Riesz Means of Dirichlet Eigenvalues for the Sub-Laplace Operator on the Engel Group

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ABSTRACT

In this paper, we are concerned with the Riesz means of Dirichlet eigenvalues for the sub-Laplace operator on the Engel group and derive different inequalities for Riesz means. The Weyl-type estimates for means of eigenvalues are given.

Keywords: Engel Group; Sub-Laplace Operator; Eigenvalues; Riesz Mean

1. Introduction

The Engel group $G$ is a Carnot group of step $r = 3$ (see [1]), its Lie algebra is generated by the left-invariant vector fields

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial x_3} + \left( \frac{-x_1 x_2}{12} - \frac{x_3}{2} \right) \frac{\partial}{\partial w},$$

$$X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2} \frac{\partial}{\partial x_3} + \frac{x_1^2}{12} \frac{\partial}{\partial w},$$

$$X_3 = \frac{\partial}{\partial x_3} + \frac{x_1}{2} \frac{\partial}{\partial x_3},$$

$$X_4 = \frac{\partial}{\partial w},$$

where $P = (x_1, x_2, x_3, w)$ is a point in $G$. It is easy to see that

$$[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_2, X_3] = 0,$$

$$[X_1, X_4] = [X_2, X_4] = 0,$$

and $[X_3, X_4] = 0$. So the Lie algebra of $G$ is $g = V_1 \oplus V_2 \oplus V_3$, where $V_1 = \text{span} \{ X_1, X_2 \}$, $V_2 = \text{span} \{ X_3 \}$ and $V_3 = \text{span} \{ X_4 \}$. The sub-Laplace operator on $G$ is of the form $\Delta_{X} = X_1^2 + X_2^2$.

In the paper, we investigate the Riesz means of the Dirichlet problem

$$\begin{cases}
-\Delta_{X} u = \lambda u, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega
\end{cases}$$

in the Engel group $G$. Here $\Omega$ is a bounded and noncharacteristics domain in $G$, with smooth boundary $\partial \Omega$. The existence of eigenvalues for (1.1) is from [2]. Let us by $R_\sigma(z)$ denote the Riesz means of order $\sigma$ of the sequence $\{ \lambda_k \}$ of eigenvalues of (1.1).

The Riesz means of Dirichlet eigenvalues for the Laplace operator in the Euclidean space have been extensively studied (see [3-5]). In recent years, E. M. Harrell II and L. Hermi in [6] treated the Riesz means $R_\sigma(z)$ of order $\sigma$ of $\{ \lambda_k \}$ on the bounded domain $\Omega \subset R^d$ and pointed out that: for $0 < \sigma \leq 2$ and $z \geq \lambda_i$,

$$R_{\sigma-1}(z) \geq \left( 1 + \frac{d}{4} \right) \frac{1}{z} R_{\sigma}(z)$$

and

$$R'_{\sigma}(z) \geq \left( 1 + \frac{d}{4} \right) \frac{\sigma}{z} R_{\sigma}(z),$$

and $\frac{R_{\sigma}(z)}{z^\sigma}$ is a nondecreasing function of $z$; for $2 < \sigma < +\infty$ and $z \geq \lambda_i$,

$$R_{\sigma-1}(z) \geq \left( 1 + \frac{d}{2\sigma} \right) \frac{1}{z} R_{\sigma}(z)$$

and

$$R'_{\sigma}(z) \geq \left( \sigma + \frac{d}{2} \right) \frac{1}{z} R_{\sigma}(z).$$

In the paper, we investigate the Riesz means of the Dirichlet problem

$$\begin{cases}
-\Delta_{X} u = \lambda u, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega
\end{cases}$$
and \( R_\sigma(z) \) is a nondecreasing function of \( z \), and then the Weyl-type estimates of means of eigenvalues is derived.

Jia et al. in [7] extended (1.2), (1.3) to the Heisenberg group.

The main results of this paper are the following.

