Univalence Conditions for Two General Integral Operators

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Abstract

Let $A$ be the class of all analytic functions which are analytic in the open unit disc $U = \{ z : |z| < 1 \}$. In this paper we study the problem of univalence for the following general integral operators:

$$ F_z(z) = \prod_{t=1}^{n} \left( \frac{f_i(t)}{t} e^{\alpha_i(t)} \right)^{\alpha_i} dt, $$

$$ G_z(z) = \prod_{t=1}^{n} \left( f_i'(t) e^{\alpha_i(t)} \right)^{\beta_i} dt, $$

in the open unit disc $U$, when $f_i, g_i \in A, \alpha_i, \beta_i \in C$.

Keywords

Analytic Functions, Integral Operators, General Schwarz Lemma

1. Introduction

Let $U = \{ z : |z| < 1 \}$ be the unit disk and $A$ be the class of all functions of the form

$$ f(z) = z + \sum_{k=2}^{n} a_k z^k, \quad z \in U $$

which are analytic in $U$ and satisfy the conditions

$$ f(0) = f'(0) - 1 = 0. $$

We denote by $S$ the class of univalent and regular functions.

In order to derive our main results, we have to recall here the following univalence conditions.

**Theorem 1.1.** [1] (Becker’s univalence criterion).

If the function $f$ is regular in unit disk $U$, $f(z) = z + a_2 z^2 + \cdots$ and

$$
\left(1 - |z|^2\right) \left|zf'(z)\right| \leq 1, \text{ for all } z \in U,
$$

then the function $f$ is univalent in $U$.

**Theorem 1.2.** [2] If the function $g$ is regular in $U$ and $|g(z)| < 1$ in $U$, then for all $\xi \in U$ the following inequalities hold

$$
\left|g(\xi) - g(z)\right| \leq \left|\frac{\xi - z}{1 - g(z)g(\xi)}\right| \left|1 - z\xi\right|
$$

and

$$
|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}.
$$

The equalities hold in case $g(z) = e^{\frac{z + u}{1 + au}}$ where $|\xi| = 1$ and $|u| < 1$.

**Remark 1.3.** [2] For $z = 0$, from inequality (3) we obtain for every $\xi \in U$

$$
\left|g(\xi) - g(0)\right| \leq \left|\frac{\xi}{1 - g(0)g(\xi)}\right| |\xi|
$$

and, hence

$$
|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)||\xi|}.
$$

Considering $g(0) = a$ and $\xi = z$, then

$$
|g(z)| \leq \frac{|z| + |a|}{1 + |a||z|},
$$

for all $z \in U$.

**2. Main Results**

In this paper we study the univalence of the following general integral operators:

$$
F_n(z) = \int_0^1 \prod_{i=1}^n \left(\frac{f_i(t)e^{\alpha_i(t)}}{t}\right)^{\alpha_i} dt,
$$

where $f_i, g_i \in A$ and $\alpha_i \in C$,

$$
G_n(z) = \int_0^1 \prod_{i=1}^n \left(f_i(t)e^{\beta_i(t)}\right)^{\beta_i} dt,
$$

where $f_i, g_i \in A$ and $\beta_i \in C$.

**Theorem 2.1.** Let $\alpha_n \in C$, $f_n \in S$, $f_n(z) = z + a_2^n z^2 + \cdots$, $n \in N^*$, $g_n \in S$, $g_n(z) = z + b_2^n z^2 + \cdots$, $n \in N^*$,

If

$$
\left|zf_n'(z) - f_n(z)\right| \leq 1,
$$

then the function $f_n$ is univalent in $U$.
for all \( n \in N^* \), for all \( z \in U \) and

\[
\left| g'_n(z) \right| \leq 1
\]

\[
|\alpha_1 + |\alpha_2| + \cdots + |\alpha_n| < 1,
\]

\( (9) \)

\[
|\alpha_1 \alpha_2 \cdots \alpha_n| \leq \frac{1}{\max_{|z| \geq 1} \left[ 2 \left(1 - |z| \right)^2 \right] |z| + |z|}
\]

\( (10) \)

where

\[
|c| = \frac{|\alpha_1 (a_1^* + 1) + \cdots + \alpha_n (a_n^* + 1)|}{2|\alpha_1 \alpha_2 \cdots \alpha_n|}
\]

then the function

\[
F_n(z) = \int_0^1 \prod_{i=1}^n \left( \frac{f_i(t)}{t} e^{\epsilon_k(t)} \right)^{\alpha_i} dt,
\]

\( (11) \)

is in the class \( S \).

