Matrices Associated with Moving Least-Squares Approximation and Corresponding Inequalities

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Abstract
In this article, some properties of matrices of moving least-squares approximation have been proven. The used technique is based on known inequalities for singular-values of matrices. Some inequalities for the norm of coefficients-vector of the linear approximation have been proven.

Keywords
Moving Least-Squares Approximation, Singular-Values

1. Statement
Let us remind the definition of the moving least-squares approximation and a basic result.
Let:
1. $\mathcal{D}$ be a bounded domain in $\mathbb{R}^d$;
2. $x_i \in \mathcal{D}$, $i=1,\cdots,m$; $x_i \neq x_j$, if $i \neq j$;
3. $f: \mathcal{D} \rightarrow \mathbb{R}$ be a continuous function;
4. $p_i: \mathcal{D} \rightarrow \mathbb{R}$ be continuous functions, $i=1,\cdots,l$. The functions $\{p_1,\cdots,p_l\}$ are linearly independent in $\mathcal{D}$ and let $\mathcal{P}_l$ be their linear span;
5. $W: (0,\infty) \rightarrow (0,\infty)$ be a strong positive function.

Usually, the basis in $\mathcal{P}_l$ is constructed by monomials. For example: $p_i(x) = x_1^{k_1} \cdots x_d^{k_d}$, where $x = (x_1,\cdots,x_d)$, $k_1,\cdots,k_d \in \mathbb{N}$, $k_1 + \cdots + k_d \leq l-1$. In the case $d=1$, the standard basis is $\{1, x, \cdots, x^{l-1}\}$.

Following [1]-[4], we will use the following definition. The moving least-squares approximation of order $l$ at
a fixed point \( x \) is the value of \( p^*(x) \), where \( p^* \in \mathcal{P} \) is minimizing the least-squares error
\[
\sum_{i=1}^{m} W(\|x - x_i\|)(p(x_i) - f(x_i))^2
\]
among all \( p \in \mathcal{P} \).

The approximation is “local” if weight function \( W \) is fast decreasing as its argument tends to infinity and interpolation is achieved if \( W(0) = \infty \). So, we define additional function \( w : [0, \infty) \to [0, \infty) \), such that:
\[
w(r) = \begin{cases} 
1 & \text{if } (r > 0) \text{ or } (r = 0 \text{ and } W(0) < \infty), \\
0 & \text{if } (r = 0 \text{ and } W(0) = \infty).
\end{cases}
\]

Some examples of \( W(r) \) and \( w(r) \), \( r \geq 0 \):
\[
2W(r) = e^{-r^2} \quad \text{exp-weight},
\]
\[
W(r) = r^{-a^2} \quad \text{Shepard weights},
\]
\[
w(x, x_i) = r^2 e^{-r^2} \quad \text{McLain weight},
\]
\[
w(x, x_i) = e^{r^2} - 1 \quad \text{see Levin’s works}.
\]

Here and below: \( \| \cdot \|_2 \) is 2-norm, \( \| \cdot \|_1 \) is 1-norm in \( \mathbb{R}^d \); the superscript \( ^t \) denotes transpose of real matrix; \( I \) is the identity matrix.

We introduce the notations:
\[
E = \begin{pmatrix}
p_1(x_1) & p_2(x_1) & \cdots & p_l(x_1) \\
p_1(x_2) & p_2(x_2) & \cdots & p_l(x_2) \\
\vdots & \vdots & \ddots & \vdots \\
p_1(x_m) & p_2(x_m) & \cdots & p_l(x_m)
\end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_l \end{pmatrix},
\]
\[
D = 2\begin{pmatrix}
w(x, x_1) & 0 & \cdots & 0 \\
0 & w(x, x_2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & w(x, x_m)
\end{pmatrix}, \quad c = \begin{pmatrix} p_1(x) \\ p_2(x) \\ \vdots \\ p_l(x) \end{pmatrix}.
\]

Through the article, we assume the following conditions (H1):
(H1.1) \( 1 \in \mathcal{P} \); 
(H1.2) \( 1 \leq l \leq m \); 
(H1.3) rank \( (E') = l \); 
(H1.4) \( w \) is smooth function.

