Relation between Two Operator Inequalities

\[ f \left( \frac{1}{B^2 AB^2} \right) \geq B^{-1} \quad \text{and} \quad A^{-1} \geq g \left( \frac{1}{A^2 BA^2} \right) \]

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Abstract

We shall show relation between two operator inequalities

\[ f \left( \frac{1}{B^2 AB^2} \right) \geq B^{-1} \quad \text{and} \quad A^{-1} \geq g \left( \frac{1}{A^2 BA^2} \right) \]

for positive, invertible operators \( A \) and \( B \), where \( f \) and \( g \) are non-negative continuous invertible functions on \((0, \infty)\) satisfying \( f(t)g(t) = t^{-1} \).

Keywords

Operator Inequality, Orthoprojection, Representing Function

1. Introduction

We denote by capital letter \( A, B \) et al. the bounded linear operators on a complex Hilbert space \( H \). An operator \( T \) on \( H \) is said to be positive, denoted by \( T \geq 0 \) if \( (Tx, x) \geq 0 \) for all \( x \in H \).

M. Ito and T. Yamazaki [1] obtained relations between two inequalities

\[ \left( \frac{r}{B^2 AB^2} \right)^{\frac{p}{p+q}} \geq B' \quad \text{and} \quad A' \geq \left( \frac{p}{A^2 BA^2} \right)^{\frac{p}{p+q}}, \quad (1.1) \]

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\[ f \left( \frac{1}{B^2 AB^2} \right) \geq B^{-1} \quad \text{and} \quad A^{-1} \geq g \left( \frac{1}{A^2 BA^2} \right). \quad \text{Advances in Pure Mathematics, 5, 93-99. http://dx.doi.org/10.4236/apm.2015.52012} \]
and Yamazaki and Yanagida [2] obtained relation between two inequalities
\[
\frac{p}{p+r} I + \frac{r}{p+r} B^2 A^p B^q \geq B^q \quad \text{and} \quad A^p \geq A^2 B^p A^q \left( \frac{p}{p+r} I + \frac{r}{p+r} A^2 B^p A^q \right)^{-1},
\]
for (not necessarily invertible) positive operators \(A\) and \(B\) and for fixed \(p \geq 0\) and \(r \geq 0\). These results led M. Ito [3] to obtain relation between two operator inequalities
\[
f \left( \frac{1}{2} B^2 A B^2 \right) \geq B \quad \text{and} \quad A \geq g \left( \frac{1}{2} A^2 BA^2 \right),
\]
for (not necessarily invertible) positive operators \(A\) and \(B\), where \(f\) and \(g\) are non-negative continuous functions on \((0, \infty)\) satisfying \(f(t) = t^1\).

Remarks (1.1): The two inequalities in (1.1) are closely related to Furuta inequalities [4].

The inequalities in (1.1) and (1.2) are equivalent, respectively, if \(A\) and \(B\) are invertibles; but they are not always equivalent. Their equivalence for invertible case was shown in [5].

Motivated by the result (1.3) of M. Ito [3], we obtain the results taking representing functions \(f\) and \(g\) as non-negative continuous invertible functions on \((0, \infty)\) satisfying \(f(t) g(t) = t^1\).

2. Main Results
We denote by \(N(T)\) the kernel of an operator \(T\).

Theorem 1: Let \(A\) and \(B\) be positive invertible operators, and let \(f\) and \(g\) be non-negative invertible continuous functions on \((0, \infty)\) satisfying \(f(t) g(t) = t^1\). Then the following hold:
1) \(f \left( \frac{1}{2} B^2 AB^2 \right) \geq B^{-1}\) ensures \(A^{-1} - g \left( \frac{1}{2} A^2 BA^2 \right) \geq A^{-1} E_{\beta^{-1}} A^{-1} - g(\infty) E_{\alpha^{-1} \beta^{-1}}\).
2) \(B^{-1} \geq f \left( \frac{1}{2} B^2 AB^2 \right)\) ensures \(g \left( \frac{1}{2} A^2 BA^2 \right) - A^{-1} \geq g(\infty) E_{\alpha^{-1} \beta^{-1}} - A^{-1} E_{\beta^{-1}} A^{-1}\).

