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Univalence Conditions for Two General Integral Operators

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Abstract

Let A be the class of all analytic functions which are analytic in the open unit disc $U=\{z:|z|<1\}$. In this paper we study the problem of univalence for the following general integral operators:

$$F_n(z)=\int_0^z\prod_{i=1}^n\left(\frac{f_i(t)}{t}e^{g_i(t)}\right)^{\alpha_i}dt,$$

$$G_n(z)=\int_0^z\prod_{i=1}^n\left(f_i'(t)e^{g_i(t)}\right)^{\beta_i}dt,$$

in the open unit disc U, when $f_i,\ g_i\in A,\ \alpha_i,\ \beta_i\in C$.

Keywords

Analytic Functions, Integral Operators, General Schwarz Lemma

1. Introduction

Let $U=\{z:|z|<1\}$ be the unit disk and A be the class of all functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k,\quad z\in U \tag{1}$$

which are analytic in U and satisfy the conditions

$$f(0)=f'(0)-1=0.$$

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We denote by S the class of univalent and regular functions.

In order to derive our main results, we have to recall here the following univalence conditions.

Theorem 1.1. [1] (Becker's univalence criterion).

If the function f is regular in unit disk U, $f(z)=z+a_2z^2+\cdots$ and

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right|\le 1,\ \text{for all}\ z\in U, \tag{2}$$

then the function f is univalent in U.

Theorem 1.2. [2] If the function g is regular in U and $|g(z)|<1$ in U, then for all $\xi\in U$ the following inequalities hold

$$\left|\frac{g(\xi)-g(z)}{1-\overline{g(z)}g(\xi)}\right|\le\left|\frac{\xi-z}{1-\overline{z}\xi}\right| \tag{3}$$

and

$$|g'(z)|\le\frac{1-|g(z)|^2}{1-|z|^2}.$$

the equalities hold in case $g(z)=\varepsilon\frac{z+u}{1+\overline{u}z}$ where $|\varepsilon|=1$ and $|u|<1$.

Remark 1.3. [2] For $z=0$, from inequality (3) we obtain for every $\xi\in U$

$$\left|\frac{g(\xi)-g(0)}{1-\overline{g(0)}g(\xi)}\right|\le|\xi| \tag{4}$$

and, hence

$$|g(\xi)|\le\frac{|\xi|+|g(0)|}{1+|g(0)||\xi|}. \tag{5}$$

Considering $g(0)=a$ and $\xi=z$, then

$$|g(z)|\le\frac{|z|+|a|}{1+|a||z|},$$

for all $z\in U$.

2. Main Results

In this paper we study the univalence of the following general integral operators:

$$F_n(z)=\int_0^z\prod_{i=1}^n\left(\frac{f_i(t)}{t}\mathrm{e}^{g_i(t)}\right)^{\alpha_i}\mathrm{d}t, \tag{6}$$

where $f_i,g_i\in A$ and $\alpha_i\in C$,

$$G_n(z)=\int_0^z\prod_{i=1}^n\left(f_i'(t)\mathrm{e}^{g_i(t)}\right)^{\beta_i}\mathrm{d}t, \tag{7}$$

where $f_i,g_i\in A$ and $\beta_i\in C$.

Theorem 2.1. Let $\alpha_n\in C$, $f_n\in S$, $f_n(z)=z+a_2^nz^2+\cdots$, $n\in N^*$, $g_n\in S$, $g_n(z)=z+b_2^nz^2+\cdots$, $n\in N^*$,

If

$$\left|\frac{zf_n'(z)-f_n(z)}{zf_n(z)}\right|\le 1, \tag{8}$$

for all $n \in N^*$, for all $z \in U$ and

$$\left|g_n'(z)\right| \le 1$$

$$\frac{|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|}{|\alpha_1 \alpha_2 \cdots \alpha_n|} < 1, \tag{9}$$

$$|\alpha_1 \alpha_2 \cdots \alpha_n| \le \frac{1}{\max\limits_{|z|\le 1}\left[2\left(1-|z|^2\right)|z|\frac{|z|+|c|}{1+|c||z|}\right]}. \tag{10}$$

where

$$|c| = \frac{\left|\alpha_1\left(a_2^1+1\right)+\cdots+\alpha_n\left(a_2^n+1\right)\right|}{2|\alpha_1\alpha_2\cdots\alpha_n|}$$

then the function

$$F_n(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t}\mathrm{e}^{g_i(t)}\right)^{\alpha_i} \mathrm{d}t, \tag{11}$$

is in the class *S*.

