Hadamard Gaps and $K_N$-type Spaces in the Unit Ball

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Received: April 3, 2017
Accepted: April 22, 2017
Published: April 25, 2017

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Abstract

In this paper, we introduce a class of holomorphic Banach spaces $N_K$ of functions on the unit ball $B$ of $\mathbb{C}^n$. We develop the necessary and sufficient condition for $N_K(B)$ spaces to be non-trivial and we discuss the nesting property of $N_K(B)$ spaces. Also, we obtain some characterizations of functions with Hadamard gaps in $N_K(B)$ spaces. As a consequence, we prove a necessary and sufficient condition for that $N_K(B)$ spaces coincides with the Beurling-type space.

Keywords

$N_K$-type Spaces, Beurling-Type Space, Hadamard Gaps

1. Introduction

Through this paper, $B$ is the unit ball of the $n$-dimensional complex Euclidean space $\mathbb{C}^n$, $S$ is the boundary of $B$. We denote the class of all holomorphic functions, with the compact-open topology on the unit ball $B$ by $H(B)$.

For any $z = (z_1, z_2, \ldots, z_n)$, $w = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$, the inner product is defined by $\langle z, w \rangle = (\overline{z_1}w_1, \overline{z_2}w_2, \ldots, \overline{z_n}w_n)$, and write $|z| = \sqrt{\langle z, z \rangle}$.

Let $dv$ be the Lebesgue volume measure on $\mathbb{C}^n$, normalized so that $v(B) \equiv 1$ and $d\sigma$ be the surface measure on $S$. Once again, we normalize $\sigma$ so that $\sigma(B) \equiv 1$. For $z \in B$ and $r > 0$ let $B_r = \{z \in B : |z| \leq r\}$.

For $\zeta \in \mathbb{B}$ the measures $v$ and $\sigma$ are related by the following formula:

$$\int_{B_r} dv = 2\pi \frac{1}{r} \int_0^{2\pi} f(r\varphi) d\varphi.$$

The identity
\[
\int_{\mathbb{B}} f \, d\sigma = \int_{\mathbb{B}} d\sigma(z) \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}z) \, d\theta,
\]

is called integration by slices, for all \(0 \leq \theta \leq 2\pi\) (see [1]).

For every point \(a \in \mathbb{B}\) the Möbius transformation \(\varphi_a : \mathbb{B} \to \mathbb{B}\) is defined by

\[
\varphi_a(z) = \frac{a - P_a(z) - S_a Q_a(z)}{1 - \langle z, a \rangle},
\]

where \(S_a = \sqrt{1 - |a|^2}, P_a(z) = \frac{a(z, a)}{|a|}, P_0 = 0\) and \(Q_a = I - P_a(z)\) (see [1] or [2]).

The map \(\varphi_a\) has the following properties that \(\varphi_a(0) = a\), \(\varphi_a(a) = 0\), \(\varphi_a = \varphi_a^{-1}\) and

\[
1 - \langle \varphi_a(z), \varphi_a(w) \rangle = \frac{(1 - |z|^2)(1 - |z|^2)}{(1 - |a|^2)(1 - |a|^2)},
\]

where \(z\) and \(w\) are arbitrary points in \(\mathbb{B}\). In particular,

\[
1 - |\varphi_a(z)|^2 = \frac{(1 - |z|^2)(1 - |z|^2)}{(1 - |a|^2)(1 - |a|^2)},
\]

For \(a \in \mathbb{B}\) the Möbius invariant Green function in the unit ball \(\mathbb{B}\) denoted by \(G(z, a) = g(\varphi_a(z))\) where \(g(z)\) is defined by:

\[
g(z) = \frac{n+1}{2n} \int_0^1 (1 - t^2)^{n-1} t^{1-2n} \, dt.
\]

For \(n > 1\), we have

\[
\frac{1}{C_n} (1 - r^2)^n r^{-q_1(n-1)} \leq C_n (1 - r^2)^n r^{-q_1(n-1)},
\]

where \(C_n\) is a constant depending on \(n\) only.

Let \(H^\infty(\mathbb{B})\) denote the Banach space of bounded functions in \(\mathcal{H}(\mathbb{B})\) with the norm \(\|f\|_\infty = \sup_{z \in \mathbb{B}} |f(z)|\).