Here \(E_{\beta^{-1}}\) and \(E_{\alpha^{-1} \beta^{-1}}\) denote orthoprojections to \(N(B^{-1})\) and \(N \left( \frac{1}{2} A^2 BA^2 \right)\) respectively.

The following Lemma is helpful in proving our results:
Lemma 2: If \(h(t)\) is a continuous function on \((0, r^2)\) and \(T\) is an invertible operator with \(\|T\| \leq r\), then
\[
\frac{1}{T^* T} h(T^* T) = \frac{1}{T} h(TT^*) \frac{1}{T^*}.
\]

Proof of Lemma: Since \(h(t)\) is a continuous function on \([0, r^2]\), it can be uniformly approximated by a sequence of polynomials on \([0, r^2]\). We may assume that \(h(t)\) itself is a polynomial \(h(T) = \sum_{k=0}^{n} \alpha_k t^k\). Then
\[
h(T^* T) = \sum_{k=0}^{n} \alpha_k (T^* T)^k = \sum_{k=0}^{n} \alpha_k T^k T^* = \sum_{k=0}^{n} \alpha_k T^k (TT^*)^k = \sum_{k=0}^{n} \alpha_k \left( (T^* T)^k \right) = \sum_{k=0}^{n} \alpha_k \left( T^k h(TT^*) T^k \right)
\]
\[
\Rightarrow \frac{1}{T^* T} h(T^* T) = \frac{1}{T} h(TT^*) \frac{1}{T^*}.
\]
Hence the result.

Proof of Theorem 1: For \(\varepsilon > 0\), let \(f_\varepsilon(t) = f(t) + \varepsilon\) and \(g_\varepsilon(t) = \frac{1}{\varepsilon f_\varepsilon(t)} = \frac{1}{t f(t) + \varepsilon}; \quad 0 < t < \infty\)
1) We suppose that \( f\left(\frac{1}{B^2AB^2}\right) \geq B^{-1} \). Then

\[
f_c\left(\frac{1}{B^2AB^2}\right) = f\left(\frac{1}{B^2AB^2}\right) + \varepsilon I \geq B^{-1} + \varepsilon I.
\]

Let \( h_c(t) = \frac{1}{f_c(t)} \) and \( T = B^2A^2 \) then

\[
A^{-1} - g_c\left(\frac{1}{A^2BA^2}\right) = A^{-1} - \frac{1}{\left(A^2BA^2\right)f_c\left(\frac{1}{A^2BA^2}\right)}
\]

\[
= A^{-1} - \frac{1}{T^TF_c\left(T^T\right)}
\]

\[
= A^{-1} \cdot \frac{1}{h_c\left(T^T\right)}
\]

\[
= A^{-1} \cdot \frac{1}{h_c\left(TT^*\right)}
\]

\[
= A^{-1} \cdot \frac{1}{T^T}
\]

\[
\geq A^{-1} \cdot \frac{1}{B^2A^2} \cdot \frac{1}{B^{-1} + \varepsilon I} \cdot \frac{1}{A^2B^2}
\]

\[
= A^{-1} \cdot \frac{1}{B^2A^2} \cdot \frac{1}{B^{-1} + \varepsilon I} \cdot \frac{1}{A^2B^2}
\]

\[
= A^{-1} \cdot \frac{1}{B^{-1} + \varepsilon I} \cdot \frac{1}{A^2}
\]

\[
= A^{-1} \cdot \frac{1}{B^{-1} + \varepsilon I} \cdot \frac{1}{A^2}
\]

\[
We have \( \lim_{t \to 0} E_{g^{-1}} = g(t) \).
\]

