Projections and Reflections in Vector Space

Kung-Kuen Tse

Department of Mathematics, Kean University, Union, NJ, USA
Email: ktse@kean.edu

Received 4 April 2016; accepted 16 May 2016; published 19 May 2016

Copyright © 2016 by author and Scientific Research Publishing Inc.
This work is licensed under the Creative Commons Attribution International License (CC BY).
http://creativecommons.org/licenses/by/4.0/

Abstract
We study projections onto a subspace and reflections with respect to a subspace in an arbitrary vector space with an inner product. We give necessary and sufficient conditions for two such transformations to commute. We then generalize the result to affine subspaces and transformations.

Keywords
Projection, Reflection, Commute, Inner Product, Affine Subspace

1. Introduction
Two lines $\ell_1$ and $\ell_2$ in $\mathbb{R}^2$ are considered. When is the reflection over $\ell_1$ followed by the reflection over $\ell_2$ the same as the reflection over $\ell_2$ followed by the reflection over $\ell_1$? It is easy to see that it is the case if and only if $\ell_1 \perp \ell_2$ or $\ell_1 = \ell_2$.

When considering subspaces of $\mathbb{R}^3$, we can ask similar questions for lines, for planes or for the mixed case of one line and one plane. Instead of addressing those cases one by one, we generalize the situation of arbitrary two linear subspaces of a vector space with an inner product.

2. Projection
Supposing that $U$ is a vector space equipped with an inner product, $V \subset U$ is a linear subspace of $U$. Given a vector $u \in U$, we know from linear algebra [1] [2] that $u$ can be decomposed uniquely as $u = p_V(u) + u'$ where $p_V(u) \in V$ is the projection of the vector $u$ onto $V$ and $u' \perp V$, i.e. $U = V \oplus V^\perp$.

Here are some elementary properties of the projection $p_V$:
1) $p_V$ is linear.
2) $u \in V$ if and only if $p_V(u) = u$.
3) $u \in V^\perp$ if and only if $p_V(u) = 0$.
4) $V \cap W \subset p_V(W)$.

5) If \( V_1 \) and \( V_2 \) are subspaces of \( U \), then \( p_{\cap V_1 V_2}(u) = p_{V_1}(u) + p_{V_2}(u) \), for all \( u \in U \).
6) If \( V_1, V_2 \) and \( W \) are subspaces of \( U \), then \( p_{V_1 \cap W}(W) \subset p_{V_1}(W) \oplus p_{V_2}(W) \).
7) If \( V_1, V_2 \) and \( W \) are subspaces of \( U \), then \( p_{V_1 \cap V_2 \cap W}(V_1 \cap V_2) \subset p_{V_1}(V_1 \cap V_2) \oplus p_{V_2}(V_1 \cap V_2) \).

**Lemma 2.1.** Supposing that \( U \) is a linear space and \( V, W \) are two linear subspaces of \( U \), if \( p_w(V) = p_r(W) \) then \( p_w(V) = p_r(W) = V \cap W \).

**Proof.** We first show that \( p_r(W) = V \cap W \). Since \( p_r(W) \subset V \cap W \) and \( p_w(V) = p_r(W) \subset W \), we have \( p_r(W) \subset V \cap W \). On the other hand, if \( u \in V \cap W \), then \( u \in V \), hence \( u = p_r(u) = p_r(W) \) and thus \( V \cap W \subset p_r(W) \). As a result, \( p_r(W) = V \cap W \). The proof of \( p_w(V) = V \cap W \) is similar.

Suppose \( U \) is a vector space and \( V, W \) are two subspaces of \( U \). Intersecting the identity \( U = (V \cap W) \oplus (V \cap W) \), with \( V \) and \( W \), we get \( V = (V \cap W) \oplus (V \cap W) \) and \( W = (V \cap W) \oplus (V \cap W) \). It is obvious that these two sums are orthogonal.

