Approximation Theorems for Exponentially Bounded $\alpha$-Times Integrated Cosine Function

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Received 26 September 2014; revised 25 October 2014; accepted 31 October 2014

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Abstract

In this paper, based on the theories of $\alpha$-times Integrated Cosine Function, we discuss the approximation theorem for $\alpha$-times Integrated Cosine Function and conclude the approximation theorem of exponentially bounded $\alpha$-times Integrated Cosine Function by the approximation theorem of $n$-times integrated semigroups. If the semigroups are equicontinuous at each point $t \in [0, \infty]$, we give different methods to prove the theorem.

Keywords

$\alpha$-Times Integrated Cosine Function, Exponentially Bounded, Approximation

1. Introduction

Integrated semigroups were introduced by Arent [1] [2] and Davies and Pang [3] in 1987. The approximation theorem is one of the fundamental theorems in the theory of operator semigroups. There have been many results on approximation [4]-[7]. Cao [8] obtained the approximation theorem for $m$-times Integrated Cosine Function, $m \in \mathbb{N}$. In this paper, we refine the theory by introducing $\alpha$-times Integrated Cosine Function for positive real numbers $\alpha$. Moreover, if the semigroups are equicontinuous at each point $t \in [0, \infty]$, we give different methods to prove the theorem.

Throughout this paper, we will denote by $X$—a Banach space with norm $\| \cdot \|$, by $B(X)$—the Banach space of all bounded linear operators from $X$ to $X$; $A$ is a linear operator in $X$, by $D(A), R(A), \rho(A), R(\lambda, A)$ respectively the domain, the range, the resolvent set, and the resolvent of $A$.

2. Preliminaries

Definition 2.1. Let $\alpha \in \mathbb{R}^+$, then a strongly continuous family $\{S(t)\}_{t \geq 0}$ in $B(X)$ is called an $\alpha$-times Integrated Cosine Function, if the following hold:

1) $S(0) = 0$;
2) For any $x \in X$, and $\forall s, t \geq 0$,

$$2S(s)S(t) = \frac{1}{\Gamma(\alpha)(1 - \alpha)} \int_0^t \left( (-1)^{\alpha - 1} \left( 1 - \alpha \right) \right)^{\alpha - 1} S(r) \, dr$$

$$+ \int_0^s \left( t + s - r \right)^{\alpha - 1} S(r) \, dr + \int_0^t \left( t - s + r \right)^{\alpha - 1} S(r) \, dr + \int_0^t \left( t - s + r \right)^{\alpha - 1} S(r) \, dr \}.$$

Definition 2.2. $A$ is a linear operator in $X$, $\alpha \in \mathbb{R}^+$, $A$ is called the generator of an $\alpha$-times Integrated Cosine Function if there are nonnegative numbers $\omega, M$ and a mapping $S : [0, \infty) \to B(X)$ such that

1) $\{S(t)\}_{t \geq 0}$ is strongly continuous and $\left\| \int_0^t S(s) \, ds \right\| \leq Me^{\omega t}$ for all $t \geq 0$;
2) $(\omega, \infty)$ is contained in the resolvent set of $A$;
3) $R(\lambda^2, A) = \lambda^{\alpha - 1} \int_0^\infty e^{-\lambda t} S(t) \, dt$ for $\lambda > \omega$.

Lemma 2.3. [9] For each $n \in N$ let $f_n \in L^1_{loc}([0, \infty), X)$, with

$$\left\| \int_0^t f_n(s) \, ds \right\| \leq Me^{\omega t}, \quad t \geq 0$$

and let

$$F_n(\lambda) = \int_0^\infty e^{-\lambda t} f_n(t) \, dt, \quad \lambda > \omega$$

Assume that

$$\lim_{n \to \infty} F_n(\lambda) \quad \text{exists for} \quad \lambda > \omega,$$

and that for a fixed $t_0 \in (0, \infty)$,

$$\sup_{n \in N} \left\| f_n(t_0) \right\| < \infty,$$

and

$$\lim_{n \to \infty} \frac{1}{H} \left( f_n(t_0 + s) - f_n(t_0) \right) \, ds = 0$$

with uniform concergence for $n \in N$. Then $\lim_{m \to \infty} f_m(t_0)$ exists.

Lemma 2.4. [10] If $A$ is a linear operator in $X$, $\alpha \geq 0$. The following assertions are equivalent:

1) There exist constant $\omega, M \geq 0$, such that $\left( \omega^2, \infty \right) \subset \rho(A)$, and

$$\left\| (\lambda - \omega)^{-k} R(\lambda^2, A) \right\| \leq Mk^!.$$

for $\lambda > \omega$, $k \in N_0 = N \cup \{0\}$.