**Theorem 1.1.** (see [2]): *Let the conditions (H1) hold true.*

Then:
1. The matrix \( E'D^{-1}E \) is non-singular;
2. The approximation defined by the moving least-squares method is
\[
\hat{L}(f) = \sum_{i=1}^{m} a_i f(x_i), \tag{1}
\]
where
\[
\mathbf{a} = A \mathbf{c} \quad \text{and} \quad A = D^{-1}E(E'D^{-1}E)^{-1}.
\tag{2}
\]
3. If \( w(\|x_i - x_j\|) = 0 \) for all \( i = 1, \cdots, m \), then the approximation is interpolatory. 

For the approximation order of moving least-squares approximation (see [2] and [5]), it is not difficult to
receive (for convenience we suppose \( d = 1 \) and standard polynomial basis, see [5]):

\[
\left| f(x) - \hat{L}(f)(x) \right| \leq \| f(x) - p^*(x) \|_\infty \left[ 1 + \sum_{i=1}^{m} |a_i| \right],
\]

and moreover (\( C = \text{const.} \))

\[
\| f(x) - p^*(x) \|_\infty \leq C h^{i+1} \max \left\{ \| f^{(i+1)}(x) \| : x \in \bar{D} \right\}.
\]

It follows from (3) and (4) that the error of moving least-squares approximation is upper-bounded from the 2-norm of coefficients of approximation (\( \| a \| \leq \sqrt{m} \| a \|_2 \)). That is why the goal in this short note is to discuss a method for majorization in the form

\[
\| a \|_2 \leq M \exp \left( N \| x - x_i \| \right).
\]

Here the constants \( M \) and \( N \) depend on singular values of matrix \( E' \), and numbers \( m \) and \( l \) (see Section 3). In Section 2, some properties of matrices associated with approximation (symmetry, positive semi-definiteness, and norm majorization by \( \sigma_{\text{max}}(E') \) and \( \sigma_{\text{max}}(E^\prime) \)) are proven.

The main result in Section 3 is formulated in the case of exp-moving least-squares approximation, but it is not hard to receive analogous results in the different cases: Backus-Gilbert weight functions, McLain weight functions, etc.

2. Some Auxiliary Lemmas

**Definition 2.1.** We will call the matrices

\[
A_1 = A_2 E = D^{-1} E \left( E' D^{-1} E \right)^{-1} E' \quad \text{and} \quad A_2 = A_1 - I
\]

\( A_1 \)-matrix and \( A_2 \)-matrix of the approximation \( \hat{L} \), respectively.

**Lemma 2.1.** Let the conditions (H1) hold true.

Then, the matrices \( A_1 D^{-1} \) and \( A_2 D^{-1} \) are symmetric.

**Proof.** Direct calculation of the corresponding transpose matrices.

**Lemma 2.2.** Let the conditions (H1) hold true.

Then:
1. All eigenvalues of \( A_1 \) are 1 and 0 with geometric multiplicity \( l \) and \( m - l \), respectively;
2. All eigenvalues of \( A_2 \) are 0 and \( -1 \) with geometric multiplicity \( l \) and \( m - l \), respectively.

**Proof.** Part 1: We will prove that the dimension of the null-space \( \dim \left( \text{null}(A_1) \right) \) is at least \( l \).

Using the definition of \( A_2 = D^{-1} E \left( E' D^{-1} E \right)^{-1} E' - I \), we receive

\[
E' A_2 = \left( E' D^{-1} E \right) \left( E' D^{-1} E \right)^{-1} E' - E' = 0.
\]

Hence,

\[
\text{im}(A_2) \subseteq \text{null}(E').
\]

Using (H1.3), \( E' \) is \((l \times m)\)-matrix with maximal rank \( l \) (\( l < m \)). Therefore, \( \dim \left( \text{null}(E') \right) = m - l \). Moreover, \( \dim \left( \text{im}(A_2) \right) = m - \dim \left( \text{null}(A_1) \right) \). That is why \( m - \dim \left( \text{null}(A_1) \right) \leq m - l \) or \( l \leq \dim \left( \text{null}(A_2) \right) \).

Part 2: We will prove that \( -1 \) is eigenvalue of \( A_2 \) with geometric multiplicity \( m - l \), or the system

\[
A_2 \eta = -\eta \iff A_2 \eta = 0
\]

has \( m - l \) linearly independent solutions.

