Argument Estimates of Multivalent Functions Involving a Certain Fractional Derivative Operator

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Abstract
The object of the present paper is to investigate various argument results of analytic and multivalent functions which are defined by using a certain fractional derivative operator. Some interesting applications are also considered.

Keywords
Multivalent Analytic Functions, Argument, Integral Operator, Fractional Derivative Operator

1. Introduction
Let \( A(p) \) denote the class of functions \( f(z) \) of the form
\[
f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}),
\]
which are analytic in the open unit disk \( \mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\} \). Also let \( A(1) = A \) denote the class of all analytic functions \( p(z) \) with \( p(0) = 1 \) which are defined on \( \mathbb{U} \).

Let \( a, b \) and \( c \) be complex numbers with \( c \neq 0, -1, -2, \ldots \). Then the Gaussian hypergeometric function \(_2 F_1 (a, b; c; z)\) is defined by
\[
_2 F_1 (a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} \frac{z^k}{k!}, \quad (1.2)
\]
where \((\eta)_k\) is the Pochhammer symbol defined, in terms of the Gamma function, by

\[
(\eta)_k = \frac{\Gamma(\eta+k)}{\Gamma(\eta)} = \begin{cases} 1, & (k = 0); \\ \eta(\eta+1)\cdots(\eta+k-1), & (k \in \mathbb{N}). \end{cases}
\]

The hypergeometric function \( _2F_1(a,b;c;z) \) is analytic in \( \mathbb{U} \) and if \( a \) or \( b \) is a negative integer, then it reduces to a polynomial.

There are a number of definitions for fractional calculus operators in the literature (cf., e.g., [1] and [2]). We use here the Saigo type fractional derivative operator defined as follows ([3]; see also [4]):

**Definition 1.** Let \( 0 \leq \lambda < 1 \) and \( \mu, \nu \in \mathbb{R} \). Then the generalized fractional derivative operator \( J_{0,z}^{\lambda,\mu,\nu} \) of a function \( f(z) \) is defined by

\[
J_{0,z}^{\lambda,\mu,\nu} f(z) = \frac{d^m}{dz^m} J_{0,z}^{\lambda,\mu,\nu} f(z), \quad (z \in \mathbb{U}; m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})
\]

The function \( f(z) \) is an analytic function in a simply-connected region of the \( z \)-plane containing the origin, with the order

\[
f(z) = O((\mathbb{U})^{\epsilon}), \quad (z \to 0)
\]

for \( \epsilon > \max\{0,\mu-\nu\} - 1 \), and the multiplicity of \((z-\zeta)^{-\lambda}\) is removed by requiring that \( \log(z-\zeta) \) to be real when \( z-\zeta > 0 \).

**Definition 2.** Under the hypotheses of Definition 1, the fractional derivative operator \( J_{0,z}^{\lambda,\mu,\nu,\nu} \) of a function \( f(z) \) is defined by

\[
J_{0,z}^{\lambda,\mu,\nu,\nu} f(z) = \frac{d^m}{dz^m} J_{0,z}^{\lambda,\mu,\nu,\nu} f(z), \quad (z \in \mathbb{U}; m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})
\]

With the aid of the above definitions, we define a modification of the fractional derivative operator \( \Delta_{z,p}^{\lambda,\mu,\nu} \) by

\[
\Delta_{z,p}^{\lambda,\mu,\nu} f(z) = \int_{\mathbb{U}} \frac{p+1}{(p+1-\mu)\Gamma(p+1-\mu+\nu)} \sum_{k=0}^\infty \frac{(p+1)_k (p+1-\mu+\nu)_k}{(p+1-\mu+\nu)_k} \left(a_k^p z^{k+p}\right) f(z) dz
\]

for \( f(z) \in \mathbb{A}(p) \) and \( \mu-\nu-p < 0 \). Then it is observed that \( \Delta_{z,p}^{\lambda,\mu,\nu} \) also maps \( \mathbb{A}(p) \) onto itself as follows:

\[
\Delta_{z,p}^{\lambda,\mu,\nu} f(z) = z^p + \sum_{k=1}^\infty \frac{(p+1)_k (p+1-\mu+\nu)_k}{(p+1-\mu+\nu)_k} a_k^p z^{k+p},
\]

\[
(z \in \mathbb{U}; 0 \leq \lambda < 1; \mu-\nu-p < 1; f \in \mathbb{A}(p)).
\]