**Theorem 1.1** For \( 0 < \sigma \leq 2 \) and \( z \geq \lambda_1 \), we have

\[
R_{\sigma-1}(z) \geq \frac{3}{2z} R_{\sigma}(z),
\]

and \( \frac{R_\sigma(z)}{z^{\alpha}} \) is a nondecreasing function of \( z \); for

\[
2 < \sigma < +\infty \quad \text{and} \quad z \geq \lambda_1,
\]

we have

\[
R_{\sigma-1}(z) \geq \left(1 + \frac{1}{\sigma}ight) \frac{1}{z} R_{\sigma}(z),
\]

and therefore

\[
R_\sigma(z) \geq \frac{4 z^3}{27 \lambda_j},
\]

Moreover, for all \( k \geq j \geq 1 \), we have the upper bound

\[
\lambda_{k+1} \leq \frac{3}{4} \frac{\lambda_k}{\lambda_j}.
\]

**Theorem 1.2** Suppose that \( z \geq 3 \lambda_j \), then

\[
R_\sigma(z) = \frac{4 z^3}{27 \lambda_j},
\]

and therefore

\[
R_\sigma(z) \geq \frac{2 z^2}{9 \lambda_j},
\]

\[
N(z) = R_\sigma(z) \geq \frac{z^2}{3 \lambda_j},
\]

**Theorem 1.3** For \( k > \frac{4 j}{3} \), we have

\[
\frac{\lambda_k}{\lambda_j} \leq \frac{9 k}{8 j}.
\]

Authors in [6] combined the Weyl-type estimates of means of eigenvalues established in [6] and the result in [8] to obtain the Weyl-type estimates of eigenvalues. But it is not easy to extend the result in [8] to the Engel group. The Weyl-type estimates of eigenvalues for (1.1) still are open questions.

This paper is arranged as follows. In Section 2 the definition of Riesz means and Lemmas are described; Section 3 is devoted to the proof of Theorem 1.1. The proof of Theorem 1.2 is appeared in Section 4. In Section 5 the proof of Theorem 1.3 is given.

### 2. Preliminaries

**Definition 2.1** For an increasing sequence \( \{\lambda_i\}_{i=1}^\infty \) of real numbers and \( z \geq 0 \), the Riesz means \( R_{\sigma}(z) \) of order \( \sigma \geq 0 \) of \( \{\lambda_i\} \) is defined by

\[
R_{\sigma}(z) = \sum_{i=1}^{\infty} (z - \lambda_i)^{\sigma},
\]

where \( (z - \lambda_i)^{\sigma} = \max\{0, z - \lambda_i\}^{\sigma} \) is the ramp function.

Similarly to Theorem 1 of [9], we immediately have

**Lemma 2.2** Denoting the \( L^2 \)-normalized eigenfunctions of (1.1) by \( \{u_j\} \), let

\[
T_{u_j m} = \left| \langle X_{u_j}, u_m \rangle \right|^2
\]

for \( \alpha = 1, 2; j, m = 1, 2, \ldots \). Then for each fixed \( \alpha \), we have

\[
R_{\sigma}(z) = 2 \sum_{j, m, \lambda_j \geq \lambda_m} \frac{(z - \lambda_j)^{\sigma} - (z - \lambda_m)^{\sigma}}{\lambda_m - \lambda_j} T_{u_j m}
\]

\[
+4 \sum_{j, m, \lambda_j \geq \lambda_m} \frac{(z - \lambda_j)^{\sigma}}{\lambda_m - \lambda_j} T_{u_j m}.
\]

**Lemma 2.3** ([10]) Let \( 0 < x < y \) and \( \sigma \geq 0 \), then

\[
\frac{y^{\sigma} - x^{\sigma}}{y - x} \leq C_{\sigma} \left( y^{\sigma-1} + x^{\sigma-1} \right),
\]

where

\[
C_{\sigma} = \begin{cases} \sigma, & 0 \leq \sigma < 1, \\ \sigma^2, & 1 \leq \sigma \leq 2, \\ \sigma^2, & 2 \leq \sigma < +\infty. \end{cases}
\]

### 3. The Proof of Theorem 1.1

In this section, we prove Theorem 1.1 and two corollaries.