Obviously the systems

\[
A_1 \eta = D^{-1} E \left( E' D^{-1} E \right)^{-1} E' \eta = 0
\]

and

\[
E' \eta = 0
\]
are equivalent. Indeed, if \( \eta_0 \) is a solution of (5), then
\[
D^{-1}E \left( E'D^{-1}E \right)^{-1} E' \eta_0 = 0 \Rightarrow E'D^{-1}E \left( E'D^{-1}E \right)^{-1} E' \eta_0 = 0 \Rightarrow E' \eta_0 = 0,
\]
\( i.e. \eta_0 \) is solution of (6).

On the other hand, if \( \eta_0 \) is a solution of (6), then
\[
\left( D^{-1}E \left( E'D^{-1}E \right)^{-1} E' \right) \eta_0 = \left( D^{-1}E \left( E'D^{-1}E \right)^{-1} \right) (E' \eta_0) = 0,
\]
\( i.e. \eta_0 \) is solution of (5). Therefore
\[
\dim(\text{im}(A)) = \dim(\text{im}(E')) = m - l.
\]

Part 3: It follows from parts 1 and 2 of the proof that 0 is an eigenvalue of \( A_2 \) with multiplicity exactly \( l \) and \( -1 \) is an eigenvalue of \( A_2 \) with multiplicity exactly \( m - l \).

It remains to prove that 1 is eigenvalue of \( A_2 \) with multiplicity at least \( l \), but this is analogous to the proven part 1 or it follows directly from the definition of \( A_i = A_2 + I \).

The following two results are proven in [6].

Theorem 2.1 (see [6], Theorem 2.2): Suppose \( U, V \) are \((m \times m)\) Hermitian matrices and either \( U \) or \( V \) is positive semi-definite. Let
\[
\lambda_1(U) \geq \cdots \geq \lambda_m(U), \quad \lambda_1(V) \geq \cdots \geq \lambda_m(V)
\]
denote the eigenvalues of \( U \) and \( V \), respectively.

Let:
1. \( \pi(U) \) is the number of positive eigenvalues of \( U \);
2. \( \nu(U) \) is the number of negative eigenvalues of \( U \);
3. \( \xi(U) \) is the number of zero eigenvalues of \( U \).

Then:
1. If \( 1 \leq k \leq \pi(U) \), then
\[
\min_{k \leq i \leq k + \nu(U)} \lambda_i(U) \geq \lambda(kU) \geq \max_{k \leq i \leq k + \nu(U)} \lambda_i(U).
\]
2. If \( \pi(U) < k \leq m - \nu(U) \), then
\[
\lambda_k(UV) = 0.
\]
3. If \( m - \nu(U) < k \leq m \), then
\[
\min_{k \leq i \leq k + \nu(U)} \lambda_i(U) \geq \lambda(kU) \geq \max_{k \leq i \leq k + \nu(U)} \lambda_i(U).
\]

Corollary 2.1. (see [6], Corollary 2.4): Suppose \( U, V \) are \((m \times m)\) Hermitian positive definite matrices. Then for any \( 1 \leq k \leq m \)
\[
\lambda_1(U) \lambda_k(V) \geq \lambda_k(UV) \geq \lambda_m(U) \lambda_m(V).
\]

As a result of Lemma 2.1, Lemma 2.2 and Theorem 2.1, we may prove the following lemma.

Lemma 2.3. Let the conditions (H1) hold true.
1. Then \( A_1D^{-1} \) and \( -A_1D^{-1} \) are symmetric positive semi-definite matrices.
2. The following inequality holds true
\[
\lambda_{\max}(A_1D^{-1}) \leq \frac{1}{\lambda_{\min}(D)}.
\]

Proof. (1) We apply Theorem 2.1, where
\[
U = D, \quad V = A_1D^{-1}.
\]

Obviously, \( U \) is a symmetric positive definite matrix (in fact it is a diagonal matrix). Moreover \( \pi(U) = m \),
\[ \mu(U) = \xi(U) = 0, \text{ if } x \neq x_i, \hspace{1em} i = 1, \ldots, m. \]

The matrix \( V \) is symmetric (see Lemma 2.1).

