Block-Transitive $4-(v,k,4)$ Designs and Ree Groups

Shaojun Dai¹, Ruihai Zhang²

¹Department of Mathematics, Tianjin Polytechnic University, Tianjin, China
²Department of Mathematics, Tianjin University of Science and Technology, Tianjin, China

Email: daishaojun1978@sina.com, zhangruihai781119@126.com

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Abstract

This article is a contribution to the study of the automorphism groups of $4-(v,k,\lambda)$ designs. Let $S=(\mathcal{P},\mathcal{B})$ be a non-trivial $4-(q^3+1,k,4)$ design where $q=3^{2n+1}$ for some positive integer $n \geq 1$, and $G \leq \text{Aut}(S)$ is block-transitive. If the socle of $G$ is isomorphic to the simple groups of lie type $2G_2(q)$, then $G$ is not flag-transitive.

Keywords

Flag-Transitive, Block-Transitive, $t$-Design, Ree Group

1. Introduction

For positive integers $t \leq k \leq v$ and $\lambda$, we define a $t-(v,k,\lambda)$ design to be a finite incidence structure $S=(\mathcal{P},\mathcal{B})$, where $\mathcal{P}$ denotes a set of points, $|\mathcal{P}|=v$, and $\mathcal{B}$ a set of blocks, $|\mathcal{B}|=b$, with the properties that each block is incident with $k$ points, and each $t$-subset of $\mathcal{P}$ is incident with $\lambda$ blocks. A flag of $S$ is an incident point-block pair $(x,B)$ with $x$ is incident with $B$, where $B \in \mathcal{B}$. We consider automorphisms of $S$ as pairs of permutations on $\mathcal{P}$ and $\mathcal{B}$ which preserve incidence structure. We call a group $G \leq \text{Aut}(S)$ of automorphisms of $S$ flag-transitive (respectively block-transitive, point $t$-transitive, point $t$-homogeneous) if $G$ acts transitively on the flags (respectively transitively on the blocks, $t$-transitively on the points, $t$-homogeneously on the points) of $S$. For short, $S$ is said to be, e.g., flag-transitive if $S$ admits a flag-transitive group of automorphisms.

For historical reasons, a $t-(v,k,\lambda)$ design with $\lambda=1$ is called a Steiner $t$-design (sometimes this is also known as a Steiner system). If $t<k<v$ holds, then we speak of a non-trivial Steiner $t$-designs.
Investigating $t$-designs for arbitrary $\lambda$, but large $t$, Cameron and Praeger proved the following result:

**Theorem 1.** (1) Let $S = (P, B)$ be a $t-\left(v, k, \lambda\right)$ design. If $G \leq \text{Aut}(S)$ acts block-transitively on $S$, then $t \leq 7$, while if $G \leq \text{Aut}(S)$ acts flag-transitively on $S$, then $t \leq 6$.

Recently, Huber (see [2]) completely classified all flag-transitive Steiner $t$-designs using the classification of the finite 2-transitive permutation groups. Hence the determination of all flag-transitive and block-transitive $t$-designs with $\lambda \geq 2$ has remained of particular interest and has been known as a long-standing and still open problem.

The present paper continues the work of classifying block-transitive $t$-designs. We discuss the block-transitive $4-\left(v, k, 4\right)$ designs and Ree groups. We get the following result:

**Main Theorem.** Let $S = (P, B)$ be a non-trivial $4-\left(q^3 + 1, k, 4\right)$ design, where $q = 3^{2n+1}$ for some positive integer $n \geq 1$, and $G \leq \text{Aut}(S)$ is block-transitive. If $\text{Soc}(G)$, the socle of $G$, is $2^2 G_2(q)$, then $G$ is not flag-transitive.

The second section describes the definitions and contains several preliminary results about flag-transitivity and $t$-designs. In Section 3, we give the proof of the Main Theorem.

### 2. Preliminary Results

The Ree groups $2^2 G_2(q)$ form an infinite family of simple groups of Lie type, and were defined in [3] as subgroups of $GL(7,q)$. Let $GF(q)$ be a finite field of $q$ elements, where $2^{2n+1} = nq + 1$ for some positive integer $n \geq 1$ (in particular, $q \geq 27$). Let $q$ be a Sylow 3-subgroup of $G$, $K$ is a multiplicative group of $GF(q)$ and $2^2 G_2(q)$ is a group of order $(2^n - 1)(2^n + 1)^2$ (see [4]-[6]). Hence $2^2 G_2(q)$ is a group of automorphisms of Steiner $3-\left(q^3 + 1, q + 1\right)$ design and acts 2-transitive on $q^3 + 1$ points (see [7]).