2) $\forall \beta \in (\alpha, \alpha + 1]$, $A$ generate a $\beta$-times Integrated Cosine Function $\{S_\beta(t)\}_{t \geq 0}$, and exist constant $k$ such that $\alpha + 1$-times Integrated Cosine Function $\{S_{\alpha + 1}(t)\}_{t \geq 0}$ hold

$$\lim_{k \to \infty} \frac{1}{H} \left\| S_{\alpha + 1}(t + h) - S_{\alpha + 1}(t) \right\| \leq ke^{\omega t} \quad (t \geq 0, h \geq 0).$$

3. Main Results

Theorem 3.1. If $A_n$ generates a $\alpha$-times Integrated Cosine Function $\{S_n(t)\}_{t \geq 0}$, and there is $M, \omega \in \mathbb{R}^+$
such that \( \|S_n(t)\| \leq Me^{\omega t} \), then the following statements are equivalent:

1) \( \lim_{n \to \infty} R(\lambda^2, A_n)x = R(\lambda^2, A_0)x, \forall x \in X, \) for some \( \lambda_0 > \omega \), and \( \{S_n(t)\}_{t \geq 0} \) is equicontinuous at each point \( t \in [0, \infty) \);

2) \( \lim_{n \to \infty} R(\lambda^2, A_n)x = R(\lambda^2, A_0)x, \forall x \in X, \lambda > \omega, \) and \( \{S_n(t)\}_{t \geq 0} \) is equicontinuous at each point \( t \in [0, \infty) \);

3) \( \lim S_n(t)x = S_0(t)x, \forall x \in X \) uniformly on compacts of \( t \geq 0 \).

Proof: 1) \( \Rightarrow \) 2) Consider the set \( \Omega = \{ \lambda : \lim R(\lambda^2, A_n)x = R(\lambda^2, A_0)x, \forall x \in X, \lambda > \omega \} \), which is nonempty by assumption.

Let \( \mu \in \Omega \), then

\[ \lambda^2 - A_n = \mu^2 - A_n + \lambda^2 - \mu^2 = \left[ I - (\mu^2 - \lambda^2)R(\mu^2, A_n) \right] (\mu^2 - A_n) \]

when \( |\mu^2 - \lambda^2| < \frac{1}{\|R(\mu^2 - A_n)\|} \)

\[ R(\lambda^2, A_n) = R(\mu^2 - A_n) \left[ I - (\mu^2 - \lambda^2)R(\mu^2, A_n) \right] = \sum_{k=0}^{\infty} (\mu^2 - \lambda^2)^k R(\mu^2, A_n)^{k+1} \]

Obviously \( R(\lambda^2, A_n) \) converges as \( n \to \infty \). Therefore, the set \( \Omega \) is open.

On the other hand, taking an accumulation point \( \lambda \) of \( \Omega \) with \( \lambda > \omega \), we can find \( \mu \in \Omega \), such that \( |\mu^2 - \lambda^2| < \frac{1}{\|R(\mu^2 - A_n)\|} \). By the above considerations, \( \lambda \) must belong to \( \Omega \), i.e., \( \Omega \) is relatively closed in \( S = \{ \lambda : \lambda > \omega \} \), which leads to the conclusion.

2) \( \Rightarrow \) 3) Let \( F_n(\lambda) = \lambda^\omega R(\lambda^2, A_n) \int_0^\infty e^{-\lambda t}S_n(t) \, dt, \) for

\[ \lim_{n \to \infty} R(\lambda^2, A_n)x = R(\lambda^2, A_0)x \]

\[ \lim_{n \to \infty} S_n(t)x \],

and \( \{S_n(t)\}_{t \geq 0} \) is equicontinuous at each point \( t \in [0, \infty) \); using Lemma 2.2, it is easy to know that \( \lim S_n(t)x \) exists. We now fix \( b > 0 \), then for each \( \varepsilon > 0, \exists K \in N \); when \( |t - s| \leq \frac{b}{K}, t, s \in [0, b] \), we have

\[ \|S_n(t) - S_n(s)\| < \frac{\varepsilon}{3} \] \hspace{1cm} (1)

Pick \( t_i = \frac{i}{k} b \in [0, b], i = 1, 2, 3, \ldots, K \), then \( \exists N_0 \in N \) such that

\[ \|S_n(I) - S_n(t_i)\| < \frac{\varepsilon}{3}, n, l \geq N_0, i = 1, 2, 3, \ldots, k. \] \hspace{1cm} (2)

From (1) (2), we have \( \|S_n(t) - S_n(t)\| < \frac{\varepsilon}{3}, n, l \geq N_0, t \in [0, b]. \)

It shows that 3) is right.