From the cited theorem, for any index \( k \) \((k = 1, \ldots, m = \pi(U))\) we have
\[ \lambda_k(A_i) = \lambda_k(A D^{-1} D) = \lambda_k(V U) \leq \min \{ \lambda(U) \lambda_{m+1-k}(V) \}. \]

In particular, if \( k = m \):
\[ \lambda_m(A_i) \leq \min \{ \lambda(U) \lambda(V) \}. \tag{7} \]

Let us suppose that there exists index \( i_0 \) \((i_0 = 1, \ldots, m-1)\) such that
\[ \lambda_{i_0}(V) \geq \cdots \geq \lambda_{m}(V) \geq 0 > \lambda_{m+1}(V) \geq \cdots \geq \lambda_m(V). \tag{8} \]

It follows from (8) and positive definiteness of \( U \), that
\[ \min \{ \lambda(U) \lambda(V) \} \leq \lambda_{m+1}(U) \lambda_{m+1}(V) < 0. \]

Therefore (see (7)), \( \lambda_m(A_i) < 0 \). This contradiction (see Lemma 2.2) proves that the matrix \( A D^{-1} \) is positive semi-definite.

If we set \( U = D, \ V = -A D^{-1} \) then by analogical arguments, we see that the matrix \( -A D^{-1} \) is positive semi-definite.

(2) From the first statement of Lemma 2.3, \( V = A D^{-1} \) is positive semi-definite. Therefore (see Corollary 2.1 and Lemma 2.2):
\[ 1 \geq \lambda_k(A) = \lambda_k(V U) \geq \max \{ \lambda_m(U) \lambda_{m+k}(V), \lambda_m(V) \lambda_k(U) \} \]
for all \( k = 1, \ldots, m \). Moreover, all numbers \( \lambda_k(U), \lambda_k(V) \) are non-negative and
\[ \lambda_{\max}(D) = \lambda_1(U) \geq \cdots \geq \lambda_m(U) = \lambda_{\min}(D), \hspace{1em} \lambda_1(V) \geq \cdots \geq \lambda_m(V). \]

Therefore
\[ 1 \geq \max \{ \lambda_m(U) \lambda_1(V), \lambda_m(V) \lambda_1(U) \}, \]
or
\[ \lambda_{\max}(A D^{-1}) = \lambda_1(V) \leq \frac{1}{\lambda_m(U)} = \frac{1}{\lambda_{\min}(D)}. \tag{13} \]

In the following, we will need some results related to inequalities for singular values. So, we will list some necessary inequalities in the next lemma.

**Lemma 2.4.** (see [7] [8]): Let \( U \) be an \((d_1 \times d_2)\)-matrix, \( V \) be an \((d_3 \times d_4)\)-matrix. Then:
\[ 2 \sigma_{\max}(U V) \leq \sigma_{\max}(U) \sigma_{\max}(V), \tag{9} \]
\[ \sigma_{\max}(U^{-1}) = \frac{1}{\sigma_{\min}(U)}, \hspace{1em} \text{if } d_1 = d_2, \det U \neq 0, \tag{10} \]
\[ \sigma_{\max}(V) \sigma_{\min}(U) \leq \sigma_{\max}(U V), \hspace{1em} \text{if } d_1 \geq d_2 = d_3, \tag{11} \]
\[ \sigma_{\max}(U) \sigma_{\min}(V) \leq \sigma_{\max}(U V), \hspace{1em} \text{if } d_4 \geq d_3 = d_2. \tag{12} \]

If \( d_1 = d_2 \) and \( U \) is Hermitian matrix, then \[ \|U\| = \sigma_{\max}(U), \hspace{1em} \sigma_i(U) = |\lambda_i(U)|, \hspace{1em} i = 1, \ldots, d_1. \]

**Lemma 2.5.** Let the conditions (H1) hold true and let \( x \neq x_i, \hspace{1em} i = 1, \ldots, m \). Then:
\[ \|A D^{-1}\| \leq \frac{1}{\lambda_{\min}(D)}. \tag{13} \]
\[
\sigma_{\text{max}}(A_i) \sigma_{\text{min}}(D^{-1}) \leq \sigma_{\text{max}}(A_i D^{-1}),
\]  
\[1 \leq \|A_i\| \leq \frac{\sigma_{\text{max}}(D)}{\sigma_{\text{min}}(D)} \]  
(14)  
(15)

Proof. The matrix \(A_i D^{-1}\) is symmetric and positive semi-definite (see Lemma 2.3 (1)). Using the second statement of Lemma 2.3 and Lemma 2.4, we receive

\[\|A_i D^{-1}\| = \sigma_{\text{max}}(A_i D^{-1}) = \lambda_{\text{max}}(A_i D^{-1}) \leq \frac{1}{\lambda_{\text{min}}(D)}.
\]

The inequality (14) follows from (12) \((d_k = d_3 = m)\).

