On Maximal Regularity and Semivariation of $\alpha$-Times Resolvent Families

Fubo Li, Miao Li
Department of Mathematics, Sichuan University, Chengdu, China
Email: lifubo@scu.edu.cn, mli@scu.edu.cn

Received October 15, 2013; revised November 15, 2013; accepted November 21, 2013

Copyright © 2013 Fubo Li, Miao Li. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

ABSTRACT
Let $A$ be the generator of an $\alpha$-times resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ on a Banach space $X$. It is shown that the fractional Cauchy problem $D_\alpha^t u(t) = Au(t) + f(t)$, $t \in (0, r]$; $u(0), u'(0) \in D(A)$ has maximal regularity on $C([0, r]; X)$ if and only if $S_\alpha(\cdot)$ is of bounded semivariation on $[0, r]$.

Keywords: $\alpha$-Times Resolvent Family; Maximal Regularity; Semivariation

1. Introduction
Many initial and boundary value problems can be reduced to an abstract Cauchy problem of the form
\begin{equation}
\begin{aligned}
&u'(t) = Au(t) + f(t), \quad t \in [0, r] \\
u(0) = x \in D(A)
\end{aligned}
\end{equation}
where $A$ is the generator of a $C_0$-semigroup. One says that (1.1) has maximal regularity on $C([0, r]; X)$ if for every $f \in C([0, r]; X)$ there exists a unique $u \in C^2([0, r]; X)$ satisfying (1.1). From the closed graph theorem it follows easily that if there is maximal regularity on $C([0, r]; X)$, then there exists a constant $C > 0$ such that
\[ \|u\|_{C([0, r]; X)} + \|Au\|_{L^1([0, r]; X)} \leq \|f\|_{C([0, r]; X)}. \]

Travis [1] proved that the maximal regularity is equivalent to the $C_0$-semigroup generated by $A$ being of bounded semivariation on $[0, r]$.

Chyan, Shaw and Piskarev [2] gave similar results for second order Cauchy problems. More precisely, they showed that the second order Cauchy problem
\begin{equation}
\begin{aligned}
&u''(t) = Au(t) + f(t), \quad t \in (0, r] \\
u(0) = x, u'(0) = y, \quad x, y \in D(A)
\end{aligned}
\end{equation}
has maximal regularity on $[0, r]$ if and only if the cosine operator function generated by $A$ is of bounded semivariation on $[0, r]$.

In this paper, we will consider the maximal regularity for fractional Cauchy problem
\begin{equation}
\begin{aligned}
&D_\alpha^t u(t) = Au(t) + f(t), \quad t \in (0, r] \\
u(0) = x, u'(0) = y, \quad x, y \in D(A)
\end{aligned}
\end{equation}
where $\alpha \in (1, 2)$, $A$ is the generator of an $\alpha$-times resolvent family (see Definition 2.2) and $D_\alpha^t u$ is understood in the Caputo sense. We show that (1.3) has maximal regularity on $C([0, r]; X)$ if and only if the corresponding $\alpha$-times resolvent family is of bounded semivariation on $[0, r]$.

2. Preliminaries
Let $1 < \alpha < 2$, $g_\alpha(t) = \delta(t)$ and
\[ g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)} \quad (\beta > 0) \]
for $t > 0$. Recall the Caputo fractional derivative of order $\alpha > 0$
\[ D_\alpha^t f(t) = \int_0^t g_{2-\alpha}(t-s) \frac{d^2}{ds^2} f(s) ds, \quad t \in [0, r] \]
for $f \in C^2([0, r]; X)$. The condition that
\[ f \in C^2([0, r]; X) \]
can be relaxed to \( f \in C^1([0, r]; X) \) and 
\[
\varrho_2 \ast (f-f(0)-f'(0)g_2) \in C^2([0, r]; X),
\]
for details and further properties see [3] and references therein. And in the above we denote by 
\[
(g_\beta \ast f)(t) = \int_0^t g_\beta(t-s)f(s)ds
\]
the convolution of \( g_\beta \) with \( f \). Note that \( g_\alpha \ast g_\beta = g_{\alpha \circ \beta} \).

