The Estimates of Diagonally Dominant Degree and Eigenvalue Inclusion Regions for the Schur Complement of Matrices

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Abstract
The theory of Schur complement plays an important role in many fields such as matrix theory, control theory and computational mathematics. In this paper, some new estimates of diagonally, $\alpha$-diagonally and product $\alpha$-diagonally dominant degree on the Schur complement of matrices are obtained, which improve some relative results. As an application, we present several new eigenvalue inclusion regions for the Schur complement of matrices. Finally, we give a numerical example to illustrate the advantages of our derived results.

Keywords
Schur Complement, Gerschgorin Theorem, Diagonally Dominant Degree, Eigenvalue

1. Introduction
Let $\mathbb{C}^{n \times n}$ denote the set of all $n \times n$ complex matrices, $N = \{1, 2, \ldots, n\}$ and $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ ($n \geq 2$). We write

\[ R_i(A) = \sum_{j \neq i} |a_{ij}|, \quad C_i(A) = \sum_{j \neq i} |a_{ji}|, \quad i \in N, \]

\[ N_i(A) = \{i \ | \ |a_{ij}| > R_i(A), i \in N \}, \quad N'_i(A) = \{i \ | \ |a_{ij}| > C_i(A), i \in N \}. \]

We know that $A$ is called a strictly diagonally dominant matrix if $|a_{ii}| > R_i(A), \ \forall i \in N$.

$A$ is called an Ostrowski matrix (see [1]) if

\[ |a_{ii}| > R_i(A), \ \forall i \in N. \]
\[ |a_{ii}| \geq R_i (A) R_j (A), \quad \forall i, j \in N, i \neq j. \]

\( SD_n \) and \( OS_n \) will be used to denote the sets of all \( n \times n \) strictly diagonally dominant matrices and the sets all \( n \times n \) Ostrowski matrices, respectively.

As shown in [2], for \( 1 \leq i \leq n \) and \( \alpha \in [0,1] \), we call \( |a_{ii}| - R_i (A), |a_{ii}| - \alpha R_i (A) - (1-\alpha) C_i (A) \) and \( \alpha \) the submatrix of \( A \), respectively.

For \( \beta \subseteq N \), denote by \( |\beta| \) the cardinality of \( \beta \) and \( \overline{\beta} = N \setminus \beta \). If \( \beta, \gamma \subseteq N \), then \( A(\beta, \gamma) \) is the submatrix of \( A \) with row indices in \( \beta \) and column indices in \( \gamma \). In particular, \( A(\beta, \beta) \) is abbreviated to \( A(\beta) \). If \( A(\beta) \) is nonsingular,

\[ A/\beta = A(\beta)^{-1} A(\beta)^T, \]

is called the Schur complement of \( A \) with respect to \( A(\beta) \).

The comparison matrix of \( A \), \( \mu(A) = (\alpha_{ij}) \), is defined by

\[ \alpha_{ij} = \begin{cases} |a_{ij}|, & \text{if } i = j, \\ -|a_{ij}|, & \text{if } i \neq j. \end{cases} \]

A matrix \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \) is called an \( M \)-matrix, if there exist a nonnegative matrix \( B \) and a real number \( s > \rho(B) \), where \( \rho(B) \) is the spectral radius of \( B \), such that \( \rho(A) = sI - B \). It is known that \( A \) is an \( h \)-matrix if and only if \( \mu(A) \) is an \( m \)-matrix, and if \( A \) is an \( m \)-matrix, then the Schur complement of \( A \) is also an \( m \)-matrix and \( \det A > 0 \) (see [3]). We denote by \( H_n \) and \( M_n \) the sets of \( h \)-matrices and \( m \)-matrices, respectively.