Further since \( g_c(t) \) increases as \( \varepsilon \) decreases and

\[
\lim_{c \to 0} g_c(t) = \begin{cases} g(t), & \text{when } t \neq 0 \\ 0, & \text{when } t \to \infty, \end{cases}
\]

we have

\[
\lim_{c \to 0} \left\{ A^{-1} - g_c\left(\frac{1}{A^2BA^2}\right) \right\} = A^{-1} - \left\{ g\left(\frac{1}{A^2BA^2}\right) - g(\varepsilon)E_{\frac{1}{A^2BA^2}} \right\}.
\]

Then

\[
A^{-1} - \left\{ g\left(\frac{1}{A^2BA^2}\right) - g(\varepsilon)E_{\frac{1}{A^2BA^2}} \right\} = \lim_{c \to 0} \left\{ A^{-1} - g_c\left(\frac{1}{A^2BA^2}\right) \right\} \geq \lim_{c \to 0} A^{-1} \left( B^{-1} + \varepsilon I \right)^{-1} A^2
\]

i.e.
\[
A^{-1} - g\left(\frac{1}{A^2BA^2}\right) \geq A^{-1} - g\left(\infty\right)E_{A^2BA^2}^{-1} - g\left(\infty\right)E_{A^2BA^2}^{-1}.
\]

2) We suppose that \(B^{-1} \geq f\left(\frac{1}{B^2AB^2}\right)\); i.e. \(f\left(\frac{1}{B^2AB^2}\right) \leq B^{-1}\), then

\[
f_c\left(\frac{1}{B^2AB^2}\right) = f\left(\frac{1}{B^2AB^2}\right) + \epsilon I \leq B^{-1} + \epsilon I.
\]

With \(h_c(t) = \frac{1}{f_c(t)}\) and \(T = B^2A^2\), we have by Lemma 2

\[
g_c\left(\frac{1}{A^2BA^2}\right) - A^{-1} = g_c\left(T^*T\right) - A^{-1}
\]

\[
= \frac{1}{T^*Tf_c(T^*T)} - A^{-1}
\]

\[
= \frac{1}{T^*T} h_c(T^*T) - A^{-1}
\]

\[
= \frac{1}{T} h_c(TT^*) \frac{1}{T} - A^{-1}
\]

\[
= \frac{1}{B^2A^2} h_c\left(\frac{1}{B^2AB^2}\right) \left[\frac{1}{A^2B^2}\right] - A^{-1}
\]

\[
= \frac{1}{B^2A^2} h_c\left(\frac{1}{B^2AB^2}\right) \left[\frac{1}{A^2B^2}\right] - A^{-1}
\]

\[
\geq \frac{1}{B^2A^2} h_c\left(\frac{1}{B^2AB^2}\right) \left[\frac{1}{A^2B^2}\right] - A^{-1}
\]

\[
= A^{-1} \left[\frac{1}{B^2AB^2} - B^{-1} + I\right] A^{-1}
\]

\[
= A^{-1} \left[\frac{1}{B^2AB^2} - B^{-1} + I\right] A^{-1}
\]

\[
= -\epsilon A^{-1} \left[\frac{1}{B^2AB^2} - B^{-1} + I\right] A^{-1}
\]

Now as \(\lim_{t \to 0} g_c(t) = E_{A^2BA^2}\) and since

\[
\lim_{t \to 0} g_c(t) = \begin{cases} g(t), & \text{when } t \neq 0 \\ 0, & \text{when } t \to \infty, \end{cases}
\]

we have

\[
\lim_{t \to 0} g_c\left(\frac{1}{A^2BA^2}\right) - A^{-1} = g\left(\frac{1}{A^2BA^2}\right) - g\left(\infty\right)E_{A^2BA^2}^{-1} - A^{-1}.
\]

Then
\[
g\left(\frac{A^2BA^2}{\lambda}\right) - g(\lambda) E_{\frac{A^2BA^2}{\lambda^2}} - A^{-1} = \lim_{t \to 0} \left(\frac{g_c\left(\frac{A^2BA^2}{\lambda}\right)}{\lambda^2} - A^{-1}\right)
\geq \lim_{t \to 0} -\frac{1}{\lambda^2} A^{-1} B\left(\frac{1}{\lambda^2} + \varepsilon I\right)^{-1} A^{-1} \frac{1}{\lambda^2}
= - A^{-1} E_{\frac{1}{\lambda^2}} A^{-1} \frac{1}{\lambda^2}
\Rightarrow g\left(\frac{A^2BA^2}{\lambda}\right) - A^{-1} \geq g(\lambda) E_{\frac{A^2BA^2}{\lambda^2}} - A^{-1} E_{\frac{1}{\lambda^2}} A^{-1} \frac{1}{\lambda^2}.
\]

thus completing the proof of 2.