Denote \( V' = V \cap (V \cap W)^\perp \) and \( W' = W \cap (V \cap W)^\perp \). Using these notations, \( V = (V \cap W) \oplus V' \) and \( W = (V \cap W) \oplus W' \).

**Lemma 2.2.** \( p_r(W') = p_w(V') = 0 \) if and only if \( p_w(V) = p_r(W) \).

**Proof.**

\[
p_r(W') = p_{(V \cap W)^\perp}(W') = p_{(V \cap W)^\perp}(V \cap W) \oplus p_{(V \cap W)^\perp}(W') = (V \cap W) \oplus (V \cap W) \oplus p_{V \cap W}(W') = (V \cap W) \oplus p_{V \cap W}(W')
\]

\[\Rightarrow\] If \( p_r(W') = 0 \), then \( p_r(W) \subset V \cap W \). On the other hand, by the fourth property of projection above, \( V \cap W \subset p_r(W) \). Similarly, \( p_w(V) = V \cap W \). Thus, \( p_r(W) = p_w(V) \).

\[\Leftarrow\] By Lemma 2.1, \( p_r(W) = V \cap W \). For \( w' \in W' \),

\[
p_r(W') = p_{(V \cap W)^\perp}(w') = p_{V \cap W}(w')
\]

Thus, \( p_r(W') = p_{V \cap W}(w') \) and \( p_r(W') \in V' \), but \( (V \cap W) \cap V' = 0 \), we must have \( p_r(w') = 0 \), i.e. \( p_r(W') = 0 \). Similarly, \( p_w(V') = 0 \).

**Theorem 2.3.** Supposing that \( U \) is a vector space and \( V, W \) are two subspaces of \( U \), then \( p_r \circ p_w = p_w \circ p_r \) if and only if \( p_w(V) = p_r(W) \).

**Proof.** \( \Rightarrow\) Assume that \( p_w(p_r(u)) = p_r(p_w(u)) \) for all \( u \in U \). In particular,

\[
p_w(v) = p_w(p_r(v)) = p_r(p_w(v)) \in p_r(W) \text{ for all } v \in V.
\]

Thus, \( p_w(V) \subset p_r(W) \). Similarly, \( p_r(W) \subset p_w(V) \).

\[\Leftarrow\] Assume \( p_w(V) = p_r(W) \). By Lemma 2.2, \( p_r(W') = p_w(V') = 0 \).

\[
p_p(p_r(u)) = p_w(p_r(u) + p_r(u)) = p_w(p_r(u)) + p_w(p_r(u)) = p_w(p_r(u)) = p_r(u)
\]

Similarly, \( p_r(p_w(u)) = p_r(u) \).

**3. Reflection over a Subspace**

Supposing that \( U \) is a vector space equipped with an inner product, \( V \subset U \) is a subspace of \( U \). We define the reflection of \( u \in U \) with respect to \( V \) as

\[
r_v(u) = 2p_v(u) - u.
\]

The above formula can be easily derived from the observation that \( p_r(u) = \frac{1}{2}(u + r_v(u)) \). Note that if \( u \in V \), then \( r_v(u) = u \).
Lemma 3.1. Supposing that $U$ is a vector space and $V$, $W$ are two vector subspaces of $U$, then $r_V \circ r_W = r_W \circ r_V$ if and only if $p_V \circ p_W = p_W \circ p_V$.

Proof. 

$$r_V (r_V (u)) = 2p_V (r_V (u)) - r_V (u) = 2p_V (2p_V (u) - u) - (2p_V (u) - u) = 4p_V (p_V (u)) - 2p_V (u) - 2p_V (u) + u.$$  

Similarly, $r_V (r_V (u)) = 4p_V (p_V (u)) - 2p_V (u) - 2p_V (u) + u$. Hence, $r_V \circ r_V = r_V \circ r_V$ if and only if $p_V \circ p_V = p_V \circ p_V$.