For \(\alpha > 0\), the Beurling-type space (sometimes also called the Bers-type space) \(H^\infty_\alpha(\mathbb{B})\) in the unit ball \(\mathbb{B}\) consists of those functions \(f \in \mathcal{H}(\mathbb{B})\) for which

\[
\|f\|_{\alpha(\mathbb{B})} = \sup_{z \in \mathbb{B}} |f(z)| (1 - |z|^2)\alpha < \infty.
\]

Let \(K : (0, \infty) \to [0, \infty)\) is a right-continuous, non-decreasing function and is not equal to zero identically. The \(N^K(\mathbb{B})\) space consists of all functions \(f \in \mathcal{H}(\mathbb{B})\) such that

\[
\|f\|_{K(\mathbb{B})} = \sup_{z \in \mathbb{B}} \left|f(z)\right| K(G(z, a)) dv(z) < \infty.
\]

Clearly, if \(K(t) = t^p\), then \(N^K(\mathbb{B}) = N^p(\mathbb{B})\). For \(K(t) = 1\) it gives the Bergman space \(A^2(\mathbb{B})\). If \(N^K(\mathbb{B})\) consists of just the constant functions, we say that it is trivial.
We assume from now that all \( K: (0, \infty) \rightarrow [0, \infty) \) to appear in this paper are right-continuous and nondecreasing function, which is not equal to 0 identically.

In [3], several basic properties of \( \mathcal{N}_K (\mathbb{B}) \) are proved, in connection with the Beurling-type space \( H^\infty_\alpha (\mathbb{B}) \). In particular, an embedding theorem for \( \mathcal{N}_K (\mathbb{B}) \) and \( H^\infty_\alpha (\mathbb{B}) \) is obtained, together with other useful properties. Hadamard gaps series and Hadamard product on \( \mathcal{N}_K \) spaces of holomorphic function in the case of the unit disk has been studied quite well in [4] and [5].

Through this, paper, given two quantities \( f_A \) and \( f_B \) both depending on a function \( f \in \mathcal{H}(\mathbb{B}) \), we are going to write \( f \sim f_{AB} \) if there exists a constant \( C \) independent of \( f \), such that \( f_A \leq C f_{AB} \) for all \( f \). When \( f_A \sim f_B \leq f_A \), we write \( f \approx f_{AB} \). If the quantities \( f_A \) and \( f_B \) are equivalent, then in particular we have \( f_A < \infty \) if and only if \( f_B < \infty \). As usual, the letter \( C \) will denote a positive constant, possibly different on each occurrence.

In this paper, we introduce \( \mathcal{N}_K (\mathbb{B}) \) spaces, in terms of the right continuous and non-decreasing function \( K: (0, \infty) \rightarrow [0, \infty) \) on the unit ball \( \mathbb{B} \). We discuss the nesting property of \( \mathcal{N}_K (\mathbb{B}) \). We prove a sufficient condition for \( \mathcal{N}_K (\mathbb{B}) = H^\infty_\alpha (\mathbb{B}) \), \( \alpha = \frac{n+1}{2} \) (the Beurling-type space). Also we generalize the necessary condition to \( \mathcal{N}_K (\mathbb{B}) \) for a kind of lacunary series. As application, we show that the sufficient condition is also a necessary to \( \mathcal{N}_K (\mathbb{B}) = H^\infty_\frac{n}{n+1} (\mathbb{B}) \).

2. \( \mathcal{N}_K \) Spaces in the Unit Ball

In this section we prove some basic Banach space properties of \( \mathcal{N}_K (\mathbb{B}) \) space. A sufficient and necessary condition for \( \mathcal{N}_K (\mathbb{B}) \) to be non-trivial is given. We discuss the nesting property of \( \mathcal{N}_K (\mathbb{B}) \) spaces and prove a sufficient condition for \( \mathcal{N}_K (\mathbb{B}) = H^\infty_\frac{n}{n+1} (\mathbb{B}) \).

**Lemma 2.1**

Let \( f(z) = \sum_{k=1}^\infty a_k z^k \) be a non-constant function, where \( k = (k_1, k_2, \ldots, k_n) \) is an \( n \)-tuple of non-negative integers and \( z^k = (z_1^{k_1}, z_2^{k_2}, \ldots, z_n^{k_n}) \).

Then, \( z^k \in \mathcal{N}_K (\mathbb{B}) \) if \( a_k \neq 0 \).