**Proof.** We have \( f_n \in S, \ f_n(z) \neq 0 \), for all \( n \in N^* \) and \( f_n(z) \to 1 \), when \( z = 0 \).

Let us consider the function:

\[
h(z) = \frac{1}{2|\alpha_1 \alpha_2 \cdots \alpha_n|} \frac{F_n(z)}{F_n'(z)}.
\]

\( (12) \)

From (6), we have:

\[
F_n'(z) = \prod_{i=1}^n \left( \frac{f_i(z)}{z} e^{\epsilon_k(z)} \right)^{\alpha_i}
\]

\( (13) \)

and

\[
F_n''(z) = \sum_{i=1}^n \alpha_i \left( \frac{f_i(z)}{z} e^{\epsilon_k(z)} \right)^{\alpha_i-1} \left( \frac{zf_i'(z) - f_i(z) e^{\epsilon_k(z)}}{z^2} \right) + \frac{f_i(z)}{z} e^{\epsilon_k(z)} g_i'(z) \prod_{i=1}^n \left( \frac{f_i(z)}{z} e^{\epsilon_k(z)} \right)^{\alpha_i}.
\]

\( (14) \)

From (13) and (14), we have:

\[
\frac{F_n'(z)}{F_n''(z)} = \sum_{i=1}^n \alpha_i \left( \frac{zf_i'(z) - f_i(z) e^{\epsilon_k(z)}}{zf_i(z)} + g_i'(z) \right).
\]

Using relations before the function \( h \) has the form:

\[
h(z) = \frac{1}{2|\alpha_1 \alpha_2 \cdots \alpha_n|} \sum_{i=1}^n \alpha_i \left( \frac{zf_i'(z) - f_i(z) e^{\epsilon_k(z)}}{zf_i(z)} + g_i'(z) \right).
\]

\( (15) \)

We have:

\[
h(0) = \frac{1}{2|\alpha_1 \alpha_2 \cdots \alpha_n|} \alpha_1 (a_1^* + 1) + \frac{1}{2|\alpha_1 \alpha_2 \cdots \alpha_n|} \alpha_2 (a_2^* + 1) + \cdots + \frac{1}{2|\alpha_1 \alpha_2 \cdots \alpha_n|} \alpha_n (a_n^* + 1).
\]

By using the relations (15), (8) and (9), we obtain:
\[ |h(z)| \leq \frac{1}{2|\alpha_1 \alpha_2 \cdots \alpha_n|} \sum_{i=1}^n |\alpha_i| \left( \frac{zf_i'(z) - f_i(z)}{zf_i'(z)} + g_i'(z) \right) \leq \frac{1}{2|\alpha_1 \alpha_2 \cdots \alpha_n|} \sum_{i=1}^n |\alpha_i| \leq 1 \]  
(16)

\[ |h(0)| = \frac{|\alpha_1 (\alpha_1' + 1) + \cdots + \alpha_n (\alpha_n' + 1)|}{2|\alpha_1 \alpha_2 \cdots \alpha_n|} = |c|. \]  
(17)

Applying Remark 1.3 for the function \( h \), we obtain:

\[ |h(z)| = \frac{1}{2|\alpha_1 \alpha_2 \cdots \alpha_n|} \left| \frac{F_n'(z)}{F_n'(z)} \right| \leq \frac{|z| + |h(0)|}{1 + |h(0)||z|} \leq \frac{|z| + |c|}{1 + |c| |z|}. \]  
(18)

From (18), we get:

\[ \left(1 - |z|^2\right) \left| \frac{F_n'(z)}{F_n'(z)} \right| \leq |\alpha_1 \alpha_2 \cdots \alpha_n| 2 \left(1 - |z|^2\right) \frac{|z| + |c|}{1 + |c| |z|}. \]  
(19)

for all \( z \in U \).

Let us consider the function: \( H : [0,1] \rightarrow R \)

\[ H(x) = 2 \left(1 - x^2\right) \frac{x + |c|}{1 + |c| x}, \quad x = |z|. \]

Since \( H\left(\frac{1}{2}\right) = \frac{1}{4} + \frac{2|c|}{2 + |c|} > 0 \), it results:

\[ \max_{x \in [0,1]} H(x) > 0. \]

Using this result and the form (19), we have:

\[ \left(1 - |z|^2\right) \left| \frac{F_n'(z)}{F_n'(z)} \right| \leq \max_{|z|} \left[ \prod_{i=1}^n |\alpha_i| \right] \left[ 2 \left(1 - |z|^2\right) \frac{|z| + |c|}{1 + |c| |z|} \right], \]  
(20)

for all \( z \in U \).