**Proof.** Let us use (2.2) and denote the first term on the right-hand side of (2.2) by \( G(\sigma, z, \alpha) \). Applying Lemma 2.3 it follows

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where we denote
\[ R(z) = 4C_\alpha \sum (z - \lambda_j)^{\sigma-1} T_{ajm} \]

and
\[ 2R(z) \leq 4C_\alpha \sum (z - \lambda_j)^{\sigma-1} \lambda_j + 4 \sum H(\sigma, z, \alpha). \] (3.3)

Since
\[ \sum (z - \lambda_j)^{\sigma-1} \lambda_j = zR_{\alpha-1}(z) - R_\alpha(z), \]

we have
\[ 2R(z) \leq 4C_\alpha (zR_{\alpha-1}(z) - R_\alpha(z)) + 4 \sum H(\sigma, z, \alpha), \]

namely,
\[ (1 + 2C_\alpha)R_\alpha(z) - 2zC_\alpha R_{\alpha-1}(z) \leq 2 \sum H(\sigma, z, \alpha). \] (3.4)

We consider three cases: 1) \( 1 \leq \sigma \leq 2 \); 2) \( 0 < \sigma < 1 \) and 3) \( \sigma > 2 \).

1) \( 1 \leq \sigma \leq 2 \). In this case, it sees \( C_\alpha = 1 \) and
\[ \frac{z - \lambda_j - C_\alpha (\lambda_q - \lambda_j)}{\lambda_q - \lambda_j} = \frac{z - \lambda_j}{\lambda_q - \lambda_j}. \]

Since \( \lambda_q > z \), it follows
\[ \frac{z - \lambda_j - C_\alpha (\lambda_q - \lambda_j)}{\lambda_q - \lambda_j} < 0, \]

and therefore
\[ H(\sigma, z, \alpha) < 0. \]

Substituting this into (3.4), we obtain
\[ (1 + 2C_\alpha)R_\alpha(z) - 2zC_\alpha R_{\alpha-1}(z) \leq 0 \]

and
\[ R_{\alpha-1}(z) \geq \frac{3}{2z} R_\alpha(z). \]

Now (1.4) is proved. Using (2.1), we have
\[ \frac{1}{\sigma} \left\| R'_\sigma(z) \right\| \geq \frac{3}{2z} R_\alpha(z), \]

and (1.5) is proved. Since
\[ \left( \frac{R_\sigma(z)}{z^{3/2}} \right)' \left( \frac{3\sigma}{2z^{3/2}} \right) - \left( \frac{3\sigma}{2z^{3/2}} \right)^{\sigma-1} \left( \frac{3\sigma}{2z^{3/2}} \right)^{\sigma-1} \leq 0, \]

it follows that \( \frac{R_\sigma(z)}{z^{3/2}} \) is a nondecreasing function of \( z \).

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2) $0 < \sigma < 1$. Now $C_\sigma = \frac{\sigma}{2} \in \left(0, \frac{1}{2}\right)$, so $1 - C_\sigma > 0$ and
\[
\frac{z - \lambda_i - C_\sigma (\lambda_j - \lambda_i)}{\lambda_j - \lambda_i} < \frac{\lambda_j - \lambda_i - C_\sigma (\lambda_j - \lambda_i)}{\lambda_j - \lambda_i} = 1 - C_\sigma \tag{3.5}
\]
Then
\[
H(\sigma, z, \alpha) \leq (1 - C_\sigma) \sum_{j < i} T_{i,j} (z - \lambda_i)^{\sigma-1}
\]
and
\[
\sum_{\sigma > 1} H(\sigma, z, \alpha) \leq (1 - C_\sigma) \sum_{j < i} (z - \lambda_i)^{\sigma-1} \lambda_j = (1 - C_\sigma) (z R_\alpha(z) - R_\alpha(z)).
\]
Substituting this into (3.4), we obtain
\[
(1 + 2C_\sigma) R_\alpha(z) - 2zC_\sigma R_{\alpha-1}(z) - \left( 2 - 2C_\sigma \right) (z R_{\alpha-1}(z) - R_\alpha(z)),
\]
namely,
\[
3R_\alpha(z) \leq 2zR_{\alpha-1}(z),
\]
and (1.4) is proved.

The remainders are discussed similarly to 1).

3) $\sigma > 2$. In this case $C_\sigma = \frac{1}{2} > 1$, so $1 - C_\sigma < 0$ and
\[
H(\sigma, z, \alpha) \leq (1 - C_\sigma) \sum_{j < i} T_{i,j} (z - \lambda_i)^{\sigma-1} < 0.
\]
Substituting this into (3.4), we have
\[
(1 + 2C_\sigma) R_\alpha(z) \leq 2zC_\sigma R_{\alpha-1}(z)
\]
and (1.6) is proved.