**Proof:**

Let \( k \) be such that \( k_1 \neq 0 \) and let \( F_k(z) = a_k z^k \). Suppose that

\[
U_\theta f(z) = f(z_1 e^{i\theta}, z_2 e^{i\theta}, \ldots, z_n e^{i\theta}) = f \circ U_\theta(z),
\]

where \( U_\theta(z) = (z_1 e^{i\theta}, z_2 e^{i\theta}, \ldots, z_n e^{i\theta}) \). Then, we have

\[
F_k(z) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(z_1 e^{i\theta}, \ldots, z_n e^{i\theta}) e^{-i\theta_1} \cdots e^{-i\theta_n} d\theta_1 \cdots d\theta_n
= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} (U_\theta f)(z) e^{-i\theta_1} \cdots e^{-i\theta_n} d\theta_1 \cdots d\theta_n.
\]
By Jensen’s inequality on convexity,
\[ |F_k(z)|^2 \leq \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} |U_{\alpha}f(z)|^2 \, d\theta_1 \cdots d\theta_n. \tag{10} \]

Consequently,
\[ \int_{\overline{B}} |F_k(z)|^2 K(G(z,a)) \, d\lambda(z) \leq \|U_{\alpha}f\|^2_k \leq \|U_{\alpha}f\|^2 \leq \|U_{\alpha}f\|^2_k. \tag{11} \]

Because \( U_{\alpha}(z) \in Aut(\mathbb{B}) \) we have \( \|U_{\alpha}f\|_k = \|f\|_k \). Therefore,
\[ \|F_kf\|_k = \|a_k z^k\|_k \leq \|f\|_k \]
and \( z^k \in N_k(\mathbb{B}) \). The lemma is proved.

**Theorem 2.1** The Holomorphic function spaces \( N_k(\mathbb{B}) \), contains all polynomials if
\[ \int_0^{2\pi} K(g(r)) \, dr < \infty. \tag{12} \]

Otherwise, \( N_k(\mathbb{B}) \) contains only constant functions.

**Proof:**
First assume that (12) holds. Let \( f(z) \) be a polynomial i.e. (there exists a \( M > 0 \) such that \( |f(z)|^2 \leq M, \forall z \in \overline{B} = \mathbb{B} \cup \mathbb{S} \)). Then,
\[ \int_{\overline{B}} |f(z)|^2 K(G(z,a)) \, dv(z) = 2n \int_0^{2\pi} r^{2n-1} K(g(r)) \, dr \int_{\overline{B}} |f(\phi_{\alpha}(r\zeta))|^2 \, d\sigma(\zeta) \tag{13} \]
\[ \leq 2nM \int_0^{1} r^{2n-1} K(g(r)) \, dr. \]

Since \( a \) is arbitrary, it follows that
\[ \|f\|_k^2 \leq 2nM \int_0^{1} r^{2n-1} K(g(r)) \, dr < \infty. \tag{14} \]

Thus, \( f \in N_k(\mathbb{B}) \) and the first half of the theorem is proved.

Now, we assume that the integral in (12) is divergent. Let \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \) is an \( n \)-tuple of non-negative integers \( |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \geq 1 \), \( f(z) = z^\alpha \).

Then, we have \( |f(\zeta)|^2 = r^{3|\alpha|} |z|^\alpha \) and
\[ \int_{\overline{B}} |f(\zeta)|^2 \, d\sigma(\zeta) \geq r^{3|\alpha|} \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!} \geq C r^{3|\alpha|}. \tag{15} \]

Thus,
\[ \|f\|_k \geq \frac{nC}{2^{3|\alpha|+\frac{1}{2}}} \int_0^{1} r^{2n-1} K(g(r)) \, dr. \tag{16} \]
There exists \( a \in \mathbb{B} \) such that \( f(a) \neq 0 \), by the subharmonicity of \( f \circ \varphi_a(r^x) \),

\[
\|f\|_K \geq \frac{3n}{2} \left| f(a) \right| \int_0^{1/2} \frac{r^{2n-1}}{(1-r^2)^{n+1}} K(g(r)) \, dr. \tag{17}
\]

Combining (17) and (18), we see that (12) implies that \( \|f\|_K = \infty \).

It is proved that \( f \notin \mathcal{N}_K(\mathbb{B}) \) and, since \( \alpha \) is arbitrary, any non-constant polynomial is not contained in \( \mathcal{N}_K(\mathbb{B}) \). Using Lemma 2.1, we conclude that \( \mathcal{N}_K(\mathbb{B}) \) contains only constant functions. The theorem is proved.

**Theorem 2.2**

Let \( K_1 \) and \( K_2 \) satisfy (12). If there exist a constant \( t_0 > 0 \) such that \( K_2(t) \leq K_1(t) \) for \( t \in (0, t_0) \), then \( \mathcal{N}_{K_1}(\mathbb{B}) \subseteq \mathcal{N}_{K_2}(\mathbb{B}) \). As a consequence, \( \mathcal{N}_{K_1}(\mathbb{B}) = \mathcal{N}_{K_2}(\mathbb{B}) \) if \( K_2(t) \approx K_1(t) \) for \( t \in (0, t_0) \).