The Schur complement of a matrix is an important part of matrix theory, which has been proved to be useful in many fields such as control theory, statistics and computational mathematics. A lot of work has been done on it (see [4]-[8]). We know that the Schur complements of strictly diagonally dominant matrices are strictly diagonally dominant matrices, and the Schur complements of Ostrowski matrices are Ostrowski matrices. These properties have been used for deriving matrix inequalities in matrix analysis and for the convergence of iterations in numerical analysis (see [9]-[12]). More importantly, studying the locations for the eigenvalues of the Schur complement is of great significance, as shown in [2] [6] [13]-[18].

The paper is organized as follows. In Section 2, we give some new estimates of diagonally dominant degree on the Schur complement of matrices. In Section 3, we present several new eigenvalue inclusion regions for the Schur complement of matrices. In Section 4, we give a numerical example to illustrate the advantages of our derived results.

2. The Diagonally Dominant Degree for the Schur Complement

In this section, we present several new estimates of diagonally, \( \alpha \)-diagonally and product \( \alpha \)-diagonally dominant degree on the Schur complement of matrices.

**Lemma 1.** [3] If \( A \in H_n \), then \( \mu(A) \geq A^{-1} \).

**Lemma 2.** [3] If \( A \in SD_n \) or \( A \in OS_n \), then \( A \in H_n \), i.e., \( \mu(A) \in M_n \).

**Lemma 3.** [6] If \( A \in SD_n \) or \( A \in OS_n \) and \( \beta \subseteq N \), then the Schur complement of \( A \) is in \( SD_{|\beta|} \) or \( OS_{|\beta|} \), where \( \overline{\beta} = N - \beta \) is the complement of \( \beta \) in \( N \) and \( |\overline{\beta}| \) is the cardinality of \( \overline{\beta} \).

**Lemma 4.** [16] Let \( a > b , \ c > b , \ b > 0 \) and \( 0 \leq \alpha \leq 1 \). Then

\[ a^\alpha c^{1-\alpha} \geq (a-b)^\alpha (c-b)^{1-\alpha} + b. \]

**Theorem 1.** Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \), \( \beta = \{i_1, i_2, \ldots, i_k\} \subseteq N \), \( \bar{\beta} = \{j_1, j_2, \ldots, j_l\} \), \( 1 \leq k < n \) and \( A/\beta = (a'_{ij}) \). Then for all \( 1 \leq i \leq l \),

\[ |a_i' - R_i (A/\beta)| \geq |a_{i,j} - R_{i,j} (A)| + \delta_i \geq |a_{i,j} - R_{i,j} (A)|, \]

and

\[ |a_i' + R_i (A/\beta)| \leq |a_{i,j} + R_{i,j} (A)| - \delta_i \leq |a_{i,j} + R_{i,j} (A)|, \]

\[ 1 \leq i \leq l. \]
where

\[
\delta_j = \sum_{i=1}^{t} a_{j,h} \left| P_{h,j}(A) \right| \frac{|a_{j,h}|}{\sum_{i=1}^{t} |a_{j,h}|}, \\
\rho = \max_{1 \leq i,j \leq k} \left[ |a_{j,h}| - \sum_{i=1}^{t} |a_{j,h}| \right] R_{h,j},
\]

\[P_{h,j}(A) = rR_{h,j}, \quad 1 \leq v \leq k.\]

**Proof.** Since \( \beta \subseteq N_r(A) \neq \emptyset \), then \( A(\beta) \in H_k \) and \( \mu(A(\beta)) \in M_k \). From Lemma 1 and Lemma 2, we have

\[
\left[ \mu(A(\beta)) \right]^{-1} \geq \left[ A(\beta) \right]^{-1}.
\]