From (14) and (10), we receive

\[\sigma_{\text{max}}(A_i) \leq \frac{\sigma_{\text{max}}(A_i D^{-1})}{\sigma_{\text{min}}(D)} = \frac{\sigma_{\text{max}}(D)}{\sigma_{\text{min}}(D)}.
\]

Therefore, the equality \(\|A_i\| = \sqrt{\sigma_{\text{max}}(A_i)}\) implies the right inequality in (15).

Using \(E' = E' A_i\) and inequality (9), we receive

\[\sigma_{\text{max}}(E') \leq \sigma_{\text{max}}(E') \sigma_{\text{max}}(A_i),
\]

or \(1 \leq \sigma_{\text{max}}(A_i) = \|A_i\|^2\), i.e. the left inequality in (15).

The lemma has been proved.

\[\square\]

3. An Inequality for the Norm of Approximation Coefficients

We will use the following hypotheses (H2):

(H2.1) The hypotheses (H1) hold true;
(H2.2) \(d = 1, x_1 < \cdots < x_m\);
(H2.3) The map \(c\) is \(C^1\)-smooth in \([x_1, x_m]\);
(H2.4) \(w([x_i - x_i]) = \exp\left(\alpha(x-x_i)^2\right), i = 1, \ldots, m\).

Theorem 3.1. Let the following conditions hold true:
1. Hypotheses (H2);
2. Let \(\dot{x} \in [x_1, x_m]\) be a fixed point;
3. The index \(k_0 \in \{1, \ldots, m\}\) is chosen such that

\[|x - x_{k_0}| = \min \{|x - x_i| : i = 1, \ldots, m\}.
\]

Then, there exist constants \(M_1, M_2 > 0\) such that

\[\|a(x)\| \leq \left(\|a(x_0)\| + M_1 |x - x_{k_0}| \right) \exp\left(M_2 |x - x_{k_0}|\right).
\]

Proof. Part 1: Let

\[H = \begin{pmatrix}
2\alpha(x-x_1) & 0 & \cdots & 0 \\
0 & 2\alpha(x-x_2) & \cdots & 0 \\
0 & 0 & \cdots & 2\alpha(x-x_m) \\
0 & 0 & \cdots & 2\alpha(x-x_m)
\end{pmatrix},
\]

then

\[\frac{dD}{dx} = HD, \quad \frac{dD^{-1}}{dx} = -HD^{-1}.
\]
We have (obviously) \( D = D(x) \), \( H = H(x) \), and \( c = c(x) \)

\[
\frac{da(x)}{dx} = \frac{d}{dx} \left( D^{-1}E \left( E'D^{-1}E \right)^{-1} c \right)
= \left( \frac{d}{dx} D^{-1} \right) E \left( E'D^{-1}E \right)^{-1} c + D^{-1}E \left( \frac{d}{dx} \left( E'D^{-1}E \right) \right)^{-1} \frac{d}{dx} c
= -HD^{-1}E \left( E'D^{-1}E \right)^{-1} c + D^{-1}E \left( \left( E'D^{-1}E \right)^{-1} \left( \frac{d}{dA} E'D^{-1}E \right) \left( E'D^{-1}E \right)^{-1} \right) c + D^{-1}E \left( E'D^{-1}E \right)^{-1} \frac{d}{dx} c
= -Ha + D^{-1}E \left( E'D^{-1}E \right)^{-1} \left( E'D^{-1}E \right)^{-1} c + D^{-1}E \left( E'D^{-1}E \right)^{-1} \frac{d}{dx} c
= \left( D^{-1}E \left( E'D^{-1}E \right)^{-1} E' - I \right) Ha + D^{-1}E \left( E'D^{-1}E \right)^{-1} \frac{d}{dx} c
= A_x Ha + A_0 \frac{d}{dx} c.
\]