Applying the condition (10) in relation (20), we obtain:

\[ \left(1 - |z|^2\right) \left| \frac{F_n'(z)}{F_n'(z)} \right| \leq 1, \]  
for all \( z \in U \) and from Theorem 1.1, we have \( F_n \in S \).

**Corollary 2.2.** Let \( \alpha \) be a complex number and the functions \( f, g \in S \), \( f(z) = z + a_2 z^2 + \cdots, \quad g(z) = z + b_2 z^2 + \cdots \).

If

\[ \left| \frac{zf'(z) - f'(z)}{zf(z)} \right| < 1 \quad \text{and} \quad \left| g'(z) \right| < 1 \]  
(21)

for all \( z \in U \) and the constant \( |\alpha| \) satisfies the condition:

\[ |\alpha| \leq \max_{|z|} \left[ \frac{1}{2|z| \left(1 - |z|^2\right)} \frac{2|z| + |a_2| + 1}{2 + |a_2| |z|} \right], \]  
(22)

then the function

\[ F_t(z) = \int_0^t \left( \frac{f(t)}{t} e^{\alpha(t)} \right)^u \, dt, \]  
(23)
is in the class $S$.

**Proof.** We consider $n = 1$ in Theorem 2.1.

**Remark 2.3.** For $n = 1$, $e^{e_1(t)} = 1$, $\alpha_1 = 1$ and $f_1 = f$ in relation (11), we obtain the integral operator

$$I(z) = \int_0^z \frac{f(t)}{t} dt,$$

introduced by J. W. Alexander in [3].

**Remark 2.4.** For $n = 1$, $e^{e_1(t)} = 1$, $\alpha = \alpha$, $f_1 = f$ in relation (6), we obtain the integral operator

$$F(z) = \int_0^z \left( f(t) \right)^{\alpha} \frac{dt}{t},$$

defined and studied by V. Pescar in [4] [5].

**Remark 2.5.** For $e^{e_1(t)} = 1$, for all $i = 1, \ldots, n$, we get the integral operator $I_n(z) = \int_0^z \prod_{i=1}^n \left( \frac{f_i(t)}{t} \right)^{\alpha_i} dt$, $z \in U$ studied by D. Breaz, N. Breaz in [6] and D. Breaz in [7].

**Theorem 2.6.** Let $\beta_n \in C$, $f_n \in S$, $f_n(z) = z + a_n^0 z^2 + \cdots$, $n \in N^*$, $g_n \in S$, $g_n(z) = z + b_n^0 z^2 + \cdots$, $n \in N^*$.

If

$$\left| \frac{f_n'(z)}{f_n'(z)} \right| \leq 1, \quad (24)$$

for all $n \in N^*$, for all $z \in U$ and $|g_n'(z)| \leq 1$,

$$\frac{|\beta_1| + |\beta_2| + \cdots + |\beta_n|}{|\beta_1 \beta_2 \cdots \beta_n|} < 1, \quad (25)$$

$$\left| \prod_{i=1}^n \beta_i \right| \leq \frac{1}{\max_{|k| \leq n} \left[ 2(1-|k|^2) \right]^{1/2} |k|^{1+|k|}}, \quad (26)$$

where

$$|k| = \frac{\beta_1 (2a_1^0 + 1) + \cdots + \beta_n (2a_n^0 + 1)}{2|\beta_1 \beta_2 \cdots \beta_n|}$$

then the function

$$G_n(z) = \int_0^z \prod_{i=1}^n \left( f_i'(t) e^{\delta_i(t)} \right)^{\beta_i} \frac{dt}{t}, \quad (27)$$

is in the class $S$.

**Proof.** We have $f_n \in S$, for all $n \in N^*$ and $(f_1'(z) e^{\delta_1(z)})^{\beta_1} \cdots (f_n'(z) e^{\delta_n(z)})^{\beta_n} = 1$, when $z = 0$. Let us consider the function:

$$p(z) = \frac{1}{2|\beta_1 \beta_2 \cdots \beta_n|} G_n'(z). \quad (28)$$

From (27), we have:

$$G_n'(z) = \prod_{i=1}^n \left( f_i'(z) e^{\delta_i(z)} \right)^{\beta_i} \quad (29)$$

and

$$G_n'(z) = \sum_{i=1}^n \beta_i \left( f_i'(z) e^{\delta_i(z)} \right)^{\beta_i-1} \left( f_i'(z) e^{\delta_i(z)} + f_i'(z) e^{\delta_i(z)} g_i'(z) \right) \prod_{i \neq i}^n \left( f_i'(z) e^{\delta_i(z)} \right)^{\beta_i}. \quad (30)$$