Corollary 3. Let \(A\) and \(B\) be positive invertible operators, and let \(f\) and \(g\) be non-negative continuous invertible functions on \((0, \infty)\) satisfying \(f(0) = 0\). Then for each \(0 \leq p \leq 1\) and \(0 \leq r \leq 1\), the following hold

1) If \(f(\lambda) = 0\) or \(N\left(\frac{A^2BA^2}{\lambda^2}\right) = \{0\}\), then \(f\left(\frac{1}{\lambda^2} B^2 A B^2\right) \geq B^2\) ensures \(A^{-1} \geq g\left(\frac{A^2BA^2}{\lambda^2}\right)\).

2) If \(N\left(B^{-1}\right) \subseteq N\left(A^{-1}\right)\), then \(B^{-1} \geq f\left(\frac{1}{\lambda^2} B^2 A B^2\right)\) ensures \(g\left(\frac{A^2BA^2}{\lambda^2}\right) \geq A^{-1}\).

Proof 1) This result follows from 1) of Theorem 1 because each of the conditions \(g(\infty) = 0\) and \(N\left(\frac{A^2BA^2}{\lambda^2}\right) = \{0\}\) implies \(g(\infty) E_{\frac{A^2BA^2}{\lambda^2}} = 0\), so that
\[
A^{-1} - g\left(\frac{A^2BA^2}{\lambda^2}\right) \geq A^{-1} E_{\frac{1}{\lambda^2}} A^{-1} \frac{1}{\lambda^2} - g(\infty) E_{\frac{A^2BA^2}{\lambda^2}} = A^{-1} E_{\frac{1}{\lambda^2}} A^{-1} \frac{1}{\lambda^2} \geq 0
\]
\[
\Rightarrow A^{-1} \geq g\left(\frac{A^2BA^2}{\lambda^2}\right).
\]

2) This result follows from 2) of Theorem 1 because \(N\left(B^{-1}\right) \subseteq N\left(A^{-1}\right)\) implies \(A^{-1} E_{\frac{1}{\lambda^2}} A^{-1} \frac{1}{\lambda^2} = 0\), so that
\[
g\left(\frac{A^2BA^2}{\lambda^2}\right) - A^{-1} \geq g(\infty) E_{\frac{A^2BA^2}{\lambda^2}} - A^{-1} E_{\frac{1}{\lambda^2}} A^{-1} \frac{1}{\lambda^2}
= g(\infty) E_{\frac{A^2BA^2}{\lambda^2}} \geq 0.
\]

Hence the proof is complete.

Remark 3.1 1) If \(f(\infty) > 0\), then automatically \(g(\infty) = 0\) since \(f(\infty) g(\infty) = \frac{1}{\infty} = 0\), so 1) of corollary 3 holds without any conditions.
2) The invertibility of positive operators \(A\) and \(B\) is necessary condition.
3) We have considered \((0, \infty)\) instead of \([0, \infty)\) because the requirement of the limit.
\[
\lim_{t \to 0} g_c(t) = 0 \quad \text{when} \quad \lambda = 0 \quad \text{is not fulfilled,} \quad \text{rather it is fulfilled when} \quad t \to \infty \quad \text{because} \quad g_c(t) = \frac{1}{t f_c(t)}.
\]

We have the following results as a consequence of corollary 3.