Thus, for any \( \varepsilon > 0 \) and \( 1 \leq t \leq l \), we obtain

\[
\left| a'_{j,h} - R_t(A/\beta) \right| \geq \left| a_{j,h} \right| - \left| \sum_{i=1}^{t} |a_{j,h}| \right| - \left( \delta - \varepsilon \right) - \left| \sum_{i=1}^{t} |a_{j,h}| \right| \left[ \mu(A(\beta)) \right]^{-1} \left( \begin{array}{c}
|a_{j,h}|
n\vdots
|a_{j,h}|
\end{array} \right)
\]

\[
\geq \left| a_{j,h} \right| - \left| \sum_{i=1}^{t} |a_{j,h}| \right| - \left( \delta - \varepsilon \right) - \left| \sum_{i=1}^{t} |a_{j,h}| \right| \left[ \mu(A(\beta)) \right]^{-1} \left( \begin{array}{c}
|a_{j,h}|
n\vdots
|a_{j,h}|
\end{array} \right)
\]

\[
= |a_{j,h}| - R_t(A) + \sum_{i=1}^{t} |a_{j,h}| - \left( \delta - \varepsilon \right) - \sum_{i=1}^{t} |a_{j,h}| \left[ \mu(A(\beta)) \right]^{-1} \left( \begin{array}{c}
|a_{j,h}|
n\vdots
|a_{j,h}|
\end{array} \right)
\]

\[
= |a_{j,h}| - R_t(A) + \delta - \varepsilon + \frac{1}{\det \left[ \mu(A(\beta)) \right]} \left( \begin{array}{c}
\sum_{i=1}^{t} |a_{j,h}| - \delta - \varepsilon \n\sum_{i=1}^{t} |a_{j,h}| \n\sum_{i=1}^{t} |a_{j,h}|
\end{array} \right)
\]

For any \( j, \in \bar{\beta} \), denote

\[
B_x = \left( \begin{array}{cccc}
x & -a_{j,h} & \cdots & -a_{j,h} \\
-\sum_{i=1}^{t} |a_{j,h}| & \cdots & \mu(A(\beta)) \\
\vdots & \cdots & \cdots & \mu(A(\beta)) \\
-\sum_{i=1}^{t} |a_{j,h}| & \cdots & \sum_{i=1}^{t} |a_{j,h}|
\end{array} \right)
\]

If

\[
x > \sum_{i=1}^{t} |a_{j,h}| \left| P_{h,j}(A) \right| \frac{|a_{j,h}|}{\sum_{i=1}^{t} |a_{j,h}|},
\]
then there exists sufficiently small positive number $\epsilon_0$ such that

$$x > \sum_{i=1}^{k} |a_{j,k}| \left( \frac{P_{v}(A)}{|a_{j,k}|} + \epsilon_0 \right).$$

(3)

Construct a positive diagonal matrix $X = \text{diag}(x_1, x_2, \ldots, x_{k+1})$, where

$$x_i = \begin{cases} 1, & \text{if } t = 1 \\ \frac{P_{v}(A)}{|a_{j,k}|} + \epsilon_0, & \text{if } t = 2, 3, \ldots, k+1. \end{cases}$$

Let $\tilde{B} = B_{X} = (\tilde{b}_{pq})$. For $p = 1$, by (3), we have

$$\left| \tilde{b}_{pp} - R_{p} (\tilde{B}) \right| = \left| \tilde{b}_{11} - \sum_{j=2}^{k+1} \tilde{b}_{jj} \right| = x - \sum_{i=1}^{k} |a_{j,k}| \left( \frac{P_{v}(A)}{|a_{j,k}|} + \epsilon_0 \right) > 0.$$

And for $p = 2, 3, \ldots, k+1$, by $\frac{P_{v}(A)}{|a_{j,k}|} \leq r$, $1 \leq v \leq k$, we obtain

$$\left| \tilde{b}_{pp} - R_{p} (\tilde{B}) \right| = \left| a_{p,p-1} \right| \left( \frac{P_{v}(A)}{|a_{p,p-1}|} + \epsilon_0 \right) - \sum_{i=p+1}^{k+1} |a_{p,i-1}| \left( \frac{P_{v}(A)}{|a_{p,i-1}|} + \epsilon_0 \right) - \sum_{i=p+1}^{k+1} |a_{p,i-1}| > 0.$$

