A Remark on Eigenfunction Estimates by Heat Flow

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Abstract

In this paper, we consider $L^\infty$ estimates of eigenfunction, or more generally, the $L^\infty$ estimates of equation $-\Delta u = fu$. We use heat flow to give a new proof of the $L^\infty$ estimates for such type equations.

Keywords

$L^\infty$ Estimates, Eigenfunction, Heat Flow

1. Introduction

Let $\Omega \subset \mathbb{R}^n \ (n > 2)$ be a bounded domain. Assume $u \in C^2 (\Omega)$, we consider the Laplacian equation

$$-\Delta u = fu,$$

where $|f| \in L^p (\Omega)$ and $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ with $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. This is a second order differential equation. If $f = \lambda$ is a constant, then $u$ is an eigenfunction with eigenvalue $\lambda$. By a standard Moser’s iteration in [1]-[5], we have $L^\infty$ interior estimates of $u$ controlled by the $L^p$ norm of $u$ for $p > 0$. In this paper, we use heat flow to consider the $L^\infty$ estimate and give a new proof of the $L^\infty$ estimates without using iteration. First, we recall the definition of the heat kernel. For any $x, y \in \mathbb{R}^n$ and $t > 0$, let

$$\rho_t (x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|y-x|^2}{4t}}.$$
be the heat kernel in \( \mathbb{R}^n \). For fixed \( y \in \mathbb{R}^n \), we know that
\[
(\partial_t - \Delta_y) \rho_t(x, y) = 0,
\]
where \( \Delta_y \) is the standard Laplacian in \( \mathbb{R}^n \) with respect to \( x \). Our main result is the following

**Theorem 1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( n > 2 \). Assume \( u \in C^2(\Omega) \) and
\[
-\Delta u = fu
\]
on \( \Omega \) with \( |f| \leq A \). Then for any \( p > n/2 \) and any compact sub-domain \( \Omega' \subset \Omega \), we have the interior \( L^p \) estimate
\[
\sup_{x \in \Omega'} |u(x)| \leq C(p, n, A, \text{dist}(\Omega', \partial \Omega))(\int_{\Omega} |u|^p(y) \, dy)^{1/p},
\]
where \( \text{dist}(\Omega', \partial \Omega) \) is the distance of \( \Omega' \) and the boundary of \( \Omega \).

**Remark 2.** Following from the proof, one can consider equation \( -\Delta u = fu + g \) or
\[
\sum_{i,j=1}^n a_{ij} \partial_{ij} u = f
\]
by choosing appropriate kernel function \( \rho_t \).

## 2. Proving the Theorem

To estimates on \( \Omega' \subset \Omega \), by the translation invariant and scaling invariant of the estimates, we only need to consider \( \Omega = B_1(0) \) and \( \Omega' = B_{t/2}(0) \). By using heat flow, we have the following lemma.

**Lemma 1.** Let \( B_1(0) \subset \mathbb{R}^n \) be a unite ball. Assume \( u \in C^2(B_1(0)) \) and
\[
-\Delta u = fu
\]
on \( B_1(0) \) with \( |f| \leq A \). Then for any \( y \in B_{t/2}(0) \), we have the interior \( L^p \) estimate
\[
|u(y)| \leq C(n, A)\int_{B_1(0)} |u(x)| \left| x-y \right|^{-n} \, dy.
\]

**Proof.** Let \( \phi(x) \) be a standard smooth cutoff function with support in \( B_1(0) \) and \( \phi \equiv 1 \) on \( B_{1/4}(0) \), moreover, \( |\Delta \phi| + |\nabla \phi| \leq C(n) \). For any \( y \in B_{t/2}(0) \), let
\[
\Psi_t(y) = \int_{B_1(0)} \phi(x) u(x) \rho_t(x, y) \, dx.
\]
By the heat equation \( (\partial_t - \Delta_y) \rho_t(x, y) = 0 \), integrating by parts, we have
\[
\partial_t \Psi_t(y) = \int_{B_1(0)} \phi(x) u(x) \partial_t \rho_t(x, y) \, dx
\]
\[
= \int_{B_1(0)} \phi(x) u(x) \Delta_y \rho_t(x, y) \, dx
\]
\[
= \int_{B_1(0)} \Delta \phi u \rho_t(x, y) \, dx
\]
\[
= \int_{B_1(0)} (\Delta \phi u + \phi \Delta u + 2 \langle \nabla \phi, \nabla u \rangle) \rho_t(x, y) \, dx
\]
\[
= \int_{B_1(0)} (\Delta \phi u + \phi \Delta u + 2 \langle \nabla \phi, \nabla \log \rho_t(x, y) \rangle u \rho_t(x, y) \, dx
\]
\[
= \int_{B_1(0)} (\Delta \phi u + \phi \Delta u + 2 \langle \nabla \phi, \nabla \rho_t(x, y) \rangle u \rho_t(x, y) \, dx
\]
where we use integrating by parts for term \( 2 \langle \nabla \phi, \nabla \rho_t(x, y) \rangle \rho_t(x, y) \) to get (7) from (6). By direct estimate, since \( \nabla \phi(x) = 0 \) for \( x \in B_{1/4}(0) \) and \( y \in B_{t/2}(0) \), then \( \|\nabla \phi, x-y\| \leq C(n) \). Therefore, for \( t \leq 1 \), we have
\[
(\|\Delta \phi + t^{-1} \|\nabla \phi, x-y\| \rho_t(x, y) \leq C(n) t^{-1-n/2} e^{-C(n) t} \leq C(n).
\]
Hence, for \( t \leq 1 \) and noting that \( |\varphi| \leq 1 \), we have