Proof. We have $f_n \in S$, $\frac{f_n(z)}{z} \ne 0$, for all $n \in N^*$ and $\left(\frac{f_1(z)}{z}\mathrm{e}^{g_1(z)}\right)^{\alpha_1} \cdots \left(\frac{f_n(z)}{z}\mathrm{e}^{g_n(z)}\right)^{\alpha_n} = 1$, when $z = 0$.

Let us consider the function:

$$h(z) = \frac{1}{2|\alpha_1\alpha_2\cdots\alpha_n|}\frac{F_n''(z)}{F_n'(z)}. \tag{12}$$

From (6), we have:

$$F_n'(z) = \prod_{i=1}^n \left(\frac{f_i(z)}{z}\mathrm{e}^{g_i(z)}\right)^{\alpha_i} \tag{13}$$

and

$$F_n''(z) = \sum_{i=1}^n \alpha_i \left(\frac{f_i(z)}{z}\mathrm{e}^{g_i(z)}\right)^{\alpha_i - 1}\left(\frac{zf_i'(z)-f_i(z)}{z^2}\mathrm{e}^{g_i(z)} + \frac{f_i(z)}{z}\mathrm{e}^{g_i(z)}g_i'(z)\right)\prod_{\substack{k=1\\k\ne i}}^n \left(\frac{f_k(z)}{z}\mathrm{e}^{g_k(z)}\right)^{\alpha_k}. \tag{14}$$

From (13) and (14), we have:

$$\frac{F_n''(z)}{F_n'(z)} = \sum_{i=1}^n \alpha_i\left(\frac{zf_i'(z)-f_i(z)}{zf_i(z)} + g_i'(z)\right).$$

Using relations before the function *h* has the form:

$$h(z) = \frac{1}{2|\alpha_1\alpha_2\cdots\alpha_n|}\sum_{i=1}^n \alpha_i\left(\frac{zf_i'(z)-f_i(z)}{zf_i(z)} + g_i'(z)\right). \tag{15}$$

We have:

$$h(0) = \frac{1}{2|\alpha_1\alpha_2\cdots\alpha_n|}\alpha_1\left(a_2^1+1\right) + \frac{1}{2|\alpha_1\alpha_2\cdots\alpha_n|}\alpha_2\left(a_2^2+1\right) + \cdots + \frac{1}{2|\alpha_1\alpha_2\cdots\alpha_n|}\alpha_n\left(a_2^n+1\right).$$

By using the relations (15), (8) and (9), we obtain:

$$\left|h(z)\right| \le \frac{1}{2\left|\alpha_1\alpha_2\cdots\alpha_n\right|}\sum_{i=1}^{n}\left|\alpha_i\left(\frac{zf_i'(z)-f_i(z)}{zf_i(z)}+g_i'(z)\right)\right| \le \frac{1}{2\left|\alpha_1\alpha_2\cdots\alpha_n\right|}2\sum_{i=1}^{n}\left|\alpha_i\right| \le 1 \tag{16}$$

$$\left|h(0)\right| = \frac{\left|\alpha_1\left(a_2^1+1\right)+\cdots+\alpha_n\left(a_2^n+1\right)\right|}{2\left|\alpha_1\alpha_2\cdots\alpha_n\right|} = |c|. \tag{17}$$

Applying Remark 1.3 for the function *h*, we obtain:

$$\left|h(z)\right| = \frac{1}{2\left|\alpha_1\alpha_2\cdots\alpha_n\right|}\left|\frac{F_n''(z)}{F_n'(z)}\right| \le \frac{|z|+|h(0)|}{1+|h(0)||z|} \le \frac{|z|+|c|}{1+|c||z|}. \tag{18}$$

From (18), we get:

$$\left|\left(1-|z|^2\right)z\frac{F_n''(z)}{F_n'(z)}\right| \le \left|\alpha_1\alpha_2\cdots\alpha_n\right| 2\left(1-|z|^2\right)|z|\frac{|z|+|c|}{1+|c||z|}, \tag{19}$$

for all $z \in U$.

Let us consider the function: $H:[0,1]\to R$

$$H(x) = 2\left(1-x^2\right)x\frac{x+|c|}{1+|c|x},\ x=|z|.$$

Since $H\left(\frac{1}{2}\right)=\frac{3}{4}\frac{1+2|c|}{2+|c|}>0$, it results:

$$\max_{x\in[0,1]} H(x) > 0.$$

Using this result and the form (19), we have:

$$\left|\left(1-|z|^2\right)z\frac{F_n''(z)}{F_n'(z)}\right| \le \left|\prod_{i=1}^{n}\alpha_i\right| \max_{|z|<1}\left[2\left(1-|z|^2\right)|z|\frac{|z|+|c|}{1+|c||z|}\right], \tag{20}$$

for all $z \in U$.