Thus, $\tilde{B} \in SD_{k+1}$, and so $B_{r} \in H_{k+1}$. Note that $B_{r} = \mu (B_{r}) \in M_{k+1}$, then

$$\det B_{r} > 0.$$  

(4)

Let $x = \sum_{i=1}^{k} |a_{j,k}| - \delta_{j,k} + \epsilon$ in $B_{r}$. Then

$$\sum_{i=1}^{k} |a_{j,k}| - \delta_{j,k} + \epsilon - \sum_{i=1}^{k} |a_{j,k}| \left( \frac{P_{v}(A)}{|a_{j,k}|} \right) = \sum_{i=1}^{k} |a_{j,k}| - \sum_{i=1}^{k} |a_{j,k}| \left( \frac{P_{v}(A)}{|a_{j,k}|} \right) - \sum_{i=1}^{k} |a_{j,k}| \left( \frac{P_{v}(A)}{|a_{j,k}|} \right) + \epsilon > 0.$$

Since $\det \left[ \mu(A(\beta)) \right] > 0$, by (4), we have

$$\left| a_{v}^{\alpha} \right| - R_{1} (A/\beta) > \left| a_{v}^{\alpha} \right| - R_{1} (A) + \delta_{j,k} - \epsilon.$$

Let $\epsilon \to 0$. Then we obtain (1). Similarly, we can prove (2). □

**Remark 1.** Note that $P_{v}(A) \leq R_{v} (A)$, $1 \leq v \leq k$.

This shows that Theorem 1 improves Theorem 2 of [17] and [2], respectively.

Next, we present some new estimates of $\alpha$-diagonally and product $\alpha$-diagonally dominant degree of the Schur complement.
Theorem 2. Let \( A = (a_{ij}) \in \mathbb{C}^{m \times n} \), \( \beta = \{i_1, i_2, \ldots, i_k\} \subseteq N_r (A) \cap N_c (A) \neq \emptyset \), \( \overline{\beta} = \{j_1, j_2, \ldots, j_l\} \), \( 1 \leq k < n \) and \( A/\beta = (a'_{ij}) \). Then for all \( 1 \leq t \leq l \), \( 0 \leq \alpha \leq 1 \),
\[
|a'_{ij} - \left( R_t (A/\beta) \right)^{\alpha} (C_1 (A/\beta))^{1-\alpha} | \geq |a_{ij} - (R_t (A) - \delta_t)^{\alpha} (C_1 (A) - \delta_t^{1-\alpha}) | \,,
\]
and
\[
|a'_{ij} + \left( R_t (A/\beta) \right)^{\alpha} (C_1 (A/\beta))^{1-\alpha} | \leq |a_{ij} + (R_t (A) - \delta_t)^{\alpha} (C_1 (A) - \delta_t^{1-\alpha}) | \,,
\]
where for any \( 1 \leq v \leq k \),
\[
\delta_t = \sum_{s=1}^{k} |a_{ij} - P_s (A) | \frac{|a_{ij} - P_s (A) |}{|a_{ij} - P_s (A) |} \,,
\]
\[
\eta = \max_{1 \leq s \leq k} \left( \sum_{t=1}^{l} |a_{ij} - Q_s (A) | \right) R_s (A) \,,
\]
\[
\delta_T = \sum_{s=1}^{k} |a_{ij} - Q_s (A) | \frac{|a_{ij} - Q_s (A) |}{|a_{ij} - Q_s (A) |} \,,
\]
\[
\xi = \max_{1 \leq s \leq k} \left( \sum_{t=1}^{l} |a_{ij} - Q_s (A) | \right) C_s (A) \,,
\]
\[
P_s (A) = \eta R_s (A) \,,
\]
\[
Q_s (A) = \xi C_s (A) \,.
\]

