta((x_{[-1]} \cdot y)x_0)
= (1 - \tau)(x_1 \otimes x_2y + x_1 \otimes y_2)
- (1 - \tau)((x_{[-1]} \cdot y_1) \otimes (x_{[-1]} \cdot y_2)x_0 + (x_{[-1]} \cdot y_1)x_0 \otimes x_02)
= x_0 \otimes x_2y + x_1 \otimes y_2 - x_{[-1]} \cdot (x_{[-1]} \cdot y_1) \otimes (x_{[-1]} \cdot y_2)x_0
+ (x_{[-1]} \cdot y_1)_0 \cdot \left((x_{[-1]} \cdot y_2)x_0 \otimes (x_{[-1]} \cdot y_1)_0
+ \left((x_{[-1]} \cdot y_1)_0 \otimes (x_{[-1]} \cdot y_2)x_0 \otimes (x_{[-1]} \cdot y_1)_0
\right).
\]

On the other hand, we have

\[
(([\cdot] \otimes id)(id \otimes \delta) + (id \otimes [\cdot])(\tau \otimes id)(id \otimes \delta))(id \otimes id \tau)(x \otimes y)
= \left( ([\cdot] \otimes id)(id \otimes \delta) + (id \otimes [\cdot])(\tau \otimes id)(id \otimes \delta) \right)(xy - (x_{[-1]} \cdot y)x_0)
= x_1 \otimes y_2 - x_{[-1]} \cdot y_1 x_0 \otimes y_2 - x_{[-1]} \cdot y_1 \otimes y_1_0
+ x_{[-1]} \cdot y_1 \otimes y_0 \cdot y_1_0 - (x_{[-1]} \cdot y_1)x_0 \otimes x_02
+ \left((x_{[-1]} \cdot y_1)_0 \otimes (x_{[-1]} \cdot y_0 \cdot x_0 \otimes (x_{[-1]} \cdot y_1)_0
+ \left((x_{[-1]} \cdot y_1)_0 \otimes (x_{[-1]} \cdot y_0 \cdot y_1_0
- x_{[-1]} \cdot y_1 \otimes x_0 \cdot y_0 \cdot x_0 \otimes x_010
+ x_{[-1]} \cdot y_1 \otimes x_0 \cdot y_2 - \left((x_{[-1]} \cdot y_1)_0 \otimes (x_{[-1]} \cdot y_2)x_0 \otimes (x_{[-1]} \cdot y_1)_0
\right) \otimes x_010
- (x_{[-1]} \cdot y_1 \cdot y_2^0 x_{[-1]} \cdot y_1 \cdot y_2 \otimes x_0 \cdot y_1_0
+ x_{[-1]} \cdot y_1 \cdot y_2 \otimes (x_{[-1]} \cdot y_2^0 x_{[-1]} \cdot y_2 \otimes x_0 \cdot y_1_0
+ \left((x_{[-1]} \cdot y_2^0 x_{[-1]} \cdot y_2 \otimes x_0 \otimes (x_{[-1]} \cdot y_1)_0
= \left((x_{[-1]} \cdot y_2^0 x_{[-1]} \cdot y_2 \otimes x_0 \otimes (x_{[-1]} \cdot y_1)_0
\right).
According to Lemma 2.2, we have
\[
x y_1 \otimes y_2 + \left( x_{(1)} y_{(1)} \cdot y_2 \right) x_0 \otimes y_{10} - \left( x_{(1)} \cdot y \right) x_{01} \otimes x_{02}
\]
\[- \left( x_{(1)} \cdot y \right)_{(-1)} x_{0q(-1)} \otimes x_{02} - x_{(1)} y_{(-1)} y_0 \otimes x_{00} - x_{(1)} y_{(-1)} y_0 \otimes x_{00}
\]
\[+ x_{(1)} \cdot y_{(-1)} \cdot x_{01} \otimes \left( x_{(-1)} \cdot y \right)_{(-1)} \cdot x_{02} \otimes \left( x_{(-1)} \cdot y \right)_{00}
\]
\[= x y_1 \otimes y_2 + \left( x_{(1)} y_{(1)} \cdot y_2 \right) x_0 \otimes y_{10} - \left( x_{(1)} \cdot y \right) x_{01} \otimes x_{02}
\]
\[= \delta\left[ x, y \right].
\]
Therefore,
\[
\left( \left[ \left[ \cdot \right] \otimes id \right](id \otimes \delta) + \left( id \otimes \left[ \cdot \right] \right)(\tau \otimes id) \left( id \otimes \delta \right) \right)\left( id \otimes id - \tau \right)(x \otimes y)
\]
\[= \delta\left[ x, y \right] - x \left( y_{(1)} \cdot y_2 \right) \otimes y_{10} + \left( x_{(1)} \cdot y \right)_{(-1)} \cdot x_{01} \otimes \left( x_{(-1)} \cdot y \right)_{(-1)} \cdot x_{02}
\]
\[+ x_{(1)} y_{(-1)} \cdot y_0 \otimes \left( x_{(-1)} \cdot y \right)_{(-1)} \cdot x_{02} \otimes \left( x_{(1)} \cdot y \right)_{00}
\]
\[= \delta\left[ x, y \right] - B\left( x, y \right) + B\left( x_{(-1)} \cdot y, x \right)
\]
\[= \delta\left[ x, y \right] - B\left( x, y \right) + B \circ \tau\left( x, y \right),
\]
as desired. We complete the proof.