Consider a closed linear operator \( A \) densely defined in a Banach space \( X \) and the fractional evolution Equation (1.3).

**Definition 2.1** A function \( u \in C([0, r]; X) \) is called a strong solution of (1.3) if 
\[
u \in C([0, r]; D(A)) \cap C^1([0, r]; X),
\]
\[
\varrho_2 \ast (u(t-x-y)) \in C^2([0, r]; X)
\]
and (1.3) holds on \([0, r] \). \( u \in C([0, r]; X) \) is called a mild solution of (1.3) if \( g_\alpha \ast u \in D(A) \) and 
\[
u(t) - x-ty = A(g_\alpha \ast u)(t) + (g_\beta \ast f)(t)
\]
for \( t \in [0, r] \).

**Definition 2.2** Assume that \( A \) is a closed, densely defined linear operator on \( X \). A family \( \{S_\alpha(t)\}_{\alpha \in (0, \rho]} \subset B(X) \) is called an \( \alpha \)-times resolvent family generated by \( A \) if the following conditions are satisfied:

(a) \( S_\alpha(0) = I \); 

(b) \( S_\alpha(t)D(A) \subset D(A) \) and \( AS_\alpha(t)x = S_\alpha(t)Ax \)

for all \( x \in D(A), t \geq 0 \); 

(c) For all \( x \in D(A) \) and \( t \geq 0 \), 
\[
S_\alpha(t)x = x + \langle g_\alpha \ast S_\alpha(t)x \rangle_Ax.
\]

**Remark 2.3** Since \( A \) is closed and densely defined, it is easy to show that for all \( x \in X \), \( g_\alpha \ast S_\alpha(t)x \in D(A) \) and 
\[
\varrho \left( (g_\alpha \ast S_\alpha(t)x) \right) = x + \langle g_\alpha \ast S_\alpha(t)x \rangle_Ax.
\]

The \( \alpha \)-times resolvent families are closely related to the solutions of (1.3). It was shown in [3] that if \( A \) generates an \( \alpha \)-times resolvent family \( S_\alpha(\cdot) \), then (1.3) has a unique strong solution given by 
\[
S_\alpha(t)x + \int_0^t S_\alpha(x)dx.
\]

Next, we recall the definition of functions of bounded semivariation (see e.g. [4]). Given a closed interval \([a, b]\) of the real line, a subdivision of \([a, b]\) is a finite sequence \( d : a = d_0 < d_1 < \cdots < d_i = b \). Let \( \Delta[d, a] \) denote the set of all subdivisions of \([a, b] \).

**Definition 2.4** For 
\[
G : [a, b] \to B(X) \quad \text{and} \quad d \in \Delta[d, a],
\]
declare 
\[
SV\left[ G \right] = \sup \left\{ \int_{[c]}^d \left( G(x) - G(d_{i+1}) \right) dx \mid x_i \in X \right\}
\]
and 
\[
SV\left[ G \right] = \sup \left\{ SV\left[ G \right] \mid d \in \Delta[d, a] \right\}
\]
We say \( G \) is of bounded semivariation if \( SV\left[ G \right] < \infty \).

**3. Main Results**

We begin with some properties on \( \alpha \)-times resolvent families which will be needed in the sequel.

**Proposition 3.1** Let \( 1 < \alpha < 2 \) and \( \{S_\alpha(t)\}_{\alpha \in (0, \rho]} \) be the \( \alpha \)-times resolvent family with generator \( A \). Define 
\[
P_\alpha(t)x = (g_\alpha \ast S_\alpha(t)x)
\]
then the following statements are true.