Proof. By Lemma 1 and Lemma 2, we have \( \left[ \mu(A(\beta)) \right]^{-1} \geq \left[ A(\beta) \right]^{-1} \). Thus, for all \( 1 \leq t \leq l \), \( 0 \leq \alpha \leq 1 \), we have
\[
|a'_{ij} - \left( R_t (A/\beta) \right)^{\alpha} (C_1 (A/\beta))^{1-\alpha} | \geq |a_{ij} - (R_t (A) - \delta_t)^{\alpha} (C_1 (A) - \delta_t^{1-\alpha}) | \,,
\]
\[
= \left| a_{ij} - \frac{1}{\left( A(\beta) \right)^{-1}} \left( a_{ij} \right) \right| - \sum_{s=1}^{k} \left| a_{ij} + \left( a_{ij} \right) \right| \left( A(\beta) \right)^{-1} \left( a_{ij} \right) \,,
\]
\[
\times \left[ \sum_{t=1}^{l} \left| a_{ij} + \left( a_{ij} \right) \right| \left( A(\beta) \right)^{-1} \left( a_{ij} \right) \right]^{1-\alpha} \,,
\]
\[
\geq |a_{ij} - \left( a_{ij} \right) \left[ \mu(A(\beta)) \right]^{-1} \left( a_{ij} \right) \right| - \sum_{s=1}^{k} \left| a_{ij} + \left( a_{ij} \right) \right| \left( A(\beta) \right)^{-1} \left( a_{ij} \right) \,,
\]
\[
\times \left[ \sum_{t=1}^{l} \left| a_{ij} + \left( a_{ij} \right) \right| \left( A(\beta) \right)^{-1} \left( a_{ij} \right) \right]^{1-\alpha} \,,
\]
\[
= \xi \left( a_{ij} \right) \left[ \mu(A(\beta)) \right]^{-1} \left( a_{ij} \right) \,,
\]
Let
\[
\xi = \left( a_{ij} \right) \left[ \mu(A(\beta)) \right]^{-1} \left( a_{ij} \right) \,.\]

Similar as the proof of Theorem 1, we can prove
Similarly, we have

\[
\sum_{i \in I} \begin{bmatrix} |a_{i,i}| + (|a_{i,k}| & \cdots & |a_{i,k}|) \mu(A(\beta)) \end{bmatrix}^{-1} \begin{bmatrix} a_{i,i} \\ \vdots \\ a_{i,k} \end{bmatrix} \leq R_{i,i}(A) - \delta_i - \zeta.
\]

By Lemma 4, we have

\[
\sum_{i \in I} \begin{bmatrix} |a_{i,i}| + (|a_{i,k}| & \cdots & |a_{i,k}|) \mu(A(\beta)) \end{bmatrix}^{-1} \begin{bmatrix} a_{i,i} \\ \vdots \\ a_{i,k} \end{bmatrix} \leq C_{i,i}(A) - \delta_{i} - \zeta.
\]

Hence, (5) holds. Similarly, we can prove (6).

**Remark 2.** Note that

\[
P_+(A) \leq R_+(A), \quad Q_+(A) \leq C_+(A).
\]

This shows that Theorem 3 improves Theorem 4 of [2].

Similar as the proof of Theorem 2, we can prove the following theorem immediately, which improves Theorem 2 of [2].

**Theorem 3.** Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \), \( \beta = \{i_1, i_2, \ldots, i_k\} \subseteq N_+(A) \cap N_-(A) \neq \emptyset \), \( \beta' = \{j_1, j_2, \ldots, j_l\} \), \( 1 \leq k < n \) and \( A/\beta = (a'_{ij}) \). Then for all \( 1 \leq t \leq l \), \( 0 \leq \alpha \leq 1 \),

\[
|a_{i,t}'| - (R_t(A/\beta))^a(C(A/\beta))^{1-a} 
\geq |a_{i,t}'| - \zeta - (R_t(A) - \delta_t - \zeta)^a(C_h(A) - \delta_t - \zeta)^{1-a}
\geq |a_{i,t}'| - \zeta - [R_{i,i}(A) - \delta_i]^{a}(C_h(A) - \delta_i)^{1-a} - \zeta
= |a_{i,t}'| - (R_t(A) - \delta_t)^a(C_h(A) - \delta_t)^{1-a}.
\]

3. Eigenvalue Inclusion Regions of the Schur Complement

In this section, based on these derived results in Section 2, we present new eigenvalue inclusion regions for the Schur complement of matrices.