**Theorem 4:** Let \(A\) and \(B\) be positive invertible operators. Then for each \(p \geq 0\) and \(r > 0\), the following hold

1) If \(B^2 A^p B^2 \geq B^r\) then \(A^p \geq \left(\frac{A^2B^r}{A^2B^2}\right)^{\frac{p}{r}}\).
2) If \[ A^p \geq \left( \frac{p}{A^2 B^+ A^*} \right)^{\frac{p}{p+r}} \] and \[ N(A^{-1}) \subseteq N(B^{-1}) \text{ then } \left( B^\frac{r}{p} A^p B^\frac{r}{p} \right)^{\frac{p}{p+r}} \geq B^r. \]

In Theorem 4 we consider that \( t^0 = 1 \) for \( t > 0 \) or \( t^0 = 0 \) when \( t \to \infty \) and we define \( T^0 = I - E_A \) for a positive invertible operator \( T \).

**Theorem 5:** Let \( A \) and \( B \) be positive invertible operators. Then for each \( p > 0 \) and \( r > 0 \), the following hold:

1) If \[ A^p \geq \frac{p}{A^2 B^+ A^*} \quad \text{and} \quad N(A^{-1}) \subseteq N(B^{-1}) \text{ then } \left( \frac{p}{A^2 B^+ A^*} \right)^{\frac{p}{p+r}} \geq B^r. \]

2) If \[ A^p \geq \frac{p}{A^2 B^+ A^*} \quad \text{and} \quad N(A^{-1}) \subseteq N(B^{-1}) \text{ then } \left( \frac{p}{A^2 B^+ A^*} \right)^{\frac{p}{p+r}} \geq B^r. \]

**Proof of Theorem 4:**

1) First we consider the case when \( p > 0 \) and \( r > 0 \). Replacing \( A \) with \( rB \) and \( B \) with \( A^p \) and putting \( f(t) = t^\frac{r}{p+r} \) and \( g(t) = t^\frac{p}{p+r} \) in 1) of Corollary 3 so that \( f(t) g(t) = t^{-1} \), we have

\[
\text{if } \left( B^\frac{r}{p} A^p B^\frac{r}{p} \right)^{\frac{p}{p+r}} \geq B^r \text{ then } A^p \geq \left( \frac{p}{A^2 B^+ A^*} \right)^{\frac{p}{p+r}}. \] (5.1)

If \( p = 0 \) and \( r > 0 \), (5.1) means that

\[
\text{if } \left[ B^\frac{r}{p} (I - E_A) B^\frac{r}{p} \right]^{-1} \geq B^r \text{ then } I - E_A \geq I - E_{(I - E_A) B^r (I - E_A)}
\]

i.e.,

\[
\text{if } \left[ B^r - B^\frac{r}{p} E_A B^\frac{r}{p} \right]^{-1} \geq B^r \text{ then } I - E_A \geq I - E_{(I - E_A) B^r (I - E_A)}
\]

i.e., if \( (I - E_A)^{-1} \geq I \) then \( I - E_A \geq I - E_{(I - E_A) B^r (I - E_A)} \)

i.e., if \( (I - E_A) \leq I \) then \( I - E_A \geq I - E_{(I - E_A) B^r (I - E_A)} \)

or in other words, \( B^\frac{r}{p} E_A B^\frac{r}{p} = 0 \) ensures \( E_{(I - E_A) B^r (I - E_A)} \geq E_A \).

But, since \( B^\frac{r}{p} E_A B^\frac{r}{p} = 0 \) implies \( (I - E_A) B^r (I - E_A) = B^r \), it follows an equivalent assertion

\[
B^\frac{r}{p} E_A B^\frac{r}{p} = 0 \text{ ensures } E_{B^r} \geq E_A, \text{ i.e., } E_{B^r} = E_{B^r} = E_A \text{ which is further equivalent to the trivial assertion } N(A) \subseteq N(B^{-1}) \text{ ensures } N(A) \subseteq N(B^{-1}).
\]

2) Again first we consider the case \( p > 0 \) and \( r > 0 \). Replacing \( A \) with \( B^r \) and \( B \) with \( A^p \) and putting \( f(t) = t^\frac{r}{p+r} \) and \( g(t) = t^\frac{p}{p+r} \) in 2) of Corollary 3.