It is easily verified from (1.6) that

\[
z \left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z)\right)' = (p-\mu)\Delta_{z,p}^{\lambda,\mu,\nu,\nu} f(z) + \mu \Delta_{z,p}^{\lambda,\mu,\nu} f(z).
\]

Note that \( \Delta_{z,p}^{0,0,\nu} f = f \), \( \Delta_{z,p}^{1,0,\nu} f = z f' / p \) and \( \Delta_{z,p}^{1,1,\nu} f = \Omega_{z}^{(1,p)} f \), where \( \Omega_{z}^{(1,p)} \) is the fractional derivative operator defined by Srivastava and Aouf [5].

In this manuscript, we drive interesting argument results of multivalent functions defined by fractional derivative operator \( \Delta_{z,p}^{\lambda,\mu,\nu} \).

2. Main Results

In order to establish our results, we require the following lemma due to Lashin [6].

**Lemma 1** [6]. Let \( h(z) \) be analytic in \( \mathbb{U} \), with \( h(0) = 1 \) and \( h(z) \neq 0 \) \( (z \in \mathbb{U}) \). Further suppose that \( \alpha, \beta \in \mathbb{R}^+ = (0, \infty) \) and

\[
\left|\arg(h(z) + \beta zh'(z))\right| < \frac{\pi}{2} \left(\alpha + \frac{2}{\pi} \arctan(\beta\alpha)\right) \quad (\alpha > 0; \beta > 0)
\]

then

...
We begin by proving the following result.

**Theorem 1.** Let \( \lambda \geq 0 \), \( \mu < \min \{v + p + 1, p\} \) and \( \alpha, \gamma, \delta \in \mathbb{R}^+ \), and let \( g(z) \in \mathcal{A}(p) \). Suppose that \( f(z) \in \mathcal{A}(p) \) satisfies the condition

\[
\arg \left( \frac{\Delta^{\lambda,\mu,v}_{z,p} f(z)}{\Delta^{\lambda,\mu,v}_{z,p} g(z)} \right) \left\{ 1 + \delta \left( \frac{\Delta^{\lambda+1,\mu+1,v+1}_{z,p} f(z)}{\Delta^{\lambda,\mu,v}_{z,p} f(z)} - \frac{\Delta^{\lambda+1,\mu+1,v+1}_{z,p} g(z)}{\Delta^{\lambda,\mu,v}_{z,p} g(z)} \right) \right\} < \frac{\pi}{2} \left( \alpha + \frac{\delta}{\gamma(p-\mu)} \right),
\]

(2.2)

then

\[
\arg \left( \frac{\Delta^{\lambda,\mu,v}_{z,p} f(z)}{\Delta^{\lambda,\mu,v}_{z,p} g(z)} \right) < \frac{\pi}{2} \alpha, \quad (z \in \mathbb{U}).
\]

(2.3)

**Proof.** If we define the function \( h(z) \) by

\[
h(z) = \left( \frac{\Delta^{\lambda,\mu,v}_{z,p} f(z)}{\Delta^{\lambda,\mu,v}_{z,p} g(z)} \right)^\gamma, \quad (\gamma \neq 0),
\]

(2.4)

then \( h(z) = 1 + c_1 z + c_2 z^2 + \cdots \) is analytic in \( \mathbb{U} \), with \( h(0) = 1 \) and \( h'(0) \neq 0 \). Making use of the logarithmic differentiation on both sides of (2.5), we have

\[
\frac{1}{\gamma h(z)} z h'(z) = \frac{z \left( \Delta^{\lambda,\mu,v}_{z,p} f(z) \right)'}{\Delta^{\lambda,\mu,v}_{z,p} f(z)} - \frac{z \left( \Delta^{\lambda,\mu,v}_{z,p} g(z) \right)'}{\Delta^{\lambda,\mu,v}_{z,p} g(z)}.
\]

(2.5)

