Value Distribution of L-Functions with Rational Moving Targets

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ABSTRACT

We prove some value-distribution results for a class of L-functions with rational moving targets. The class contains Selberg class, as well as the Riemann-zeta function.

Keywords: Value Distribution; Moving Target; L-Function; Selberg Class

1. Introduction

We define the class \( \mathcal{M} \) to be the collection of functions \( L(s) = \sum_{n=1}^{\infty} a(n)/n^s \), satisfying Ramanujan hypothesis, Analytic continuation and Functional equation. We also denote the degree of a function \( L \in \mathcal{M} \) by \( d_L \) which is a non-negative real number. We refer the reader to Chapter six of [1] for a complete definitions. Obviously, the class \( \mathcal{M} \) contains the Selberg class. Also every function in the class \( \mathcal{M} \) is an L-function and the Riemann-zeta function is in the class. In this paper, we prove a value-distribution theorem for the class \( \mathcal{M} \) with rational moving targets. The theorem generalizes the value-distribution results in Chapter seven of [1] from fixed targets to moving targets.

Theorem. Assume that \( L \in \mathcal{M} \) and \( R \) is a rational function with \( \lim_{s \to \infty} R(s) \neq 1 \). Let the roots of the equation \( L(s) - R(s) = 0 \) be denoted by \( \rho_R = \beta_R + i \gamma_R \). Then

(I) For any \( b > \max \left\{ \frac{1}{2}, 1 - \frac{1}{d_L} \right\} \),
\[ \sum_{\rho_R \not\in [T,T+2T]} \left( \beta_R - b \right) = O(T), \quad \text{as} \quad T \to \infty. \]

(II) For sufficiently large negative \( b \),
\[ 2\pi \sum_{T < \gamma_R < 2T} \left( \beta_R - b \right) = (-b)d_r T \log \frac{4T}{\nu} + O(\log T), \quad \text{as} \quad T \to \infty. \]

Proof of (I). It is known that if \( L \in \mathcal{M} \), then
\[ L(s) = \sum_{n=1}^{\infty} a(n)/n^s = 1 + O(\kappa_0^{-\sigma}), \quad \text{as} \quad \sigma \to \infty; \]
where \( \kappa_0 \) is the index of the first non-zero term of the sequence of \( \{a(n)\}_{n=1}^{\infty} \), \( s = \sigma + it \) with \( \sigma, t \in \mathbb{R} \). Since \( \lim_{s \to \infty} L(s) - R(s) = 0 \), there exists \( \sigma_0 > 0 \) such that \( L(s) - R(s) \neq 0 \) for \( \operatorname{Re} s = \sigma > \sigma_0 \). It follows that \( \beta_R < \sigma_0 \) for all real part of zeros of the function \( L(s) - R(s) \). We set \( R(z) = P(z)/Q(z) \) where the degrees of \( P, Q \) are \( p, q \), respectively; and define
\[ \tilde{\ell}(s) = (s - R(s)). \]

Thus, there is \( \theta > 1 \) such that \( \tilde{\ell} \) is analytic in the region \( |s| > \theta \) since \( L \) is a meromorphic function in \( \mathbb{C} \) with the only pole at \( s = 1 \). We apply Littlewood’s argument principle [3] to \( \tilde{\ell} \) in the rectangle \( \mathcal{R} = \{ \sigma + it : b \leq \sigma \leq c, T \leq t \leq 2T \} \) where \( c, T \) are parameters satisfying \( c > \max \{ \sigma_0 + 1, b \}, T > \theta \). Thus,
\[ \int_{\mathcal{R}} \log \tilde{\ell}(s) \, ds = -2\pi \int_b^\infty \nu(\sigma) \, d\sigma \]
where the given logarithm is defined as in Littlewood’s argument principle [3]. To prove our result, however, we first decompose our auxiliary function by...
\[
\tilde{e}(s) = \begin{cases} 
P(s)\left(\frac{L(s)}{P(s)} - \frac{1}{Q(s)}\right) = P(s)\ell_1(s) & \text{for } p \leq q \\
R(s)\left(\frac{L(s)}{R(s)} - 1\right) = R(s)\ell_2(s) & \text{for } p > q 
\end{cases}
\]

(1)

Without loss of generality, we may assume that \(p, q \geq 1\) whenever \(p \leq q\) since we can always write, \(pq \leq \frac{pq}{pq}\), \(pq \leq \frac{pq}{pq}\).

\[
L_{s_{ps}} = R_{s_{ps}} = \frac{L_{s_{ps}}}{R_{s_{ps}}} = 1
\]