Noting (2.1), it implies
\[
\frac{1}{\sigma} R'_\alpha(z) \geq \left(1 + \frac{1}{\sigma}\right) \frac{1}{z} R_\alpha(z)
\]
and (1.7) is proved.

Similarly,
\[
\left( \frac{R_\alpha(z)}{z^{\sigma+1}} \right)' = \frac{R'_\alpha(z) z^{\sigma+1} - R_\alpha(z) (\sigma+1) z^\sigma}{z^{2(\sigma+1)}}
\]
\[
= \frac{z^\sigma [z R'_\alpha(z) - (\sigma+1) R_\alpha(z)]}{z^{2(\sigma+1)}} \geq 0,
\]
thus $\frac{R_\alpha(z)}{z^{\sigma+1}}$ is a nondecreasing function of $z$.

**Corollary 3.1** For all $\sigma \geq 2$ and $z \geq (1+\sigma) \lambda_i$,
\[
\sigma^{\sigma-1} \lambda_i^{1-\sigma} \left( \frac{z}{1+\sigma} \right)^{1-\sigma} \leq R_\alpha(z) \leq L_{\sigma, z}^{cl} [Q] z^{\sigma+1},
\]
where $L_{\sigma, z}^{cl} = \frac{\Gamma(\sigma+1)}{4n^{\Gamma(\sigma+2)}}$.

**Proof.** 1) Noting $R_\alpha(z) = \sum_k (z_0 - \lambda_i)^{\alpha} \geq (z_0 - \lambda_i)^{\alpha}$, for any $z_0 > \lambda_i$, it follows from Theorem 1.1 that for all $z \geq z_0$,
\[
\frac{R_\alpha(z)}{z^{\sigma+1}} \geq \frac{R_\alpha(z_0)}{z_0^{\sigma+1}} \geq \frac{(z_0 - \lambda_i)^{\alpha}}{z_0^{\sigma+1}}.
\]
So
\[
R_\alpha(z) \geq (z_0 - \lambda_i)^{\alpha} \left( \frac{z}{z_0} \right)^{\sigma+1}.
\]
Since (3.7) holds for arbitrary $z_0 > \lambda_i$, it yields
\[
R_\alpha(z) \geq \max_{z_0 > \lambda_i} \left[ (z_0 - \lambda_i)^{\alpha} \left( \frac{z}{z_0} \right)^{\sigma+1} \right].
\]
Due to
\[
\left( \frac{z_0 - \lambda_i}{z_0} \right)^{\sigma+1} = \frac{(z_0 - \lambda_i)^{\sigma+1} - (\sigma+1)(z_0 - \lambda_i)^{\sigma} z_0\sigma}{z_0^{2(\sigma+1)}} = \frac{(z_0 - \lambda_i)^{\sigma-1} \left[ \sigma z_0 - (\sigma+1)(z_0 - \lambda_i) \right]}{z_0^{\sigma+2}}
\]
we see that when $z_0 = (\sigma+1) \lambda_i$, it gets
\[
\max_{z_0 > \lambda_i} \left[ (z_0 - \lambda_i)^{\alpha} \left( \frac{z}{z_0} \right)^{\sigma+1} \right] = \sigma^\sigma \lambda_i^{1-\sigma} \left( \frac{z}{1+\sigma} \right)^{1-\sigma}.
\]
For $z \geq z_0 = (\sigma+1) \lambda_i$, we have
\[
R_\alpha(z) \geq \sigma^\sigma \lambda_i^{1-\sigma} \left( \frac{z}{1+\sigma} \right)^{1-\sigma}
\]
and the inequality in the left-hand side of (3.6) is valid.

2) By the Berezin-Lieb inequality (see [11]), we have
\[
R_\alpha(z) \rightarrow L_{\sigma, z}^{cl} [Q], z \rightarrow \infty.
\]
Notice that $\frac{R_\alpha(z)}{z^{\sigma+1}}$ is nondecreasing to $z$, it follows
\[
R_\alpha(z) \leq L_{\sigma, z}^{cl} [Q]
\].
and the inequality in the right-hand side of (3.6) is proved.