Since \( N(A^p) = N(A^{-1}) \subseteq N(B^{-1}) = N(B^r) \), we have

\[
A^p \geq \left( \frac{p}{A^2 B^+ A^*} \right)^{\frac{p}{p+r}} \text{ ensures } \left[ B^\frac{r}{p} A^p B^\frac{r}{p} \right]^{\frac{p}{p+r}} \geq B^r. \] (5.2)

If \( p = 0 \) and \( r > 0 \), (5.2) means that \( (I - E_A) \geq I - E_{(I - E_A) B^r (I - E_A)} \) ensures \( \left[ B^\frac{r}{p} (I - E_A) B^\frac{r}{p} \right]^{-1} \geq B^r \) i.e.,
\((I - E_d) \geq I - E_{(I-E_d)B^{-r}(I-E_d)}\)

ensures \(B^{\frac{r}{2}} E_d B^{\frac{r}{2}} = 0\), \hspace{1cm} (5.3)

which implies that \((I - E_d)B^{-r}(I - E_d) = B^{-r}\).

Hence (5.3) means that \(E_{\tilde{y}} = E_{\tilde{y}} \geq E_d\) ensures \(B^{\frac{r}{2}} E_d B^{\frac{r}{2}} = 0\), i.e. \(N(A) \subseteq N(B^{-1})\) ensures \(N(A) \subseteq N(B^{-1})\).

Hence the result.

Proof of Theorem 5: We can prove by the similar way to Theorem 4 for \(p > 0\) and \(r > 0\), replacing \(A\) with \(A^p\) and \(B\) with \(B^{-r}\) and putting \(f(t) = \frac{-p}{p+r} - \frac{r}{p+r} - t\) and \(g(t) = t\left(-\frac{r}{p+r} - \frac{p}{p+r}\right)^{-1}\) for 1) in 1) of Corollary 3 and replacing \(A\) with \(B^{-r}\) and \(B\) with \(A^p\) and putting \(f(t) = t\left(-\frac{r}{p+r} - \frac{p}{p+r}\right)^{-1}\) and \(g(t) = -\frac{p}{p+r} - \frac{r}{p+r} - t\) for 2) in 2) of Corollary 3.

**Corollary 4**: Let \(A\) and \(B\) be positive invertible operators, and let \(f\) and \(g\) be non-negative continuous invertible functions on \((0, \infty)\) satisfying \(f(t) g(t) = t^{-1}\). If \(N\left(\frac{1}{A^2BA^2}\right) = \{0\}\), then

\[
\begin{align*}
    f\left(\frac{1}{B^{\frac{r}{2}}AB\frac{1}{2}}\right) \geq B^{-1} \Rightarrow A^{-1} \geq g\left(\frac{1}{A^2BA^2}\right).
\end{align*}
\]

Proof: The proof \((\Rightarrow)\) follows directly by applying the condition \(N\left(\frac{1}{A^2BA^2}\right) = \{0\}\), in 1) of Corollary 3 and for the proof \((\Leftarrow)\) we have only to interchange the roles of \(A\) and \(B\) and those of \(f\) and \(g\) in 2) of Corollary 3. Since \(\{0\} = N\left(A^{-1}\right) \subseteq N\left(B^{-1}\right)\) if \(N\left(\frac{1}{A^2BA^2}\right) = \{0\}\).

**References**

[1] Ito, M. and Yamazaki, T. (2002) Relations between Two Inequalities \(\left(B^{\frac{r}{2}}A^pB^{\frac{1}{2}}\right)^{\frac{1}{p+r}} \geq B^r\) and \(A^r \geq \left(A^\frac{r}{2}B^{\frac{1}{2}}A^{\frac{1}{2}}\right)^{\frac{p}{p+r}}\) and Their Applications. *Integral Equations and Operator Theory*, 44, 442-450. [http://dx.doi.org/10.1007/BF01193670](http://dx.doi.org/10.1007/BF01193670)


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