(2)

\[\mathcal{O}(T) \text{ terms are the integrals of the maximum contribution from writing } \tilde{\ell}(s) \text{ as a sum of logarithms. By our choice of } T, \text{ both } \log P \text{ and } \log R \text{ are analytic in } \mathcal{R}. \text{ Hence, Cauchy's Theorem gives}
\]

\[
\int_{\partial \mathcal{R}} \log \tilde{\ell}(s) \, ds = \int_{\partial \mathcal{R}} \log \ell_1(s) + \log P(s) \, ds + \mathcal{O}(T) \quad \text{for } p \leq q
\]

\[
\int_{\partial \mathcal{R}} \log \tilde{\ell}_2(s) + \log R(s) \, ds + \mathcal{O}(T) \quad \text{for } p > q
\]

(3)

To connect this integral with Littlewood's argument principle [3], we note that the definition of \(c\) guarantees that \(\mathcal{O}(T)\) is imaginary-valued, we get for \(k = 1, 2\)

\[
-2\pi i \sum_{\beta_k \leq T} \beta_k - b
\]

for instance.

We now estimate \(I_{1,k}\). For \(T\) large enough, we have

\[
\log \left| \ell_1(b + it) \right| = \log \left| \frac{L(b + it)}{P(b + it)} - \frac{1}{Q(b + it)} \right| \leq \log \left| \frac{L(b + it)}{P(b + it)} + \frac{1}{Q(b + it)} \right| 
\]

\[
\leq \log \left( |L(b + it)| + 1 \right) = \log^* \left( |L(b + it)| + 1 \right) \leq \log^* \left| L(b + it) \right| + \log 2.
\]

Then for \(T\) large enough, \(t \geq T, k = 2\), we find in a similar fashion that

\[
\log \left| \ell_2(b + it) \right| = \log \left| \frac{L(b + it)}{R(b + it)} - 1 \right| 
\]

\[
\leq \log^* \left( |b + it| \right) + \log 2.
\]

Since we have the same estimate for \(k = 1, 2\), we find that

\[
I_{1,k}(T,b) = I_{1,k} \leq \int_T^{2T} \log^* \left| b + it \right| \, dt + \mathcal{O}(T)
\]

\[
= \frac{T}{2} \int_T^{2T} \log^* \left| b + it \right| \, dt + \mathcal{O}(T)
\]

\[
\leq \frac{T}{2} \log^* \left( \frac{1}{T} \int_T^{2T} \left| L(b + it) \right| \, dt \right) + \mathcal{O}(T)
\]

where the final bound follows from Jensen’s inequality.
It is known [2] that for \( b > \max \left\{ \frac{1}{2}, 1 - \frac{1}{d_L} \right\} \),
\[
\lim_{T \to \infty} \frac{1}{T} \int_{\gamma} |L(b + it)| dt = \sum_{n=1}^{\infty} \frac{\rho(n)^2}{n^{2\sigma}} = O(1).
\]
Hence, \( I_{1,k}(T, b) \leq O(T) \) uniformly in
\[
b > \max \left\{ \frac{1}{2}, 1 - \frac{1}{d_L} \right\}.
\]

We next move to estimate \( I_{2,k} \). For sufficiently large positive real number \( c \), we have
\[
\frac{L(c + it)}{P(c + it)} \leq 1 \quad \text{and} \quad \frac{L(c + it)}{R(c + it)} \leq 1,
\]
so
\[
\log \left| \ell_1^k(c + it) \right| \leq \log \left| 1 - \frac{L(c + it)}{P(c + it)} \right|
\]
for \( T \) large enough, since \( q \geq 1 \). Furthermore,
\[
\log \left| \ell_2^k(c + it) \right| = \log \left| 1 - \frac{L(c + it)}{R(c + it)} \right|
\]
so that \( \ell_2^k(c + it) = 0 \) for \( c \in [h, c] \), then \( g(c) = 0 \).

We next move to estimate \( I_{2,k} \). For sufficiently large \( T \) and some constant \( M \) we have
\[
\int_{\gamma} \frac{T}{(P(c + it))^{k}} (n_1 \cdots n_k)^{\alpha} \leq \frac{T}{|P(c + it)|^{k}} \leq M^{\alpha + 1} \leq 1,
\]
for \( k \in \mathbb{N} \) and
\[
\limsup_{k \to \infty} \left( \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \right)^k = \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} < 1
\]
for sufficiently large \( c \). In light of these bounds and the definition of \( M \), we have (6)
\[
\text{where the last equality holds because \( c \) could be sufficiently large. Replacing \( P \) by \( R \) in the above computations, we see analogously that} \quad I_{2,k} = O(1).
\]
Finally, we estimate \( I_{3,k} \) and \( I_{4,k} \). We show the computation for \( I_{3,k} \) explicitly and note that the bound for
\[
\text{for \( N \) large enough. Replacing \( \ell_k^k \) by \( \ell_k^k \) in the above computations, we see analogously that} \quad I_{2,k} = O(1).
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\]
By (5), \( \log |g_z(c)| \) is bounded. Further, it is clear from a property of \( L \) functions that we have
\[
|L(s)| \leq A|t|^\theta, \quad \text{as } t \to \infty, n, \text{ as } t \to \infty;
\]
for some positive absolute numbers \( A, B \) in any vertical strip of bounded width. The same estimate must hold for \( g_z(z) \) as well. Thus, the integral in (8) is \( O(\log T) \), implying that \( \hat{n}_k(R) = O(\log T) \). Since the interval \([b,c] \subseteq \mathbb{D}(c, R) \subseteq D(c, R')\), it follows that
\[
N \leq \hat{n}_k(R') = O(\log T).
\]