3) \( \Rightarrow \) 2) fix \( t_0 \in [0, \infty) \), for each \( \varepsilon > 0, \exists N_0 \in N \), when \( n \geq N_0 \).
We have
\[ \| S_n(s) - S_{N_0}(s) \| < \frac{\varepsilon}{3}, \quad s \in [0, t+1]. \]

For \( S_n(t) \) is continuous on \([0, t+1]\), then \( \exists \delta_0 > 0, \| s - t \| < \delta_0 \), when \( s \in [0, t+1] \)
We have
\[ \| S_n(s) - S_n(t) \| < \frac{\varepsilon}{3}, \quad n = 1, 2, 3, \ldots, N_0 \]

Therefore, if \( n \geq N_0, \quad s \in [0, t+1] \), then \( \exists \delta > 0 \), \( \delta_0 > 0 \), \( \delta < \delta_0 \), when \( \| s - t \| < \delta_0, \quad t \in [0, 1] \)
In conclusion \( \{ S_n(t), n \in \mathbb{N} \} \) is equicontinuous at \( t \).

By using the dominated convergence theorem, we obtain
\[ \lim_{n \to \infty} F_n(\lambda) = \lambda^{\alpha-1} R(\lambda^2, A_n) = \int_0^\infty e^{-\lambda t} S_n(t) dt = \int_0^\infty e^{-\lambda t^2} S_n(t) dt \]
So 2) is right.

2) \( \Rightarrow 1) \) the proof is obvious.
The proof is completed.

**Corollary 3.2.** If \( A_n \) is the generator of \( \alpha \)-times Integrated Cosine Function \( \{ S_n(t) \}_{t \geq 0} \) satisfying:
\[ \| S_n(t) - S_n(t+h) \| \leq Me^{\omega h} h^\gamma, \quad n \in \mathbb{N}, t, h \geq 0, \gamma \in (0, 1] \]
Then (1)-(3) are equivalent:
1) \( \lim_{n \to \infty} R(\lambda^2, A_n)x = R(\lambda^2, A_n)x, \quad \forall x \in X, \quad \lambda > \omega. \)
2) \( \lim_{n \to \infty} R(\lambda^2, A_n)x = R(\lambda^2, A_n)x, \quad \forall x \in X, \quad \lambda > \omega. \)
3) \( \lim_{n \to \infty} S_n(t)x = S_0(t)x, \quad \forall x \in X, \quad \text{uniformly on compacts of } t \geq 0. \)

**Theorem 3.3.** If \( A_n \) is the generator of \( \alpha \)-times Integrated Cosine Function \( \{ S_n(t) \}_{t \geq 0} \), and there is \( M, \omega \in \mathbb{R}^+ \) such that \( \| S_n(t) \| \leq Me^{\omega t}, \quad \forall x \in X, \quad \lambda > \omega, \quad \{ S_n(t) \}_{t \geq 0} \) is equicontinuous at each point \( t \in [0, \infty] \).
\[ \lim_{n \to \infty} R(\lambda^2, A_n)x = R(\lambda^2) x \quad \text{exist, for some } \lambda_0 > \omega, \quad \ker R(\lambda_0^2) = \{ 0 \}, \quad \text{then there is a linear operator } A - \text{generator of } \alpha \times \text{times Integrated Cosine Function } S(t), \quad \text{such that } \lim_{n \to \infty} S_n(t)x = S(t)x, \quad \forall x \in X, \quad \text{and uniformly on compacts of } t \geq 0. \]
Proof: By \( \lim_{n \to \infty} R(\lambda^2, A_n)x = R(\lambda^2)x, \quad \text{from the resolvent identity, we have} \)
\[ R(\lambda^2, A_n) - R(\mu^2, A_n) = (\mu^2 - \lambda^2) R(\lambda^2, A_n) R(\mu^2, A_n) \]
then \( R(\lambda^2) - R(\mu^2) = (\mu^2 - \lambda^2) R(\lambda^2) R(\mu^2), \quad \lambda, \mu > \omega \) hence \( \ker R(\lambda^2) \) and \( \text{Rang} R(\lambda^2) \) independent \( \lambda \). Since \( \ker R(\lambda_0^2) = \{ 0 \} \), then there is a linear operator \( A, \quad D(A) = \text{Rang} R(\lambda^2), \quad R(\lambda^2)x = (\lambda^2 I - A)^{-1} x. \)
By Definition 2.2, we know that
\[ \lambda^{\alpha} R(\lambda^2, A_n)x = \int_0^\infty e^{-\lambda t} S_n(t) x dt, \quad \forall x \in X, \lambda > \omega, \]
for \( \lim_{n \to \infty} R(\lambda^2, A_n)x = R(\lambda^2)x \) exist, by the proof of the Theorem 3.1, we obtain that
\[ \lim_{n \to \infty} S_n(t)x = S(t)x \quad \text{exist,} \]
hence $\lambda^{1-\alpha} R(\lambda^2, A)x = \int_0^\infty e^{-\lambda t} S(t)x dt, \quad \forall x \in X, \quad \lambda > \omega$.

then $A$ generates a $\alpha$-times Integrated Cosine Function $\{S(t)\}_{t \geq 0}$, such that $\lim_{n \to \infty} S_n(t)x = S(t)x, \quad \forall x \in X$, and uniformly on compacts of $t \geq 0$.

References


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