**Corollary 3.2** 1) For $1 \leq \sigma \leq 2$ and $z \geq (\sigma + 2)\lambda_i$, 
$$R_{\sigma}(z) \geq \left(\frac{\sigma+1}{\sigma+2}\right)^{\sigma+1} \lambda_i^{-1} z^{\sigma+1}. \quad (3.8)$$

2) For $0 \leq \sigma < 1$ and $z \geq (\sigma + 3)\lambda_i$, 
$$R_{\sigma}(z) \geq \frac{3(\sigma+2)^{3\sigma+1}}{2(\sigma+3)^{3\sigma+2}} \lambda_i^{-1} z^{\sigma+1}. \quad (3.9)$$

**Proof.** 1) By Corollary 3.1, we know that for $1 \leq \sigma \leq 2$ and $z \geq (\sigma + 2)\lambda_i$, it holds
$$R_{\sigma+1}(z) \geq (\sigma+1)^{\sigma+1} \lambda_i^{-1} \left(\frac{z}{\sigma+2}\right)^{\sigma+2}. \quad (3.10)$$

Using Theorem 1.1, we have
$$R_{\sigma}(z) \geq \left(1 + \frac{1}{\sigma+1}\right)^{\frac{1}{z}} R_{\sigma+1}(z), \text{ for } 1 \leq \sigma \leq 2. \quad (3.11)$$

Combining (3.10) and (3.11), it follows
$$R_{\sigma}(z) \geq \left(1 + \frac{1}{\sigma+1}\right)^{\frac{1}{z}} (\sigma+1)^{\sigma+1} \lambda_i^{-1} \left(\frac{z}{\sigma+2}\right)^{\sigma+2}$$
$$= \left(\frac{\sigma+1}{\sigma+2}\right)^{\sigma+1} \lambda_i^{-1} z^{\sigma+1}$$

and (3.8) is proved.

2) By Corollary 3.1, it shows that for $0 \leq \sigma < 1$ and $z \geq (\sigma + 3)\lambda_i$, it holds
$$R_{\sigma+2}(z) \geq (\sigma+2)^{3\sigma+1} \lambda_i^{-1} \left(\frac{z}{\sigma+3}\right)^{3\sigma+3}. \quad (3.12)$$

From Theorem 1.1, we see that for $0 \leq \sigma < 1$,
$$R_{\sigma}(z) \geq \frac{3}{2z} R_{\sigma+2}(z), \text{ for } 0 \leq \sigma < 1. \quad (3.13)$$

In the light of (3.12) and (3.13), it obtains
$$R_{\sigma}(z) \geq \frac{9}{4z^2} R_{\sigma+2}(z) \geq \frac{9}{4z^2} (\sigma+2)^{3\sigma+2} \lambda_i^{-1} \left(\frac{z}{\sigma+3}\right)^{3\sigma+3}$$
$$= \frac{3(\sigma+2)^{3\sigma+1}}{2(\sigma+3)^{3\sigma+2}} \lambda_i^{-1} z^{\sigma+1}.$$

Noting that
$$\frac{3(\sigma+2)}{2(\sigma+3)} = \frac{3}{2} \left(1 - \frac{1}{\sigma+3}\right) \geq 1, \text{ for } 0 \leq \sigma < 1,$$ we have
$$R_{\sigma}(z) \geq \frac{3(\sigma+2)^{\sigma+1}}{2(\sigma+3)^{\sigma+2}} \lambda_i^{-1} z^{\sigma+1},$$
and (3.9) is proved.

**Remark 3.3** Specially, we have
$$R_{\sigma}(z) \geq \frac{3}{2z} R_{\sigma}(z) \geq \frac{2}{9} \lambda_i^{-1} z^{\sigma+1}, \quad (3.14)$$
$$N(z) = R_{\sigma}(z) \geq \frac{3}{2z} R_{\sigma}(z) \geq \frac{9}{4z^2} R_{\sigma}(z) \geq \frac{z}{3\lambda_i}. \quad (3.15)$$

### 4. Proof of Theorem 1.2

Denote
$$\overline{\lambda}_j = \frac{1}{N} \sum_{i=1}^{N} \lambda_i \quad \text{and} \quad \overline{\lambda}_j = \frac{1}{N} \sum_{i=1}^{N} \lambda_i^2,$$

and let $\text{ind}(z)$ be the greatest integer $i$ such that $\lambda_i \leq z$.