**Proof:** Let \( f \in \mathcal{N}_{K_1}(\mathbb{B}) \). We note that from the property of \( g(z) \), there exists a constant \( \delta > 0 \), such that \( g(z) < t_0 \) if \( |z| > \delta \). Then, we have

\[
\int_{\mathbb{B}} \left( f(z) \right)^2 K_2(G(z, a)) \, dv(z) = \int_{\mathbb{B}} \left( f(a) \right)^2 K_2(g(z)) \, dv(z)\tag{18}
\]

where

\[
\int_{\mathbb{B}} \left( f(a) \right)^2 K_2(g(z)) \, dv(z) \leq \|f\|_K^2 \int_{\mathbb{B}} \left( 1-|z|^2 \right)^n K_2(g(z)) \, dv(z)
\]

\[
\leq 2n \|f\|_K^2 \int_{0}^{\delta} r^{2n-1} K_2(g(r)) \, dr < \infty,
\]

and

\[
\int \left( f(a) \right)^2 K_1(g(z)) \, dv(z)
\]

\[
\leq \int \left( f(a) \right)^2 K_1(g(z)) \, dv(z) \leq \|f\|_K^2 < \infty.
\]

This shows that \( \|f\|_{K_2} < \infty \) and, consequently, \( f \in \mathcal{N}_{K_2}(\mathbb{B}) \).

**Theorem 2.3**

Let \( K : (0, \infty) \rightarrow [0, \infty) \) be nondecreasing function, then \( \mathcal{N}_K(\mathbb{B}) \subset H_\infty^\omega(\mathbb{B}) \).

**Proof:** The theorem proved in [3].

**Theorem 2.4**

\( \mathcal{N}_K(\mathbb{B}) = H_\infty^\omega(\mathbb{B}) \) if

\[
\int_{0}^{\delta} r^{2n-1} K(g(r)) \, dr < \infty. \tag{19}
\]

**Proof:** Let \( f \in H_\infty^\omega(\mathbb{B}) \). Then,
Thus, \( \|f\|_k < \infty \) and \( f \in \mathcal{N}_K(\mathbb{B}) \). This shows that \( H^\infty_{\frac{n+1}{2}}(\mathbb{B}) \subset \mathcal{N}_K(\mathbb{B}) \). By Theorem 2.3, we have \( \mathcal{N}_K(\mathbb{B}) \subset H^\infty_{\frac{n+1}{2}}(\mathbb{B}) \). The proof of theorem is complete.

3. Hadamard Gaps in \( \mathcal{N}_K \) Spaces in the Unit Ball

In this section we prove a necessary condition for a lacunary series defined by a normal sequence to belong to \( \mathcal{N}_K(\mathbb{B}) \) space. As an implication of Theorem 2.4, we prove that (19) is also necessary for \( \mathcal{N}_K(\mathbb{B}) = H^\infty_{\frac{n+1}{2}}(\mathbb{B}) \).

Recall that an \( f \in \mathcal{H}(\mathbb{B}) \) written in the form \( f(z) = \sum_{k=0}^{\infty} P_n(z) \) where \( P_n \) is a homogeneous polynomial of degree \( n \), is said to have Hadamard gaps (also known as lacunary series) if there exists a constant \( c > 1 \) such that (see e.g. [6])

\[
\frac{n_{k+1}}{n_k} \geq c, \forall k \geq 0.
\]

Let \( \Lambda_n \subset \mathbb{S} \) for \( n = n_0, n_0 + 1, \ldots \). The sequence of homogeneous polynomials

\[
P_n(z) = \sum_{\zeta \in \Lambda_n} \langle z, \zeta \rangle^n,
\]

is called a normal sequence if it possesses the following property (see [7]):

- \( |P_n(z)| \leq C |z|^n \) for \( z \in \mathbb{B} \);
- \( \sum_{\xi \in \Lambda_n} \xi \zeta \xi^n \geq \frac{n_{k+1}}{C} \).

In what following, we will consider all lacunary series defined by normal sequences of homogeneous polynomials. To formulate our main result, we denote

\[
L_j = \int_\mathbb{S} |P_{n_j}(\zeta)|^2 d\sigma(\zeta).
\]

**Theorem 3.1**

Let \( P_n(z) \) be a normal sequence and let \( I_k = \{ n \in \mathbb{N} : 2^k \leq n \leq 2^{k+1} \} \). Then a lacunary series \( f(z) = \sum_{k=0}^{\infty} P_n(z) \) belongs to \( \mathcal{N}_K(\mathbb{B}) \) if

\[
\sum_{k=0}^{\infty} \frac{n_{k+1}}{2^k} K(n_k^{-m}) L_j < \infty.
\]