\[
|\partial_t \Phi_t (y)| \leq C(n) \int_{\mathbb{R}^n} |f|(x) + C(n, A) \int_{\mathbb{R}^n} \rho(x, y) \Phi_t(x, y) dx.
\]

Since \( |f| \leq A \), then we have

\[
|\partial_t \Phi_t (y)| \leq C(n) \int_{\mathbb{R}^n} |f|(x) + C(n, A) \int_{\mathbb{R}^n} \rho(x, y) \Phi_t(x, y) dx.
\]

By the property of heat kernel, we have \( \Phi_t(u) = u(y) \). Then we have

\[
|u(y) - \Phi_t(y)| \leq \int_0^t |\partial_t \Phi_t(y)| dt \leq C(n) \int_{\mathbb{R}^n} |f|(x) + C(n, A) \int_{\mathbb{R}^n} \rho(x, y) \partial_t \Phi_t(x, y) dx dt.
\]

On the other hand, as \( n > 2 \), we have

\[
\int_0^1 \rho_t(x, y) dt = \int_0^1 (4\pi t)^{-n/2} e^{-|x-y|^2/4t} dt = (4\pi)^{-n/2} \int_0^\infty e^{-x^2/4s} \frac{1}{s^{n/2}} ds \leq C(n)|x-y|^{-n}.
\] (9)

Combining with \( |\Phi_t(y)| \leq C(n) \int_{\mathbb{R}^n} |f|(x) dx \), we have

\[
|\Phi_t(y)| \leq C(n, A) \int_{\mathbb{R}^n} \frac{\rho(x, y)}{|x-y|^{n/2}} dx.
\]

Hence we finish the proof.

The following lemma is fundamental.

**Lemma 2.** For any \( y \in B_i(0) \) and any \( 0 < p < n \), we have

\[
\int_{B_i(0)} \frac{1}{|x-y|^p} dx \leq C(n, p).
\]

**Proof.** Let \( r_i = 2^{-i} \) and \( A_i = B_{r_i}(y) \setminus B_{r_i}(y) \). Then

\[
\int_{B_i(0)} \frac{1}{|x-y|^p} dx \leq \sum_{i=0}^\infty \int_{A_i} \frac{1}{|x-y|^p} dx \leq \sum_{i=0}^\infty r_i^{-p} \int_{A_i} dx.
\]

(10)

\[
\leq \sum_{i=0}^\infty r_i^{-p} C(n) r_i^{-p} \leq C(n) \sum_{i=0}^\infty r_i^{-p} \leq C(n, p).
\] (11)

Now we are ready to prove Theorem 1.

**Proof of Theorem 1.** Refmaintheory. For any compact subset \( \Omega' \subset \Omega \), let \( 2r = \text{dist}(\Omega', \partial \Omega) \). For any \( x \in \Omega' \), we have \( B_r(x) \subset \Omega \). Consider equation

\[
-\Delta u = fu,
\]
on \( B_r(x) \). By Lemma 1, since the estimates are scaling invariant, we have

\[
|u(x)| \leq C(r, n, A) \int_{B_1(x)} \frac{|f(y)|}{|x-y|^{n/2}} dy \leq C(r, n, A) \left( \int_{B_1(x)} |f|^p(y) dy \right)^{1/p} \left( \int_{B_1(x)} |x-y|^{p(n-2)/(p-1)} \right)^{1/p}.
\]

If \( p > n/2 \), then \( p(n-2)/(p-1) < n \). By Lemma 2, we have

\[
|u(x)| \leq C(r, n, A, p) \left( \int_{B_1(x)} |f|^p(y) dy \right)^{1/p} \leq C(r, n, A, p) \left( \int_{\Omega} |f|^p(y) dy \right)^{1/p}.
\]

Hence we finish the proof.

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References


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