**Theorem 4.** Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \), \( \beta = \{i_1, i_2, \ldots, i_k\} \subseteq N_+(A) \neq \emptyset \), \( \beta' = \{j_1, j_2, \ldots, j_l\} \), \( 1 \leq k < n \) and \( A/\beta = (a'_{ij}) \) and \( \lambda \) be eigenvalue of \( A/\beta \). Then there exists \( 1 \leq t \leq l \) such that

\[
|\lambda - a_{i,t}'| \leq R_{i,i}(A) - \delta_i \leq R_{i,i}(A).
\]

**Proof.** By Gerschgorin Circle Theorem, we know that there exists \( 1 \leq t \leq l \) such that \( |\lambda - a_{i,t}'| \leq R_{i,i}(A/\beta) \). Thus, by Lemma 1 and Lemma 2, we have
\[ 0 \geq |\lambda - a'_{h,h}| - R_{h}(A/\beta) \]
\[ = |\lambda - a_{h,h} + (a_{j,k}, \ldots, a_{j,k})[A(\beta)]^{-1} \left( \begin{array}{c} a_{j,k} \\
\vdots \\
a_{j,k} 
\end{array} \right) - \sum_{s=1}^{l} a_{j,k} - (a_{j,k}, \ldots, a_{j,k})[A(\beta)]^{-1} \left( \begin{array}{c} a_{j,k} \\
\vdots \\
a_{j,k} 
\end{array} \right) | \]
\[ \geq |\lambda - a_{h,h}| - \sum_{s=1}^{l} |a_{j,k}| + \sum_{s=1}^{l} |(a_{j,k}, \ldots, a_{j,k})[\mu(A(\beta))]^{-1} \left( \begin{array}{c} a_{j,k} \\
\vdots \\
a_{j,k} 
\end{array} \right) | \]
\[ = |\lambda - a_{h,h}| - R_{h}(A) + \sum_{s=1}^{l} |a_{j,k}| + \delta_{h} - \delta_{h} - \sum_{s=1}^{l} [(a_{j,k}, \ldots, a_{j,k})[\mu(A(\beta))]^{-1} \left( \begin{array}{c} a_{j,k} \\
\vdots \\
a_{j,k} 
\end{array} \right) | \]
\[ \geq |\lambda - a_{h,h}| - R_{h}(A) + \delta_{h}, \]
i.e.,
\[ |\lambda - a_{h,h}| \leq R_{h}(A) - \delta_{h} \leq R_{h}(A). \]

Thus, (7) holds.

**Lemma 5.** [2] Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \) and \( 0 \leq \alpha \leq 1 \). Then for any eigenvalue \( \mu \) of \( A \), there exists \( 1 \leq t \leq n \) such that
\[ |\mu - a_{tt}| \leq (R_{h}(A))^{\alpha} \left( C_{t}(A) \right)^{1-\alpha}. \]

**Theorem 5.** Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \), \( \beta = \{i_1, i_2, \ldots, i_k\} \subseteq N_{i1}(A) \cap N_{i2}(A) \neq \phi \), \( \overline{\beta} = \{j_1, j_2, \ldots, j_l\} \), \( 1 \leq k < n \), \( A/\beta = (a_{ij}') \) and \( \lambda \) be eigenvalue of \( A/\beta \). Then for any \( 0 \leq \alpha \leq 1 \), there exists \( 1 \leq t \leq l \) such that
\[ |\lambda - a_{h,h}'| \leq \left( R_{h}(A) - \delta_{h}' \right)^{\alpha} \left( C_{h}(A) - \delta_{h}' \right)^{1-\alpha} \leq \left( R_{h}(A) \right)^{\alpha} \left( C_{h}(A) \right)^{1-\alpha}. \]