Applying the condition (10) in relation (20), we obtain:

$$\left(1-|z|^2\right)\left|\frac{zF_n''(z)}{F_n'(z)}\right| \le 1,$$

for all $z \in U$ and from Theorem 1.1, we have $F_n \in S$.

Corollary 2.2. Let α be a complex number and the functions $f \in S$, $f(z)=z+a_2z^2+\cdots$, $g \in S$, $g(z)=z+b_2z^2+\cdots$.

If

$$\left|\frac{zf'(z)-f(z)}{zf(z)}\right| < 1 \text{ and } \left|g'(z)\right| < 1 \tag{21}$$

for all $z \in U$ and the constant $|\alpha|$ satisfies the condition:

$$|\alpha| \le \frac{1}{\max_{|z|\le 1}\left[2|z|\left(1-|z|^2\right)\frac{2|z|+|a_2+1|}{2+|a_2+1||z|}\right]}, \tag{22}$$

then the function

$$F_1(z) = \int_0^z \left(\frac{f(t)}{t}\mathrm{e}^{g(t)}\right)^{\alpha} \mathrm{d}t, \tag{23}$$

is in the class *S*.

Proof. We consider $n=1$ in Theorem 2.1.

Remark 2.3. For $n=1$, $e^{g_1(t)}=1$, $\alpha_1=1$ and $f_1=f$ in relation (11), we obtain the integral operator $I(z)=\int_0^z \frac{f(t)}{t}\mathrm{d}t$, introduced by J. W. Alexander in [3].

Remark 2.4. For $n=1$, $e^{g_1(t)}=1$, $\alpha_1=\alpha$, $f_1=f$ in relation (6), we obtain the integral operator $F(z)=\int_0^z \left(\frac{f(t)}{t}\right)^{\alpha}\mathrm{d}t$, defined and studied by V. Pescar in [4] [5].

Remark 2.5. For $e^{g_i(t)}=1$, for all $i=1,\cdots,n$, we get the integral operator $I_n(z)=\int_0^1 \prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\alpha_i}\mathrm{d}t$, $z\in U$ studied by D. Breaz, N. Breaz in [6] and D. Breaz in [7].

Theorem 2.6.

Let $\beta_n\in C$, $f_n\in S$, $f_n(z)=z+a_2^n z^2+\cdots$, $n\in N^*$, $g_n\in S$, $g_n(z)=z+b_2^n z^2+\cdots$, $n\in N^*$.

If

$$\left|\frac{f_n''(z)}{f_n'(z)}\right|\le 1, \tag{24}$$

for all $n\in N^*$, for all $z\in U$ and $\left|g_n'(z)\right|\le 1$

$$\frac{|\beta_1|+|\beta_2|+\cdots+|\beta_n|}{|\beta_1\beta_2\cdots\beta_n|}<1, \tag{25}$$

$$\left|\prod_{i=1}^{n}\beta_i\right|\le \frac{1}{\max\limits_{|z|\le 1}\left[2\left(1-|z|^2\right)|z|\frac{|z|+|c|}{1+|c||z|}\right]}, \tag{26}$$

where

$$|c|=\frac{\left|\beta_1\left(2a_2^1+1\right)+\cdots+\beta_n\left(2a_2^n+1\right)\right|}{2|\beta_1\beta_2\cdots\beta_n|}$$

then the function

$$G_n(z)=\int_0^z\prod_{i=1}^{n}\left(f_i'(t)e^{g_i(t)}\right)^{\beta_i}\mathrm{d}t, \tag{27}$$

is in the class *S*.

Proof. We have $f_n\in S$, for all $n\in N^*$ and $\left(f_1'(z)e^{g_1(z)}\right)^{\beta_1}\cdots\left(f_n'(z)e^{g_n(z)}\right)^{\beta_n}=1$, when $z=0$.

Let us consider the function:

$$p(z)=\frac{1}{2|\beta_1\beta_2\cdots\beta_n|}\frac{G_n''(z)}{G_n'(z)}. \tag{28}$$

From (27), we have:

$$G_n'(z)=\prod_{i=1}^{n}\left(f_i'(z)e^{g_i(z)}\right)^{\beta_i} \tag{29}$$

and

$$G_n''(z)=\sum_{i=1}^{n}\beta_i\left(f_i'(z)e^{g_i(z)}\right)^{\beta_i-1}\left(f_i''(z)e^{g_i(z)}+f_i'(z)e^{g_i(z)}g_i'(z)\right)\prod_{\substack{k=1\\k\ne i}}^{n}\left(f_k'(z)e^{g_k(z)}\right)^{\beta_k}. \tag{30}$$

From (29) and (30), we get:

$$\frac{G_n''(z)}{G_n'(z)}=\sum_{i=1}^{n}\beta_i\left(\frac{f_i''(z)}{f_i'(z)}+g_i'(z)\right). \tag{31}$$