(a) For every \( x \in X \), 
\[
\int_0^t P_\alpha(s)ds \in D(A)
\]
and 
\[
A\int_0^t P_\alpha(s)xds = S_\alpha(t)x - x;
\]

(b) For every \( x \in X \), 
\[
0 \leq a, b \leq t,
\]
\[
\int_a^b P_\alpha(s)xds \in D(A)
\]
and 
\[
A\int_a^b P_\alpha(s)xds = aS_\alpha(t-a)x - bS_\alpha(t-b)x
\]
\[
+ \int_a^b S_\alpha(t-s)xds;
\]

(c) For every \( x \in X \), 
\[
\int_0^t S_\alpha(s)xds \in D(A)
\]
and 
\[
A\int_0^t S_\alpha(s)xds = \alpha (S_\alpha(t)x) + \alpha P_\alpha(t)x;
\]

(d) If \( f \in C([0, r]; X) \), then \( g_\alpha \ast S_\alpha \ast f \in D(A) \) and 
\[
A(g_\alpha \ast S_\alpha \ast f) = (S_\alpha^{-1}) \ast f.
\]

**Proof.** (a) follows from the fact that 
\[
\int_0^t P_\alpha(s)xds = (g_\alpha \ast S_\alpha(t)x) x
\]
\[
= (g_\alpha \ast S_\alpha(t)x) e_D(A)
\]
and 
\[
A(g_\alpha \ast S_\alpha(t)x) x = S_\alpha(t)x - x
\]
by Remark 2.3. (b) By integration by parts we have 
\[
\int_a^b P_\alpha(t-s)xds = \int_a^b \left[ \int_s^t P_\alpha(t)dx \right]t_{s}d_{s}r
\]
\[
= \int_0^t \left[ (g_\alpha \ast S_\alpha(t)x) \right] d_{s}r
\]
\[
= -s (g_\alpha \ast S_\alpha(t-s)x) x + \int_0^t (g_\alpha \ast S_\alpha(t-s)x ds
\]
\[
= a(g_\alpha \ast S_\alpha(t-a)x - b(g_\alpha \ast S_\alpha(t-b)x
\]
\[
+ \int_a^b (g_\alpha \ast S_\alpha(t-s)x ds.}
since \((g_a \ast S_a)(t)\) \(\in D(A)\) by Remark 2.3, operating \(A\) on both sides of the above identity gives (b).

(c) follows from the fact that
\[
\int_0^t g_a(t-s)(s-t)P_a(s)\,dx + \int_0^t g_a(t-s)P_a(s)\,dx = -\alpha \int_0^t g_{a+1}(t-s)P_a(s)\,dx + t(g_a \ast P_a)(t)\]
\[
= -\alpha (g_a \ast (A \ast S_a)) + t(g_a \ast P_a)(t) = -\alpha (g_a \ast (S_a - 1))(t) + t(g_a \ast P_a)(t)
\]
\[
= -\alpha (g_a \ast S_a)(t)x + \alpha g_{a+1}(t)x + t(g_a \ast P_a)(t) = -\alpha (g_a \ast S_a)(t) + tP_a(t).
\]

(d) (3.1) is true for step functions, and then for continuous functions by the closedness of \(A\).

The following two lemmas can be proved similarly as that in [1, 2].

**Lemma 3.2** If \(f \in C([0, r]; X)\) and the \(\alpha\)-times resolvent family \(S_a(t)\) is of bounded semivariation on \([0, r]\), then \((P_a \ast f)(t)\) \(\in D(A)\) and

\[
A(P_a \ast f)(t) = -\int_0^t d\left(S_a(t-s)\right)f(s).
\]

**Lemma 3.3** If \(f \in C([0, r]; X)\) and the \(\alpha\)-times resolvent family \(S_a(t)\) is of bounded semivariation on \([0, r]\), then \(D^\alpha u(t)\) is continuous in \(t\) on \([0, r]\).