Therefore, the function \( a(x) \) satisfies the differential equation

\[
\frac{da(x)}{dx} = A_x Ha + A_0 \frac{d}{dx} c.
\] (16)

Part 2: Obviously

\[
\|A_x H\| = \|A_x - I\| H \leq (\|A\| + 1) \|H\|.
\]

It follows from (15) that

\[
\|A_x\| \leq \sqrt{\frac{\sigma_{\max}(D)}{\sigma_{\min}(D)}}.
\]

Here \( \sigma_{\max}(D) \leq 2 \exp(\alpha r^2) \), \( r = x_m - x_1 \), and \( \sigma_{\min}(D) \geq 2 \). Hence

\[
\|A_x\| \leq \sqrt{\exp(\alpha r^2)}.
\]

For the norm of diagonal matrix \( H \), we receive

\[
\|H\| \leq 2ar.
\]

Therefore \( \|A_x H\| \leq M_2 \), where

\[
M_2 = 2ar \left( 1 + \sqrt{\exp(\alpha r^2)} \right).
\]

We will use Lemma 2.4 to obtain the norm of \( A_0 \).

Obviously, \( A_0 E' = A_1 \). Therefore by (12) \( (m = d_4 \geq d_3 = l) \), we have

\[
\sigma_{\max}(A_0) \sigma_{\min}(E') \leq \sigma_{\max}(A_1),
\]

i.e.

\[
\|A_0\| \leq \frac{1}{\sigma_{\min}(E')} \sqrt{\frac{\sigma_{\max}(D)}{\sigma_{\min}(D)}}.
\]

Therefore, if we set \( M_{11} = \frac{M_2}{\sigma_{\max}(E')} \), then \( \|A_0\| \leq M_1 \).

Let the constant \( M_{12} \) be chosen such that

\[
\left| \frac{d}{dx} c(x) \right| \leq M_{12}, \quad x \in [x_1, x_m]
\]
and let $M_1 = M_{11}, M_{12}.$

Part 3: On the end, we have only to apply Lemma 4.1 form [9] to the Equation (16):

$$|a(x)| \leq \left| a(x_0) \right| + \left| \int_{x_0}^x A_0 \frac{d}{dx} e^M \right| \left| \exp \left( \int_{x_0}^x A_2 dx \right) \right|$$

$$\leq \left( |a(x_0)| + M_1 |x-x_0| \right) \exp \left( M_2 |x-x_0| \right).$$

**Remark 3.1.** Let the hypotheses (H2) hold true and let moreover

$$p_1(x) = 0, p_2(x) = x, \ldots, p_l(x) = x^{l-4}, \quad l \geq 1.$$ 

In such a case, we may replace the differentiation of vector-function

$$c(x) = \begin{pmatrix} p_1(x) \\ p_2(x) \\ \vdots \\ p_l(x) \end{pmatrix}$$

by left-multiplication:

$$\frac{dc(x)}{dx} = \begin{pmatrix} 0 \\ 1 \\ 2x \\ 3x^2 \\ \vdots \\ (l-2)x^{l-3} \\ (l-1)x^{l-2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 2 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 3 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -2 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & -1 \end{pmatrix} = \bar{c}c(x).$$

The singular values of the matrix $\bar{c}$ are: $0, 1, \ldots, l-1.$ Therefore $\|\bar{c}\| = \sqrt{l-1}.$ That is why, we may chose

$$M_{22} = \sqrt{(l-1)} \max_{1 \leq i \leq l} \left( \max_{a_i < c_i < a_0} |p_i(x)| \right).$$

Additionally, if we suppose $|x_i| \leq |x_m|,$ then

$$\max_{a_i < c_i < a_0} |p_i(x)| = |p_i(x_m)|, \quad i = 1, \ldots, l.$$ 

Therefore, in such a case:

$$M_{22} = \sqrt{(l-1)} \max_{1 \leq i \leq l} \left( |p_i(x_m)| \right).$$

If we suppose $-1 \leq x_i \leq x \leq x_m \leq 1,$ then obviously, we may set

$$M_{22} = \sqrt{l-1}.$$ 

**References**