Using relation (31) the function p has the form:

$$p(z)=\frac{1}{2|\beta_1\beta_2\cdots\beta_n|}\sum_{i=1}^{n}\beta_i\left(\frac{f_i''(z)}{f_i'(z)}+g_i'(z)\right).$$

We have:

$$p(0)=\frac{\beta_1\left(2a_2^1+1\right)+\beta_2\left(2a_2^2+1\right)+\cdots+\beta_n\left(2a_2^n+1\right)}{2|\beta_1\beta_2\cdots\beta_n|}.$$

By using the relations (24), (25) and (28), we obtain:

$$|p(z)|\le\frac{1}{2|\beta_1\beta_2\cdots\beta_n|}\sum_{i=1}^{n}\left|\beta_i\left(\frac{f_i''(z)}{f_i'(z)}+g_i'(z)\right)\right|\le\frac{1}{2|\beta_1\beta_2\cdots\beta_n|}2\sum_{i=1}^{n}|\beta_i|\le 1 \tag{32}$$

and

$$|p(0)|=\frac{\left|\beta_1\left(2a_2^1+1\right)+\beta_2\left(2a_2^2+1\right)+\cdots+\beta_n\left(2a_2^n+1\right)\right|}{2|\beta_1\beta_2\cdots\beta_n|}=|c|. \tag{33}$$

Applying Remark 1.3 for the function p, we obtain:

$$|p(z)|=\frac{1}{2|\beta_1\beta_2\cdots\beta_n|}\left|\frac{G''(z)}{G'(z)}\right|\le\frac{|z|+|p(0)|}{1+|p(0)||z|}\le\frac{|z|+|c|}{1+|c||z|}. \tag{34}$$

From (34), we get:

$$\left|\left(1-|z|^2\right)z\frac{G_n''(z)}{G_n'(z)}\right|\le|\beta_1\beta_2\cdots\beta_n|2\left(1-|z|^2\right)|z|\frac{|z|+|c|}{1+|c||z|}, \tag{35}$$

for all $z\in U$.

Let us consider the function $Q:[0,1]\to R$

$$Q(x)=2\left(1-x^2\right)x\frac{x+|c|}{1+|c|x},\ x=|z|.$$

Since $Q\left(\frac{1}{2}\right)=\frac{3}{4}\frac{1+2|c|}{2+|c|}>0$, it results:

$$\max_{x\in[0,1]}Q(x)>0.$$

Using this result and the form (35), we have:

$$\left|\left(1-|z|^2\right)z\frac{G_n''(z)}{G_n'(z)}\right|\le\left|\prod_{i=1}^{n}\beta_i\right|\max_{|z|<1}\left[2\left(1-|z|^2\right)|z|\frac{|z|+|c|}{1+|c||z|}\right], \tag{36}$$

for all $z\in U$.

Applying the condition (26) in relation (36), we obtain:

$$\left(1-|z|^2\right)\left|\frac{zF_n''(z)}{F_n'(z)}\right|\le 1,$$

for all $z\in U$ and from Theorem 1.1, we have $G_n\in S$.

Corollary 2.7. Let β be a complex number and the functions $f \in S$, $f(z) = z + a_2 z^2 + \cdots$, $g \in S$, $g(z) = z + b_2 z^2 + \cdots$.

If

$$\left|\frac{f''(z)}{f'(z)}\right| < 1 \text{ and } \left|g'(z)\right| < 1 \tag{37}$$

for all $z \in U$ and the constant $|\beta|$ satisfies the condition:

$$|\beta| \le \frac{1}{\max\limits_{|z|\le 1}\left[2|z|\left(1-|z|^2\right)\frac{2|z|+|2a_2+1|}{2+|2a_2+1||z|}\right]}, \tag{38}$$

then the function

$$G_1(z) = \int_0^z \left(f'(t)e^{g(t)}\right)^{\beta} dt, \tag{39}$$

is in the class *S*.

Proof. We consider $n = 1$ in Theorem 2.6.

Remark 2.8. For $n = 1$, $e^{g_1(t)} = 1$, $\beta_1 = \beta$, $f_1 = f$ in relation (27), we obtain the integral operator $G_\beta(z) = \int_0^z \left(f'(t)\right)^{\beta} dt$, defined and studied by V. Pescar in [8] [9].

Remark 2.9. For $n = 1$ and $\beta = \alpha$ in relation (27), we obtain the integral operator $I_1(f,g)(z) = \int_0^z \left(f'(t)e^{g(t)}\right)^{\alpha} dt$, introduced and studied by N. Ularu and D. Breaz in [10] and [11].

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