**Proof:** By Lemma 5, we know that there exists \( 1 \leq t \leq l \) such that
\[ |\lambda - a_{tt}'| \leq (R_{h}(A/\beta))^{\alpha} \left( C_{t}(A/\beta) \right)^{1-\alpha}. \]

Therefore,
\[ 0 \geq |\lambda - a_{tt}'| - \left( R_{h}(A/\beta) \right)^{\alpha} \left( C_{t}(A/\beta) \right)^{1-\alpha} \]
\[ = |\lambda - a_{h,h} - (a_{j,k}, \ldots, a_{j,k})[A(\beta)]^{-1} \left( \begin{array}{c} a_{j,k} \\
\vdots \\
a_{j,k} 
\end{array} \right) - \sum_{s=1}^{l} a_{j,k} - (a_{j,k}, \ldots, a_{j,k})[A(\beta)]^{-1} \left( \begin{array}{c} a_{j,k} \\
\vdots \\
a_{j,k} 
\end{array} \right) | \]
\[ \geq |\lambda - a_{h,h} - (a_{j,k}, \ldots, a_{j,k})[A(\beta)]^{-1} \left( \begin{array}{c} a_{j,k} \\
\vdots \\
a_{j,k} 
\end{array} \right) - \sum_{s=1}^{l} a_{j,k} - (a_{j,k}, \ldots, a_{j,k})[A(\beta)]^{-1} \left( \begin{array}{c} a_{j,k} \\
\vdots \\
a_{j,k} 
\end{array} \right) | \]
\[ \times \left( \sum_{s=1}^{l} a_{j,k} + (a_{j,k}, \ldots, a_{j,k})[A(\beta)]^{-1} \left( \begin{array}{c} a_{j,k} \\
\vdots \\
a_{j,k} 
\end{array} \right) \right)^{1-\alpha} \]
\[ \leq |\lambda - a_{h,h} - (a_{j,k}, \ldots, a_{j,k})[A(\beta)]^{-1} \left( \begin{array}{c} a_{j,k} \\
\vdots \\
a_{j,k} 
\end{array} \right) - \sum_{s=1}^{l} a_{j,k} - (a_{j,k}, \ldots, a_{j,k})[A(\beta)]^{-1} \left( \begin{array}{c} a_{j,k} \\
\vdots \\
a_{j,k} 
\end{array} \right) | \]
\[ \times \left( \sum_{s=1}^{l} a_{j,k} + (a_{j,k}, \ldots, a_{j,k})[A(\beta)]^{-1} \left( \begin{array}{c} a_{j,k} \\
\vdots \\
a_{j,k} 
\end{array} \right) \right)^{1-\alpha}. \]
Similar as the proof of Theorem 2, we can prove

\[
\left( a_{j_1}, \ldots, a_{j_k} \right) \left( A(\beta) \right)^{-1} \left( a_{i_1}, \ldots, a_{i_h} \right) + \sum_{t=1}^{l} \left( a_{j_1}, \ldots, a_{j_k} \right) \left( A(\beta) \right)^{-1} \left( a_{i_1}, \ldots, a_{i_h} \right) \right)^{\alpha}
\times \sum_{t=1}^{l} \left( a_{j_1}, \ldots, a_{j_k} \right) \left( A(\beta) \right)^{-1} \left( a_{i_1}, \ldots, a_{i_h} \right) \right)^{1-\alpha}
\leq \left( R_j (A) - \delta \right)^{\alpha} \left( C_j (A) - \delta_i \right)^{1-\alpha}.
\]

Thus, we have

\[
0 \geq | \lambda - a_{j_k} | - (R_j (A/\beta))^\alpha (C_j (A/\beta))^{1-\alpha}
\geq | \lambda - a_{i_h} | - \left( R_j (A) - \delta \right) \left( C_j (A) - \delta_i \right)^{1-\alpha}.
\]

Further, we obtain (8).