Theorem 3.2. Supposing that $U$ is a vector space and $V$, $W$ are two subspaces of $U$, then $r_V \circ r_W = r_W \circ r_V$ if and only if $p_V \circ p_W = p_W \circ p_V$.

Proof. By Lemma 3.1, $p_V \circ p_W = p_W \circ p_V$. By Theorem 2.3, $p_V \circ p_W = p_W \circ p_V$ if and only if $p_V (W) = p_W (V)$.

4. Projection onto a Translated Subspace

Define the projection of $u \in U$ onto a translated subspace $\hat{V} = V + v_0$ as

$$p_V (u) = p_V (u - v_0) + v_0 = p_V (u) - p_V (v_0) + v_0.$$  

$p_V$ is well defined: supposing $V + v_0 = V' + v_0$, then $v_0 = v_0' \in V$. Hence $p_V (v_0) - p_V (v_0') = p_V (v_0 - v_0') = v_0 - v_0'$ and thus $p_V (u) = p_V (v_0) + v_0 = p_V (u) - p_V (v_0') + v_0$.

Theorem 4.1. $p_V \circ p_W = p_W \circ p_V$ if and only if $\hat{V} \cap \hat{W} \neq \phi$ and $p_V (W) = p_W (V)$.

Proof. 

By Theorem 2.3, the first equation implies $p_V (W) = p_W (V)$. The second equation simply means that $\hat{V} \cap \hat{W} \neq \phi$.

$\Rightarrow$ By Theorem 2.3, the first equation is satisfied. To show the second equation, since $\hat{V} \cap \hat{W} \neq \phi$, we have $\hat{v} + v_0 = \hat{v} + w_0$, for some $\hat{v} \in V$ and $\hat{w} \in W$, or $v_0 = \hat{w} + w_0 - \hat{v}$:

$$-p_W (p_V (v_0)) + p_W (v_0) = -p_W (p_V (v_0)) + p_W (v_0) = -p_W (p_V (v_0)) + p_W (v_0) - p_V (v_0) + v_0 = -p_W (p_V (v_0)) + p_W (v_0) - p_W (w_0) + w_0.$$  

which is the second equation.

5. Reflection over a Translated Subspace

We next discuss the reflection over a translated subspace. Let $V \subset U$ be a subspace. A translated subspace is $\hat{V} = V + v_0$ for some $v_0 \in U$. We define the reflection of $u \in U$ over $\hat{V}$ as
\[ r_{\tilde{v}}(u) = r_{v}(u - v_0) + v_0 = r_{v}(u) - r_{v}(v_0) + v_0. \]

\( r_{\tilde{v}} \) is well-defined: supposing \( V + v_0 = V + v'_0 \), then \( v_0 - v'_0 \in V \) and hence \( r_{v}(v_0) - r_{v}(v'_0) = r_{v}(v_0 - v'_0) = v_0 - v'_0. \)

As a result,
\[ r_{v}(u) - r_{v}(v_0) + v_0 = r_{v}(u) - r_{v}(v'_0) + v'_0. \]

Supposing \( \hat{W} = W + w_0 \) for some \( w_0 \in W \), is another translated subspace.
\[ r_{w}(r_{\tilde{v}}(u)) = r_{w}(r_{v}(u)) - r_{w}(w_0) + w_0 \]
\[ = r_{w}(r_{v}(u)) - r_{w}(r_{v}(v_0) + v_0) - r_{w}(w_0) + w_0 \]
\[ = r_{w}(r_{v}(u)) - r_{w}(r_{v}(v_0)) + r_{w}(v_0) - r_{w}(w_0) + w_0. \]

Similarly, \( r_{w}(r_{\tilde{v}}(u)) = r_{w}(r_{v}(u)) - r_{v}(r_{w}(w_0)) - r_{v}(w_0) + v_0. \)

**Theorem 5.1.** \( r_{w} \circ r_{\tilde{v}} = r_{\tilde{v}} \circ r_{w} \) if and only if \( V \cap \hat{W} \neq \phi \) and \( p_{\tilde{v}}(W) = p_{w}(V) \).