Next we turn to the solution of
\[
D^\alpha u(t) = Au(t) + f(t), \quad t \in (0, r],
\]
\[
u(0) = 0, u'(0) = 0,
\]
where \(A\) is the generator of an \(\alpha\)-times resolvent family. If \(v(t)\) is a mild solution of (3.2), then by Definition 2.1 \((g_a \ast v)(t) \in D(A)\) and
\[
v(t) = (g_a \ast v)(t) + (g_a \ast f)(t).
\]

It then follows from the properties of \(\alpha\)-times resolvent family that
\[
1 \ast v = (S_a - A(g_a \ast S_a)) \ast v
\]
\[
= S_a \ast v - S_a \ast A(g_a \ast v)
\]
\[
= S_a \ast (v - A(g_a \ast v))
\]
\[
= S_a \ast g_a \ast f,
\]

which implies that \((g_a \ast S_a \ast f)\) is differentiable and
\[
v(t) = \frac{d}{dt}(g_a \ast S_a \ast f)(t)
\]
\[
= (g_a \ast S_a \ast f)(t)
\]
\[
= (P_a \ast f)(t).
\]

Therefore, the mild solution of (1.3) is given by
\[
u(t) = S_a(t)x + \int_0^t S_a(s)dx + (P_a \ast f)(t) \quad \text{(3.3)}
\]

**Proposition 3.4** Let \(A\) be the generator of an \(\alpha\)-times resolvent family \(S_a(t)\), and let \(f \in C([0, r]; X)\) and \(x, y \in D(A)\). Then the following statements are equivalent:

(a) (1.3) has a strong solution;

(b) \((S_a \ast f)(\cdot) \in C^1([0, r]; X)\);

(c) \((P_a \ast f)(t)\) \(\in D(A)\) for \(0 \leq t \leq r\) and \(A(P_a \ast f)(t)\) is continuous in \(t\) on \([0, r]\).

**Proof.** (a) If \(u(t)\) is a strong solution of (1.3), then it is given by (3.3) since every strong solution is a mild solution, therefore, by the definition of strong solutions,
\[
g_{2-\alpha} \ast P_a \ast f = g_{\alpha} \ast S_a \ast f \in C^1([0, r]; X);
\]

it then follows that \(S_a \ast f \in C^1([0, r]; X)\), this is (b).

(b) \(\Rightarrow\) (c). Suppose that \(S_a \ast f \in C^1([0, r]; X)\). Since \(g_{\alpha} \ast P_a \ast f = g_{\alpha} \ast S_a \ast f\), by Proposition 3.1(d),
\[
g_{\alpha} \ast P_a \ast f \in D(A)
\]

and
\[
A(g_{\alpha} \ast P_a \ast f) = A(g_{\alpha} \ast S_a \ast f) = (S_a - 1) \ast f \quad \text{(3.4)}
\]

Since \(A\) is closed and \(S_a \ast f \in C^1([0, r]; X)\), we have \(P_a \ast f \in D(A)\) and \(A(P_a \ast f) = (S_a - 1) \ast f \) is continuous.

(c) \(\Rightarrow\) (a). By (3.4),
\[
g_{\alpha} \ast A(P_a \ast f) = A(g_{\alpha} \ast P_a \ast f) = (S_a - 1) \ast f
\]

therefore \(S_a \ast f\) is differentiable and thus
\[
g_{2-\alpha} \ast P_a \ast f = g_{\alpha} \ast S_a \ast f
\]

is in \(C^1([0, r]; X)\). It is easy to check that \(u(t)\) defined by (3.3) is a strong solution of (1.3).

Now we are in the position to give the main result of this paper. The proof is similar to that of Proposition 3.1 in [1] or Theorem 4.2 in [2], we write it out for completeness.
Theorem 3.5 Suppose that \( A \) generates an \( \alpha \)-times resolvent family \( \{S_t(\cdot)\}_{t \geq 0} \). Then the function (3.3) is a strong solution of the Cauchy problem (1.3) for every pair \( x, y \in D(A) \) and continuous function \( f \) if and only if \( S_\alpha(\cdot) \) is of bounded semivariation on \([0, r]\).