Let $\text{ind}(z) = i$, it implies that $\lambda_i \leq z$ and $\lambda_{i+1} > z$, so
$$R_{\sigma}(z) = \sum_{k} (z - \lambda_k)^2$$
$$= (z - \lambda_1)^2 + (z - \lambda_2)^2 + \cdots + (z - \lambda_J)^2$$
$$= i z^2 - 2iz(\lambda_1 + \lambda_2 + \cdots + \lambda_j + \lambda_{j+1} + \lambda_{j+2} + \cdots + \lambda_J)$$
$$= i z^2 - 2iz \overline{\lambda}_j + \lambda_{j+1} + \lambda_{j+2} + \cdots + \lambda_J.$$

(4.1)

For any integer $j$ and $z \geq \lambda_j$, it implies $\text{ind}(z) \geq j$, and
$$R_{\sigma}(z) \geq Q(z, j) := j \left( z^2 - 2z \overline{\lambda}_j + \lambda_j^2 \right).$$

Using Theorem 1.1, we have that for $z \geq \lambda_j$,
$$R_{\sigma}(z) \geq \frac{Q(z, j)}{z_j} \quad \text{or}$$
$$R_{\sigma}(z) \geq Q(z, j) \left( \frac{z}{z_j} \right)^{j}. \quad (4.2)$$

By the Cauchy-Schwarz inequality, it follows
$$\overline{\lambda}_j^2 \leq \lambda_j^2$$

and
$$Q(z, j) = j \left( z^2 - 2z \overline{\lambda}_j + \lambda_j^2 \right)$$
$$= j \left[ z^2 - 2z \overline{\lambda}_j + \overline{\lambda}_j^2 + \lambda_j^2 - \overline{\lambda}_j^2 \right]$$
$$= j \left[ (z - \overline{\lambda}_j)^2 + \lambda_j^2 - \overline{\lambda}_j^2 \right] \geq j \left( z - \overline{\lambda}_j^2 \right)^2.$$

(4.3)
Proof of Theorem 1.2 1) Substituting $z_j = 3\lambda_j$ into (4.2) and noticing (4.3), we have

$$R_2(z) \geq j(z_j - 3\lambda_j)^2 \frac{z_j^3}{z_j^3} = \frac{4jz^3}{27\lambda_j}$$

and (1.8) is proved.

2) We take (1.8) into (3.14) to obtain

$$R_k(z) \geq \frac{3}{2z} \frac{4jz^3}{9\lambda_j} = \frac{2jz^3}{9\lambda_j}$$

and (1.9) is proved.

3) Combining (1.8) and (3.15), it implies

$$N(z) = R_k(z) \geq \frac{9}{4z^2} \frac{4jz^3}{27\lambda_j} = \frac{jz}{3\lambda_j}$$

and (1.10) is proved.

4) If $\lambda_{k+1} \leq 3\lambda_j$, then (1.11) is clearly valid; if $\lambda_{k+1} > 3\lambda_j$, then (1.10) shows by letting $z \to \lambda_{k+1}$ that

$$\frac{\lambda_{k+1}}{\lambda_j} \leq \frac{3k}{j}.$$ 

So (1.11) is proved and Theorem 1.2 is proved. □

Corollary 4.1 We have

$$\lambda_{k+1} \leq 3\lambda_k$$

and

$$\lambda_{k+1} \leq 3k\lambda_k.$$ (4.4)

5. Proof of Theorem 1.3

We first recall the following definition before proving Theorem 1.3.

Definition 5.1 If $f(z)$ is superlinear in $z$ as $z \to \infty$, then its Legendre transform is defined by

$$L[f](w) = \sup_z \{|wz - f(z)|\}. \quad (5.1)$$

Remark 5.2 If $f(z) \geq g(z)$ for all $z$, then $L[f](w) \leq L[g](w)$ for all $w$; Since the maximizing value of $z$ in (5.1) is a nondecreasing function of $w$, it follows that for $w$ sufficiently large, the maximizing $z$ exceeds $z_j = 3\lambda_j$.