### 4. A Numerical Example

In this section, we present a numerical example to illustrate the advantages of our derived results.

**Example 1.** Let

\[
A = \begin{pmatrix} 20 & 2 & 5 & 1 & 4 \\ 2 & 15 & 2 & 4 & 1 \\ 2 & 3 & 17 & 2 & 1 \\ 4 & 3 & 4 & 8 & 1 \\ 5 & 1 & 3 & 3 & 12 \end{pmatrix}, \quad \beta = \{1, 3\}.
\]

By calculation with Matlab 7.1, we have that

\[
R_1 (A) = 12; \quad R_2 (A) = 9; \quad R_3 (A) = 8; \quad R_4 (A) = 12; \quad R_5 (A) = 12;
\]

\[
C_1 (A) = 13; \quad C_2 (A) = 9; \quad C_3 (A) = 14; \quad C_4 (A) = 10; \quad C_5 (A) = 7;
\]

\[
\delta_2 = 2.1800; \quad \delta_4 = 4.3600; \quad \delta_5 = 4.2500; \quad \delta_2^{\beta} = 1.4550; \quad \delta_4^{\beta} = 0.8404; \quad \delta_5^{\beta} = 1.7813.
\]

Since \( \beta \in N_c (A) \), by Theorem 4, the eigenvalue inclusion set of \( A/\beta \) is

\[
\Gamma_1 = \{ \lambda | \lambda - 15 \leq 6.8200 \} \cup \{ \lambda | \lambda - 8 \leq 7.6400 \} \cup \{ \lambda | \lambda - 12 \leq 7.7500 \}.
\]

From Theorem 4 of [2], the eigenvalue inclusion set of \( A/\beta \) is

\[
\Gamma_1^{\ast} = \{ \lambda | \lambda - 15 \leq 7.1412 \} \cup \{ \lambda | \lambda - 8 \leq 8.2824 \} \cup \{ \lambda | \lambda - 12 \leq 8.4118 \}.
\]

We use **Figure 1** to illustrate \( \Gamma_1 \subseteq \Gamma_1^{\ast} \). And the eigenvalues of \( A/\beta \) are denoted by “+” in **Figure 1**. The blue dotted line and green dashed line denote the corresponding discs \( \Gamma_1 \) and \( \Gamma_1^{\ast} \) respectively.

Meanwhile, since \( \beta \in N_r (A) \cap N_c (A) \), by taking \( \alpha = 0.5 \) in Theorem 5, the eigenvalue inclusion set of \( A/\beta \) is
Figure 1. The blue dotted line and green dashed line denote the corresponding discs $\Gamma_1$ and $\Gamma'_1$, respectively.

Figure 2. The blue dotted line and green dashed line denote the corresponding discs $\Gamma_2$ and $\Gamma'_2$, respectively.

$$\Gamma_2 = \{ \lambda \mid \lambda - 15 \leq 7.1733 \} \cup \{ \lambda \mid \lambda - 8 \leq 8.3654 \} \cup \{ \lambda \mid \lambda - 12 \leq 6.3596 \}.$$

From Theorem 5 of [2], the eigenvalue inclusion set of $A/\beta$ is

$$\Gamma'_2 = \{ \lambda \mid \lambda - 15 \leq 7.4492 \} \cup \{ \lambda \mid \lambda - 8 \leq 8.7751 \} \cup \{ \lambda \mid \lambda - 12 \leq 6.7544 \}.$$

We use Figure 2 to illustrate $\Gamma_2 \subseteq \Gamma'_2$. And the eigenvalues of $A/\beta$ are denoted by “+” in Figure 2. The blue dotted line and green dashed line denote the corresponding discs $\Gamma_2$ and $\Gamma'_2$ respectively. It is clear that $\Gamma_1 \not\subseteq \Gamma_2$ and $\Gamma_2 \not\subseteq \Gamma_1$.

References