With this bound, we integrate (7) to deduce that
\[
\int_3^4 \arg \ell_k(\sigma + it) d\sigma \leq \int_3^4 (N + \pi) d\sigma = O(\log T).
\]
As previously noted, we may bound \( I_{4,k} \) in the same way. Thus, we attain the desired bounds for \( j = 1, \ldots, 4 \) and \( k = 1, 2 \). Consequently, the first part of the theorem is proved by using (4).

**Proof of (II).** As in the proof of the first part of the theorem, we conclude that there exists a real number \( \sigma_o \) for which the real parts \( \beta_k \) of all \( R \)-values satisfy \( \beta_k < \sigma_o \); and also, there exist \( B, T' \) for each rational function \( R \) such that no zeros of \( L(s) - R(s) = 0 \) lie in the quarter-plane \( \sigma < -B, t > T' \).

As before, we define the rectangle
\[
\mathcal{R} = \{ s = \sigma + it : b \leq \sigma \leq c, T \leq t \leq T' \}
\]
where \( b, c, T \) are parameters satisfying \( b < -B - 1, c > \max \{ \sigma_o + 1, b \}, T > \max \{ c, T' + 1 \} \).

Proceeding as in the proof of the first part of the theorem, we see that

\[
2\pi i \sum_{r > \gamma > 0} \beta_k^r b = -i \left[ \int_{L} \log |t(b + it)| dt - \int_{L} \log |\ell_k(c + it)| dt \right]
\]
\[
= \int_{L} \arg \ell_k(\sigma + it) d\sigma + \int_{L} \arg \ell_k(\sigma + 2it) d\sigma + O(T)
\]
\[
= I_1 + \sum_{j \neq k} I_{j,k} + O(T)
\]

for \( k = 1, 2 \) where \( \ell_k \) is defined as in (1). In the equation above, we note that we have chosen to compute \( I_1 \) separately. Indeed, this is the only estimate that we will need. For the integrals \( I_{j,k} \), \( j = 2, 3, 4 \) and \( k = 1, 2 \), the bounds given as in the proof of the first part of the theorem still hold. First, integral \( I_{2,k} \) is unchanged. On the other hand, the integrals \( I_{3,k}, I_{4,k} \) have changed by our choice of \( b \), but, as we have done as before, we still have the desired bound since the only requirement is that we consider \( L \) in a vertical strip of fixed width, which we have in this case.

We now bound \( I_4 \). Since \( b < -B \), we have by the functional equation in the definition of \( L \) function,
We now consider the last term in (9). Since,

\[ \limsup_{t \to \pm \infty} \frac{\log|L(b + it)|}{\log|t|} = \left( \frac{1}{2} - b \right) d_L, \]

and noting \( b < 0 \), we have for any \( \delta > 0 \) and \( t \geq T \)

\[ |L(b + it)| \geq \left| \left( \frac{1}{2} - b \right) d_L - \delta \right| \]

for sufficiently large \( T \). Then we see the quotient

\[ \frac{R(b + it)}{L(b + it)} \leq \frac{R(b + it)}{\left| \left( \frac{1}{2} - b \right) d_L - \delta \right|} = O\left( \frac{1}{t} \right) \]

when \( -b \) is large enough so that

\[ \text{deg } R < \left( \frac{1}{2} - b \right) d_L - \delta + 1. \]

Therefore, we find that

\[ \log \left| 1 - \frac{R(s)}{L(s)} \right| = O\left( \frac{1}{t} \right). \]

Integrating in light of these estimates, we see

\[ \int_{T}^{2T} \log|L(b + it) - R(b + it)| dt \]

\[ = \left( \frac{1}{2} - b \right) \int_{T}^{2T} \left( d_L \log t + \log \left( \lambda Q^2 \right) \right) dt \]

\[ + \int_{T}^{2T} \log|L(1 - b - it)| dt + O(\log T). \]

The first integral is \( d_L T \log \frac{4T}{e} + T \log \left( \lambda Q^2 \right) \), and the second integral is \( O(1) \) for sufficiently large and negative \( b \) by the method used to derive (6). Hence,

\[ I_i = \left( \frac{1}{2} - b \right) \left( d_L T \log \frac{4T}{e} + T \log \left( \lambda Q^2 \right) \right) + O(\log T). \]

With the estimates for the \( I_{j,k} \)'s, we have proved the second part of the theorem.

REFERENCES