Proof of Theorem 1.3 From (1.9), we have

$$L[R_k](w) \preceq L \left[ \frac{2jz^2}{9\lambda_j} \right](w). \quad (5.2)$$

Now let us calculate $L[R_k](w)$. Since

$$R_k(z) = \sum \{z - \lambda_k\},$$

is piecewise linear function of $z$, it implies that the maximizing value of $z$ in the Legendre transform of $R_k$ is attained at one of the critical values.

In fact if $\lambda_k < z \leq \lambda_{k+1}$, then

$$L[R_k](w) = \sup_z \{|wz - R_k(z)|\} = \sup_z \{|wz - \sum \{z - \lambda_k\}\} = \sup_z \{|(w-k)z + \lambda_1 + \lambda_2 + \cdots + \lambda_k\}.$$ Noting that the maximizing value of $z$ is a non-decreasing function of $w$, we see $w-k \geq 0$, therefore the critical value $z_* = \lambda_{k+1}$.

It is easy to check $k = [w]$ and

$$L[R_k](w) = \sup_z \{|(w-k)z + \lambda_1 + \lambda_2 + \cdots + \lambda_k\} = \left( w - [w] \right) \lambda_{[w]+1} + [w] \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_{[w]}}{[w]}.$$ (5.3)

Next we calculate $L \left[ \frac{2jz^2}{9\lambda_j} \right](w)$. Noting

$$L \left[ \frac{2jz^2}{9\lambda_j} \right](w) = \sup_z \left\{ wz - \frac{2jz^2}{9\lambda_j} \right\}$$

and letting

$$f(z) = wz - \frac{2jz^2}{9\lambda_j},$$

we know $f'(z) = w - \frac{4jz}{9\lambda_j}$. By $f'(z) = 0$, it solves

$$z_* = \frac{9w\lambda_j}{4j}. \quad (5.4)$$

Therefore

$$L \left[ \frac{2jz^2}{9\lambda_j} \right](w) = \sup_z \left\{ wz - \frac{2jz^2}{9\lambda_j} \right\} = \frac{9w\lambda_j}{4j} - \frac{2j}{9\lambda_j} \left( \frac{9w\lambda_j}{4j} \right)^2 = \frac{9\lambda_j}{8j} w^2.$$ (5.5)

Taking (5.3) and (5.5) into (5.2), we have

$$\left( w - [w] \right) \lambda_{[w]+1} + [w] \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_{[w]}}{[w]} = \frac{9\lambda_j}{8j} w^2.$$ (5.6)

By (5.4), it has
From Theorem 1.2, \( z_i \geq 3\lambda_j \), so \( w \geq \frac{4j}{9\lambda_j} \cdot 3\lambda_j = \frac{4j}{3} \).

Then it follows that if \( w \) is restricted to the value \( w \geq \frac{4j}{3} \), then \((5.6)\) is valid.

Meanwhile, for any \( w \), we can always find an integer \( k \) such that \( k-1 \leq w < k \) and

\[
[w] = k - 1.
\]

If \( k > \frac{4j}{3} \) and \( w \) approaches to \( k \) from below, then we obtain from \((5.5)\) that

\[
\lambda_1 + \lambda_2 + \cdots + \lambda_k = \lambda_k + (k-1)\overline{\lambda_{k-1}} \leq \frac{9\lambda_j}{8j} k^2.
\]

Therefore

\[
\frac{\overline{\lambda_1}}{\lambda_1} \leq \frac{9k}{8j}.
\]

and Theorem 1.3 is proved. \( \square \)

**Remark 5.3** If we let \( j = 1 \), then

\[
\frac{\overline{\lambda_1}}{\lambda_1} \leq \frac{9}{8} k.
\] (5.7)

We point out that (5.7) is sharper than (4.4). In fact, we get from (4.4) that

\[
\sum_{j=0}^{k-1} \overline{\lambda_1} \leq 3\sum_{j=0}^{k-1} \overline{j} = \frac{3k(k-1)}{2} \leq \frac{3}{2} k^2
\]

and

\[
\frac{\overline{\lambda_k}}{\lambda_1} \leq \frac{3}{2} k.
\]

But \( \frac{9k}{8} < \frac{3k}{2} \) is always valid, so (5.7) is sharper than (4.4).

**REFERENCES**