**Corollary 2.4.** Let \( \left( A, m, \Delta \right) \) be a braided infinitesimal bialgebra. Assume that the braiding \( \tau \) on \( A \) is symmetric and the balanceator \( B = 0 \). Then \( \left( A, \left[ \cdot \right] = m - m \tau, \delta = \Delta - m \Delta \right) \) is a braided Lie bialgebra.

Proof. Straightforward from Theorem 2.3.

**Example 2.5.** Let \( q \) be an 2th root of unit of \( k \) and \( G \) the cyclic group of order 2 generated by \( g \). \( \mathbf{H} = kG \) be the group algebra in the usual way. We consider the algebra \( A_4 = k\left[ x^j \right] / \left( x^j \right)^2 \). By [8], \( A_4 \) is a infinitesimal bialgebra equipped with the comultiplication:
\[
\Delta(1) = 0, \Delta(x) = x \otimes x^2 - 1 \otimes x^3, \Delta\left( x^2 \right) = x^2 \otimes x^2, \Delta\left( x^3 \right) = x^3 \otimes x^3.
\]
Define the left-\( H \)-module action and the left-\( H \)-comodule coaction of \( A \) by
\[
g^i \cdot x^j = q^i x^j, \rho\left( x^j \right) = g^j \otimes x^j, \quad i = 0, 1, \quad j = 0, 1, 2, 3.
\]
It is not hard to check that the multiplication and the comultiplication are
both $H$-linear and $H$-colinear, therefore $A$ is a braided infinitesimal bialgebra. Since $B(x,x) = 2x^2 \otimes x^2 - qx \otimes x^2 - qx^2 \otimes x - x \otimes x^3$ and 
\[ \tau(x \otimes x) = (x_{(-1)} \cdot x)x_0 = (g \cdot x)x = qx \otimes x, \]
it is clear that $B(x,x) = B\tau(x,x)$ if and only if $q = 1$. If $q = 1$, it is not hard to check that the bialgebra is symmetric on $A$. By Theorem 2.3, $(A,\tau) = m - mr, \delta = \Delta - \tau\Delta$ is a braided Lie bialgebra.

**Example 2.6.** Let $q$ be a 4th root of unit of $k$. Consider the Hopf algebra $H = kG$, where $G$ is a cyclic group of order 4 generated by $g$. Recall from [1] that $A = M_2(k)$ is a braided infinitesimal bialgebra in $\mathcal{YD}$ equipped with the comultiplication:
\[ \Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \]
and the $H$-module action, the $H$-comodule coaction:
\[ g^k \cdot E_{ij} = q^{2\delta(i+j)}E_{ij}, \quad \rho(E_{ij}) = g^{2\delta(i+j)} \otimes E_{ij}, \quad k = 0,1,2,3, i,j = 1,2. \]

Since
\[ B(E_{11},E_{21}) = 2(E_{12} \otimes E_{22} - E_{11} \otimes E_{12}), \]
\[ B(E_{11},E_{11}) = B(E_{21},E_{11}) = 2(E_{22} \otimes E_{12} - E_{11} \otimes E_{11}), \]
we claim that the balanceator is not symmetric. By Theorem 2.3, $(A,\tau) = m - mr, \delta = \Delta - \tau\Delta$ is not a braided Lie bialgebra, where $m$ is the multiplication of $A$.

Let $A = \{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a,b \in k \} \subset M_2(k)$. It is clear that $A$ is both $H$-stable and $H$-costable, hence $A$ is also a braided infinitesimal bialgebra contained in $A$. One can check easily that the balanceator $B = 0$ on $A$. By Corollary 2.4, $(A,\tau) = m - mr, \delta = \Delta - \tau\Delta$ is a braided Lie bialgebra.

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