**Proof:** Let \( f \in \mathcal{N}_K(\mathbb{B}) \). Then, we have
\[\int_{\mathbb{B}} |f(z)|^2 K(G(z,a)) \, dv(z) \geq \int_{\mathbb{B}} \left( \sum_{j=0}^{n} P_{n_j}(z) \right)^2 K(G(z)) \, dv(z) \]
\[\geq \sum_{k=0}^{n} \frac{n_k}{2^k} \sum_{n_j \in I_k} L_j \int_0^{r^{2^{m-1}}} K(g(r)) \, dr,\]
where
\[\sum_{j=0}^{n} P_{n_j}(z) \int \leq \sum_{k=0}^{n} \frac{1}{2^k} \sum_{n_j \in I_k} |P_{n_j}(z)|^2.\]
By (6) for \(1/2 \leq r \leq 1\), we have
\[K(g(r)) \geq K(c^{-1}(1-r)^m).\]
Consequently,
\[\int_0^{r^{2^{m-1}}} K(g(r)) \, dr \geq \int_0^{r^{2^{m-1}}} K(c^{-1}(1-r)^m) \, dr \geq \int_0^{\log_2 n_k} e^{-2^m} \, dr \geq n_k \int_0^{c_1} e^{-2^m} \, dr.
\]
Let \(k'\) be sufficiently large such that \(n_k \log 2 > c_1 + 1\). Then, for \(k \geq k'\),
\[\int_0^{r^{2^{m-1}}} K(g(r)) \, dr \geq n_k^{m+1} K(n_k^{-m}).\]
And
\[\int_{\mathbb{B}} |f(z)|^2 K(G(z,a)) \, dv(z) \geq C \sum_{k=0}^{n} \frac{n_k^m}{2^k} K(n_k^{-m}) \sum_{n_j \in I_k} L_j.\]
This shows (24) and the theorem is proved.

**Theorem 3.2**
\[\mathcal{N}_K(\mathbb{B}) = H_{m+1}^{\infty}(\mathbb{B})\] if and only if (18) holds.

**Proof:** The sufficient condition was proved by Theorem 2.4. Now we prove the necessary condition, assume that \(\mathcal{N}_K(\mathbb{B}) = H_{m+1}^{\infty}(\mathbb{B})\). Among lacunary series defined by normal sequences, we consider
\[f(z) = \sum_{k=0}^{n} P_{k+1}(z),\]
where \(P_{k+1} = \sum_{\zeta \in \Lambda_k} (\zeta - \zeta')^{2^k}\) and \(|P_{k+1}| = C|z|^{2^k}\) for \(k \geq k_0, 2^{k_0} \geq n_0\) and \(z \in \mathbb{B}\).
Thus
\[|f(z)| \leq (1-|z|^{2^k})^{\sum_{k=0}^{n_0}} |P_{k+1}(z)| \leq C \sum_{k=0}^{n_0} |z|^{2^k} \leq C.\]
This shows that \(f \in H_{m+1}^{\infty}(\mathbb{B})\) and, consequently, \(f \in \mathcal{N}_K(\mathbb{B})\). By Theorem 3.1, we have
\[
\sum_{k=1}^{\infty} 2^{(1+m-1)} K \left( 2^{-mk} \right) < \infty.
\] (33)

By (6), we have
\[
\int_0^1 \frac{r^{2m-1}}{(1-r^2)^{m+1}} K \left( g(r) \right) dr \leq \int_0^1 \frac{t^{2m-1}}{t^{m-1}} K \left( r^n \right) dr.
\] (34)

On the other hand,
\[
\int_0^{r^{2m-1}} t^{m-1} K \left( t^n \right) dt = \sum_{k=1}^{\infty} \int_0^{r^{2m-1}} t^{m-1} K \left( t^n \right) dt
\]
\[= \sum_{k=1}^{\infty} 2^{-2(4k+1)} 2^{-m-1} K \left( 2^{-mk} \right),
\] (35)

since \( K \) is non-decreasing. Thus,
\[
\int_0^{1/2} \frac{r^{2m-1}}{(1-r^2)^{m+1}} K \left( g(r) \right) dr < \infty.
\] (36)

Combining this, we obtain (18). The theorem is proved.

4. Conclusion

Our aim of the present paper is to characterize the holomorphic functions with Hadamard gaps in \( N_K \)-type spaces on the unit ball, where \( K \) is the right continuous and non-decreasing function. Our main results will be of important uses in the study of operator theory of holomorphic function spaces.

Acknowledgements

The authors are thankful to the referee for his/her valuable comments and very useful suggestions.

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