Here we gather notation which are used throughout this paper. For a $t$-design $S = (P, B)$ with $G \leq \text{Aut}(S)$, let $r$ denotes the number of blocks through a given point, $x \in P$; $B \in B$; $x \in P$. We define $G_{\text{aut}} = G \cap G_B$. For integers $m$ and $n$, let $(m, n)$ denotes the greatest common divisor of $m$ and $n$, and $m \mid n$ if $m$ divides $n$.

**Lemma 1.** (2) Let $G$ act flag-transitively on $t-\left(v, k, \lambda\right)$ design $S = (P, B)$. Then $G$ is block-transitive and the following cases hold:

1) $|G| = |G_{\text{aut}}| = |G_{\text{st}}|$, where $x \in P$;
2) $|G| = |G_{\text{aut}}| = |G_{\text{bt}}|$, where $B \in B$;
3) $|G| = |G_{\text{aut}}| = |G_{\text{at}}|$, where $x \in B$.

**Lemma 2.** (8) Let $S = (P, B)$ is a non-trivial $t-\left(v, k, \lambda\right)$ design. Then $

\lambda(v-t+1) \geq (k-t+2)(k-t+1)$.

**Lemma 3.** (8) Let $S = (P, B)$ is a non-trivial $4-\left(v, k, \lambda\right)$ design. Then $bk = vr$.

**Corollary 1.** Let $S = (P, B)$ is a non-trivial $4-\left(v, k, 4\right)$ design. If $v = q^3 + 1$, Then $k < 3 + 2q\sqrt{q}$.

**Proof.** By Lemma 2, we have $4(v-3) \geq (k-2)(k-3)$. If $v = q^3 + 1$, then $4(q^3 - 2) \geq (k-2)(k-3)$.

Hence $k^2 - 5k - 4q^3 + 14 \leq 0$.

We get $k \leq \frac{5 + \sqrt{16q^3 - 31}}{2} < 3 + 2q\sqrt{q}$. 

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3. Proof of the Main Theorem

Suppose that $G$ acts flag-transitively on $4-(v,k,4)$ design and $v=q^3+1$. Then $G$ is block-transitive and point-transitive. Since $T = \mathcal{G}_2(q) \leq G \leq \text{Aut}(T)$, we may assume that $G = \langle \alpha \rangle$ and $G = T \setminus \langle \alpha \rangle$ by Dedekind’s theorem, where $\alpha : x \to x', x \in GF(q)$ and $\alpha$ is an automorphism of field $GF(q)$. Let $q = 3^f$, $f = 2n + 1$ is odd, and $|\langle \alpha \rangle| = m$, then $m | f$. Obviously, $|G| = q^3(q^3+1)(q-1)m$.

First, we will prove that if $g \in G$ fixes three different points of $\mathcal{P}$, then $g$ must fix at least four points in $\mathcal{P}$.

Suppose that $g \in G$. Let $P$ be a normal Sylow 3-subgroup of $G$. Then $P$ acts regularly on $\mathcal{P}$. By Lemma 3(2) and Lemma 1(3),

$$G_{x} = \left[ G_{x} \cap \mathcal{P} \right] \cap \mathcal{P} = \left[ G_{x} \cap \mathcal{P} \right] \cap \mathcal{P}.$$

Thus

$$|G_{x}| = \frac{(k-1)(k-2)(k-3)|G_x|}{4(v-1)(v-2)(v-3)} = \frac{(k-1)(k-2)(k-3)|G_x|}{4q^3(q^3+1)(q-1)^2} = \frac{(k-1)(k-2)(k-3)}{4q^3+q+1}(q^3+2).$$

By Lemma 2,

$$4|G_{x}|(q^3+2) = (k-1)(k-2)(k-3) \leq (k-1)(k-2)(k-3) \leq (k-1)(k-2)(k-3) \leq 4(q^3+2) = 4(k-1)(q^3+2),$$

Again by Corollary 1,

$$1 \leq |G_{x}| \leq \frac{k-1}{q^3+2} \leq \frac{2+2q\sqrt{q}}{q^3+2} < 1 (q \geq 27).$$

This is impossible.

This completes the proof the Main Theorem.

References