By applying the identity (1.7) in (2.6), we observe that

\[
h(z) + \frac{\delta}{\gamma(p-\mu)} z h'(z) = \left( \frac{\Delta^{\lambda,\mu,v}_{z,p} f(z)}{\Delta^{\lambda,\mu,v}_{z,p} g(z)} \right)^\gamma \left\{ 1 + \delta \left( \frac{\Delta^{\lambda+1,\mu+1,v+1}_{z,p} f(z)}{\Delta^{\lambda,\mu,v}_{z,p} f(z)} - \frac{\Delta^{\lambda+1,\mu+1,v+1}_{z,p} g(z)}{\Delta^{\lambda,\mu,v}_{z,p} g(z)} \right) \right\}.
\]

(2.6)

Hence, by using Lemma 1, we conclude that

\[
\arg h(z) < \frac{\pi}{2} \alpha, \quad (z \in \mathbb{U}),
\]

which completes the proof of Theorem 1.

**Remark 1.** Putting \( \lambda = \mu = 0 \), \( \delta = p = 1 \) and \( g(z) = z \) in Theorem 1, we obtain the result due to Lashin ([6], Theorem 2.2).

Taking \( \gamma = 1 \) and \( g(z) = z^\nu \) in Theorem 1, we have the following corollary.

**Corollary 1.** Let \( \lambda \geq 0 \), \( \mu < \min \{v + p + 1, p\} \) and \( \alpha, \delta \in \mathbb{R}^+ \). Suppose that \( f(z) \in \mathcal{A}(p) \) satisfies the condition

\[
\arg \left( 1 - \delta \frac{\Delta^{\lambda,\mu,v}_{z,p} f(z)}{z^\nu} + \delta \frac{\Delta^{\lambda+1,\mu+1,v+1}_{z,p} f(z)}{z^\nu} \right) < \frac{\pi}{2} \left( \alpha + \frac{2}{\pi} \arctan \left( \frac{\delta \alpha}{p-\mu} \right) \right),
\]

then

\[
\arg \left( \frac{\Delta^{\lambda,\mu,v}_{z,p} f(z)}{z^\nu} \right) < \frac{\pi}{2} \alpha, \quad (z \in \mathbb{U}).
\]

**Theorem 2.** Let \( \lambda \geq 0 \), \( \mu < \min \{v + p + 1, p\} \), \( 0 < \delta \leq 1 \) and \( \alpha, \delta \in \mathbb{R}^+ \). Suppose that \( f(z) \in \mathcal{A}(p) \) sat-
tisfies the condition
\[
\left| \arg \left( \frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^p} \right) \right| < \frac{\pi}{2} \left( \alpha + \frac{2}{\pi} \arctan \left[ \frac{\delta}{\gamma(p-\mu)} \right] \right), \quad (z \in \mathbb{U}). \tag{2.7}
\]
then
\[
\left| \arg \left( \frac{\gamma(p-\mu)}{\delta} \frac{\gamma(p-\mu) - \delta(p+1)}{\delta} \Delta_{z,p}^{\lambda,\mu,\nu} f(t) dt \right) \right| < \frac{\pi}{2} \alpha. \tag{2.8}
\]

Proof. If we set
\[
h(z) = \frac{\gamma(p-\mu)}{\delta} \int \frac{\gamma(p-\mu) - \delta(p+1)}{\delta} \Delta_{z,p}^{\lambda,\mu,\nu} f(t) dt,
\]
then \( h(z) = 1 + c_1z + c_2z^2 + \cdots \) is analytic in \( \mathbb{U} \), with \( h(0) = 1 \) and \( h'(0) \neq 0 \). By using the logarithmic differentiation on both sides of (2.9), we obtain
\[
h(z) + \frac{\delta}{\gamma(p-\mu)} z h'(z) = \frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^p}.
\]
Thus, in view of Lemma 1, we have
\[
\left| \arg h(z) \right| < \frac{\pi}{2} \alpha, \quad (z \in \mathbb{U}),
\]
which evidently proves Theorem 2.

Remark 2. Setting \( \lambda = \mu = 0 \) and \( \gamma = \delta = p = 1 \) in Theorem 2, we get the result obtained by Goyal and Goswami ([7], Corollary 3.6). Putting \( \lambda = \mu = \gamma = \delta = 1 \) in Theorem 2, we obtain the following result.