From (29) and (30), we get:
Using relation (31) the function $p$ has the form:

$$p(z) = \frac{1}{2|\beta_1 \beta_2 \cdots \beta_n|} \sum_{i=1}^{n} \beta_i \left( \frac{f(z)}{f'(z)} + g'_i(z) \right).$$

We have:

$$p(0) = \frac{\beta_1 (2a_1^2 + 1) + \beta_2 (2a_2^2 + 1) + \cdots + \beta_n (2a_n^2 + 1)}{2|\beta_1 \beta_2 \cdots \beta_n|}.$$

By using the relations (24), (25) and (28), we obtain:

$$|p(z)| \leq \frac{1}{2|\beta_1 \beta_2 \cdots \beta_n|} \left| \sum_{i=1}^{n} \beta_i \left( \frac{f(z)}{f'(z)} + g'_i(z) \right) \right| \leq \frac{1}{2|\beta_1 \beta_2 \cdots \beta_n|} \sum_{i=1}^{n} |\beta_i| \leq 1$$

and

$$|p(0)| = \frac{\beta_1 (2a_1^2 + 1) + \beta_2 (2a_2^2 + 1) + \cdots + \beta_n (2a_n^2 + 1)}{2|\beta_1 \beta_2 \cdots \beta_n|} = |c|.$$ (32)

Applying Remark 1.3 for the function $p$, we obtain:

$$|p(z)| = \frac{1}{2|\beta_1 \beta_2 \cdots \beta_n|} \left| G^*(z) \right| \leq \frac{|z| + |p(0)|}{1 + |p(0)||z|} \leq \frac{|z| + |c|}{1 + |c||z|}.$$ (33)

From (34), we get:

$$\left| (1-|z|^2) \frac{G^*_u(z)}{G'_u(z)} \right| \leq |\beta_1 \beta_2 \cdots \beta_n| \left| 2(1-|z|^2) \right| \left| \frac{|z| + |c|}{1 + |c||z|} \right|. $$ (35)

for all $z \in U$.

Let us consider the function $Q : [0,1] \to R$

$$Q(x) = 2 \left( 1 - x^2 \right) x \frac{x + |c|}{1 + |c||x|}, x = |z|.$$  

Since $Q\left( \frac{1}{2} \right) = \frac{3}{4} \frac{1 + 2|c|}{2 + |c|} > 0$, it results:

$$\max_{x \in [0,1]} Q(x) > 0.$$  

Using this result and the form (35), we have:

$$\left| (1-|z|^2) \frac{G^*_u(z)}{G'_u(z)} \right| \leq \prod_{i=1}^{n} \beta_i \left| \max_{|z| \in [0,1]} \left[ 2\left( 1 - |z|^2 \right) \frac{|z| + |c|}{1 + |c||z|} \right] \right|. $$ (36)

for all $z \in U$.

Applying the condition (26) in relation (36), we obtain:

$$\left| (1-|z|^2) \frac{x F^*_u(z)}{F'_u(z)} \right| \leq 1,$$

for all $z \in U$ and from Theorem 1.1, we have $G_u \in S$.
Corollary 2.7. Let $\beta$ be a complex number and the functions $f \in S$, $f(z) = z + a_2z^2 + \cdots$, $g \in S$, $g(z) = z + b_2z^2 + \cdots$. If
\[
\left| \frac{f''(z)}{f'(z)} \right| < 1 \quad \text{and} \quad \left| g'(z) \right| < 1
\]
for all $z \in U$ and the constant $|\beta|$ satisfies the condition:
\[
|\beta| \leq \frac{1}{\max \left| z \right| \left( 1 + \left| z \right|^2 \right) \left( 2 \left| z \right| + 2a_2 + 1 \right) / \left( 2 + 2a_2 + 1 \right) \left| z \right|^2}.
\]
then the function
\[
G(z) = \int_0^z \left( f'(t) e^{\beta t} \right)^\beta dt,
\]
is in the class $S$.

Proof. We consider $n = 1$ in Theorem 2.6.

Remark 2.8. For $n = 1$, $e^{\alpha_1 t} = 1$, $\beta_1 = \beta$, $f_1 = f$ in relation (27), we obtain the integral operator $G_\beta(z) = \int_0^z \left( f'(t) e^{\beta t} \right)^\beta dt$, defined and studied by V. Pescar in [8] [9].

Remark 2.9. For $n = 1$ and $\beta = \alpha$ in relation (27), we obtain the integral operator $I_\alpha(f, g)(z) = \int_0^z \left( f'(t) e^{\alpha t} \right)^\alpha dt$, introduced and studied by N. Ularu and D. Breaz in [10] and [11].

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References


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