Proof. The sufficiency follows from Lemmas 3.2 and 3.3.

Conversely, suppose that for \( x, y \in D(A) \) and continuous function \( f \), \( u(t) \) given by (3.3) is a strong solution for (1.3). Define the bounded linear operator \( L: C([0, r]; X) \rightarrow X \) by \( L(f) = (P_\alpha * f)(r) \). By Proposition 3.1,

\[
AL(f_{d, \epsilon}) = A\left[\int_0^r P_\alpha(r-s) f_{d, \epsilon}(s) \, ds \right] = \sum_{j=1}^n \left[ A \int_{d_{j-1}}^{d_j} P_\alpha(r-s) x_{i_j} \, ds + A \int_{d_{j-1}}^{d_j} P_\alpha(r-s) x_{i_j} \, ds \right] + A \int_{d_{n-1}}^{d_n} s-\frac{d_j - d_{j-1}}{\epsilon} P_\alpha(r-s) (x_{i_j} - x_{i_{j-1}}) \, ds
\]

\[
= \sum_{j=1}^n \left[ S_{\alpha}(r-d_{j-1} + \epsilon)(x_{i_j} - x_{i_{j-1}}) - S_{\alpha}(r-d_j)(x_{i_j} - x_{i_{j-1}}) \right]
\]

it then follows that

\[
\left\| \sum_{j=1}^n \left[ S_{\alpha}(r-d_{j-1} + \epsilon)(x_{i_j} - x_{i_{j-1}}) - S_{\alpha}(r-d_j)(x_{i_j} - x_{i_{j-1}}) \right] x \right\| \leq \| AL(f_{d, \epsilon}) \| + \sum_{j=1}^n \left\| S_{\alpha}(r-d_j)(x_{i_j} - x_{i_{j-1}}) - \frac{1}{\epsilon} \int_{d_{j-1}}^{d_j} S_{\alpha}(r-s)(x_{i_j} - x_{i_{j-1}}) \, ds \right\|
\]

By letting \( \epsilon \to 0 \), we obtain that \( S_\alpha(\cdot) \) is of bounded semivariation on \([0, r]\).

Corollary 3.6 Suppose that \( \{S_t(\cdot)\}_{t \geq 0} \) is an \( \alpha \)-times resolvent family with generator \( A \) and \( S_\alpha(\cdot) \) is of bounded semivariation on \([0, r]\) for some \( r > 0 \). Then \( R(P_\alpha(t)) \subseteq D(A) \) for \( t \in [0, r] \) and \( \|t AP_\alpha(t)\| \)

is bounded on \([0, r]\).

Proof. For \( x \in X \), consider \( f(t) = \alpha S_t(x) \). By Proposition 3.1(e), \( tP_\alpha(t)x \) is a mild solution of (3.2). Moreover, it follows from Proposition 3.4 that \( P_\alpha * f \) is a strong solution of (3.2). Since a strong solution must be a mild solution, we have \( \langle P_\alpha * f \rangle(t) = tP_\alpha(t) \). Thus our claim follows from Proposition 3.4.

Remark 3.7 Let \( \alpha = 1 \). If \( A \) generates a \( C_0 \)-semigroup \( T(\cdot) \), then the condition that \( tAT(\cdot) \) is bounded on \([0, r]\) implies that \( T(\cdot) \) is analytic (see [5]). When \( \alpha = 2 \) and \( A \) generates a cosine function \( C(\cdot) \), then the condition that \( tAC(\cdot) \) is bounded on \([0, r]\) implies that \( A \) is bounded ([3]). However, since there is no semigroup property for \( \alpha \)-times resolvent family, it is not clear that one can get the analyticity of \( S_\alpha(\cdot) \) from the local boundedness of \( tAP_\alpha(t) \).

REFERENCES