**Proof.** \( r_{w}(r_{\tilde{v}}(u)) = r_{\tilde{v}}(r_{w}(u)) \) if and only if
\[
\begin{align*}
  r_{w}(r_{\tilde{v}}(u)) &= r_{w}(r_{v}(u)) - 4p_{w}(v_0) + 4p_{w}(v_0) + 2p_{v}(v_0) - 2v_0 - 2p_{w}(w_0) + 2w_0, \\
  r_{\tilde{v}}(r_{w}(u)) &= r_{v}(r_{w}(u)) - 4p_{v}(v_0) + 4p_{w}(w_0) + 2p_{w}(w_0) - 2w_0 - 2p_{v}(v_0) + 2v_0.
\end{align*}
\]

(\( \Rightarrow \)) By Theorem 3.2, \( r_{v} \circ r_{w} = r_{w} \circ r_{\tilde{v}} \) implies \( p_{\tilde{v}}(W) = p_{w}(V) \). The second equation simply means \( V \cap \hat{W} \neq \phi \).

(\( \Leftarrow \)) We express \( r_{w}(r_{\tilde{v}}(u)) \) and \( r_{\tilde{v}}(r_{w}(u)) \) in terms of projections:
\[
\begin{align*}
  r_{w}(r_{\tilde{v}}(v_0)) &= r_{v}(r_{w}(v_0)) - 4p_{v}(v_0) + 4p_{w}(v_0) + 2p_{v}(v_0) - 2v_0 - 2p_{w}(w_0) + 2w_0, \\
  r_{\tilde{v}}(r_{w}(v_0)) &= r_{w}(r_{v}(v_0)) - 4p_{w}(v_0) + 4p_{v}(v_0) + 2p_{v}(v_0) - 2v_0 - 2p_{w}(w_0) + 2w_0.
\end{align*}
\]

By Theorem 3.2, \( p_{\tilde{v}}(W) = p_{w}(V) \) implies \( r_{v} \circ r_{w} = r_{w} \circ r_{\tilde{v}} \). By Lemma 3.1, we also have \( p_{v} \circ p_{w} = p_{w} \circ p_{v} \).

To show \( r_{w} \circ r_{\tilde{v}} = r_{\tilde{v}} \circ r_{w} \), it suffices to verify the second equation
\[ -p_{w}(p_{v}(v_0)) + p_{w}(v_0) + p_{v}(v_0) - v_0 = -p_{v}(p_{w}(w_0)) + p_{v}(w_0) + p_{w}(w_0) - w_0. \]

Since \( V \cap \hat{W} \neq \phi \), we must have \( \tilde{v} + v_0 = \tilde{w} + w_0 \) for some \( \tilde{v} \in V \) and \( \tilde{w} \in W \), or \( v_0 = \tilde{w} + w_0 - \tilde{v} \):
\[
\begin{align*}
  &-p_{w}(p_{v}(v_0)) + p_{w}(v_0) + p_{v}(v_0) - v_0 = -p_{w}(p_{v}(\tilde{w} + w_0 - \tilde{v})) + p_{w}(\tilde{w} + w_0 - \tilde{v}) - p_{w}(\tilde{w} + w_0 - \tilde{v}) - v_0 + \tilde{v} \\
  &= -p_{w}(p_{v}(\tilde{w})) - p_{w}(p_{v}(w_0)) + p_{w}(\tilde{w} + w_0 - \tilde{v}) + p_{v}(\tilde{w} + w_0 - \tilde{v}) - p_{v}(\tilde{v}) + \tilde{w} + w_0 - \tilde{v} \\
  &= -p_{w}(p_{v}(\tilde{w})) - p_{w}(p_{v}(w_0)) + p_{w}(w_0) + p_{v}(\tilde{w} + w_0 - \tilde{v}) - p_{v}(\tilde{v}) + \tilde{w} + w_0 - \tilde{v} \\
  &= -p_{w}(p_{v}(w_0)) + p_{w}(w_0) + p_{v}(\tilde{w} + w_0 - \tilde{v}) - p_{v}(\tilde{v}) + \tilde{w} + w_0 - \tilde{v} \\
  &= -p_{v}(p_{w}(w_0)) + p_{v}(w_0) + p_{w}(w_0) - w_0.
\end{align*}
\]