Corollary 2. Let \( \alpha \in \mathbb{R}^+ \). Suppose that \( f(z) \in \mathcal{A}(p) \) \( (p \neq 1) \) satisfies the condition
\[
\left| \arg \left( \frac{f(z)}{p^{\alpha-1}} \right) \right| < \frac{\pi}{2} \left( \alpha + \frac{2}{\pi} \arctan \left( \frac{\alpha}{p-1} \right) \right),
\]
then
\[
\left| \arg \left( \frac{p-1}{p^{\alpha-1}} \int \frac{f'(t)}{t} dt \right) \right| < \frac{\pi}{2} \alpha, \quad (z \in \mathbb{U}).
\]

Finally, we consider the generalized Bernardi-Libera-Livingston integral operator \( \mathcal{L}_\sigma(f) \) \( (\sigma > -p) \) defined by (cf. [8] [9] and [10])
\[
\mathcal{L}_\sigma(f) = \int_{\sigma}^{\frac{\sigma+p}{z^\sigma}} f(t) dt, \quad (f \in \mathcal{A}(p); \sigma > -p).
\]

Theorem 3. Let \( \lambda \geq 0, \ \mu < \min \{p+1,1\}, \ \sigma > -p \) and \( \alpha, \gamma, \delta \in \mathbb{R}^+ \), and let \( g(z) \in \mathcal{A}(p) \). Suppose that \( f(z) \in \mathcal{A}(p) \) satisfies the condition
\[
\left| \arg \left( \frac{\Delta_{z,p}^{\lambda,\mu,\nu} \mathcal{L}_\sigma(f)(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} \mathcal{L}_\sigma(g)(z)} \right) \right| < \frac{\pi}{2} \left( \alpha + \frac{2}{\pi} \arctan \left[ \frac{\delta}{\gamma(\sigma+p)} \right] \right), \tag{2.11}
\]
then
\[
\left| \arg \left( \frac{\Delta_{z,p}^{\lambda,\mu,\nu} \mathcal{L}_\sigma(f)(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} \mathcal{L}_\sigma(g)(z)} \right) \right| < \frac{\pi}{2} \alpha, \quad (z \in \mathbb{U}). \tag{2.12}
\]

Proof. From (2.10) we observe that
\[
  z\left(\Delta_{z,p}^{\Lambda,\mu,\nu} \mathcal{L}_{\sigma}(f)(z)\right)' = (\sigma + p)\Delta_{z,p}^{\Lambda,\mu,\nu} f(z) - \sigma\Delta_{z,p}^{\Lambda,\mu,\nu} \mathcal{L}_{\sigma}(f)(z).
\]  

(2.13)

If we let

\[
  h(z) = \frac{\Delta_{z,p}^{\Lambda,\mu,\nu} \mathcal{L}_{\sigma}(f)(z)}{\Delta_{z,p}^{\Lambda,\mu,\nu} \mathcal{L}_{\sigma}(g)(z)}, \quad (\gamma \neq 0),
\]

(2.14)

then \( h(z) = 1 + c_1 z + c_2 z^2 + \cdots \) is analytic in \( \mathbb{U} \), with \( h(0) = 1 \) and \( h'(0) \neq 0 \). Differentiating both sides of (2.14) logarithmically, it follows that

\[
  \frac{1}{\gamma} \frac{zh'(z)}{h(z)} = \frac{z\left(\Delta_{z,p}^{\Lambda,\mu,\nu} \mathcal{L}_{\sigma}(f)(z)\right)'}{\Delta_{z,p}^{\Lambda,\mu,\nu} \mathcal{L}_{\sigma}(f)(z)} - \frac{z\left(\Delta_{z,p}^{\Lambda,\mu,\nu} \mathcal{L}_{\sigma}(g)(z)\right)'}{\Delta_{z,p}^{\Lambda,\mu,\nu} \mathcal{L}_{\sigma}(g)(z)}.
\]

(2.15)

Hence, by applying the same arguments as in the proof of Theorem 1 with (2.13) and (2.15), we obtain

\[
  \left| \arg h(z) \right| < \frac{\pi}{2} \alpha, \quad (z \in \mathbb{U}),
\]

which proves Theorem 3.

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**References**


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