**6. Mixed Transformations**

**Theorem 6.1.** \( p_{\tilde{v}} \circ r_{w} = p_{w} \circ r_{\tilde{v}} \) if and only if \( V \cap \hat{W} \neq \phi \) and \( p_{\tilde{v}}(W) = p_{w}(V) \).

**Theorem 6.2.** \( p_{\tilde{v}} \circ r_{w} = p_{w} \circ r_{\tilde{v}} \) if and only if \( V \cap \hat{W} \neq \phi \) and \( V = W \).

**Theorem 6.3.** \( p_{\tilde{v}} \circ r_{w} = p_{w} \circ r_{\tilde{v}} \) if and only if \( V \cap \hat{W} \neq \phi \) and \( V = W \).

**7. Generalizations**

If we denote \( \Sigma_n \), the permutation group of order \( n \), then
Theorem 7.1.

\[ p_{r_{(1)}} \circ \cdots \circ p_{r_{(a)}} = p_{r_{(1)}} \circ \cdots \circ p_{r_{(a)}} \text{ for all } \sigma, \tau \in \Sigma_n \]

if and only if

\[ p_{r_{(1)}} \cdots p_{r_{(n-1)}} (V_{\alpha(a)}) = p_{r_{(1)}} \cdots p_{r_{(n-1)}} (V_{\tau(a)}) \text{ for all } \sigma, \tau \in \Sigma_n. \]

Theorem 7.2.

\[ r_{r_{(1)}} \circ \cdots \circ r_{r_{(a)}} = r_{r_{(1)}} \circ \cdots \circ r_{r_{(a)}} \text{ for all } \sigma, \tau \in \Sigma_n \]

if and only if

\[ p_{r_{(1)}} \cdots p_{r_{(n-1)}} (V_{\alpha(a)}) = p_{r_{(1)}} \cdots p_{r_{(n-1)}} (V_{\tau(a)}) \text{ for all } \sigma, \tau \in \Sigma_n. \]

Theorem 7.3.

\[ p_{r_{(1)}} \circ \cdots \circ p_{r_{(a)}} = p_{r_{(1)}} \circ \cdots \circ p_{r_{(a)}} \text{ for all } \sigma, \tau \in \Sigma_n \]

if and only if \( \hat{V}_i \cap \hat{V}_j \neq \phi \) for all \( i, j \) and

\[ p_{r_{(1)}} \cdots p_{r_{(n-1)}} (V_{\alpha(a)}) = p_{r_{(1)}} \cdots p_{r_{(n-1)}} (V_{\tau(a)}) \text{ for all } \sigma, \tau \in \Sigma_n. \]

Theorem 7.4.

\[ r_{r_{(1)}} \circ \cdots \circ r_{r_{(a)}} = r_{r_{(1)}} \circ \cdots \circ r_{r_{(a)}} \text{ for all } \sigma, \tau \in \Sigma_n \]

if and only if \( \hat{V}_i \cap \hat{V}_j \neq \phi \) for all \( i, j \) and

\[ p_{r_{(1)}} \cdots p_{r_{(n-1)}} (V_{\alpha(a)}) = p_{r_{(1)}} \cdots p_{r_{(n-1)}} (V_{\tau(a)}) \text{ for all } \sigma, \tau \in \Sigma_n. \]

References
