Primes in Arithmetic Progressions to Moduli with a Large Power Factor

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ABSTRACT

Recently Elliott studied the distribution of primes in arithmetic progressions whose moduli can be divisible by high-powers of a given integer and showed that for integer $a \geq 2$ and real number $A > 0$. There is a $B = B(A) > 0$ such that

$$\sum_{d \leq x^{\frac{1}{2}L^{-1}Q^{-1}}} \max_{y \leq x^{\frac{1}{2}L^{-1}Q^{-1}}} \left| \pi(y; qd, r) - \frac{\text{Li}(y)}{\phi(qd)} \right| \ll \frac{x}{\phi(q)L^4},$$

holds uniformly for moduli $q \leq x^3 \exp\left(- \left(\log \log x\right)^3\right)$ that are powers of $a$. In this paper we are able to improve his result.

Keywords: Primes; Arithmetic Progressions; Riemann Hypothesis

1. Introduction and Main Results

Let $p$ denote a prime number. For integer $a, q$ with $(a, q) = 1$, we introduce

$$\pi(x; q, a) = \sum_{\substack{p \leq x \mod q}} 1$$

to count the number of primes in the arithmetic progression $a \mod q$ not exceeding $x$. For fixed $q$, we have

$$\pi(x; q, a) \sim \frac{1}{\phi(q)} \pi(x)$$

as $x$ tends to infinity. However the most important thing in this context is the range uniformity for the moduli $q$ in terms of $x$. The Siegel-Walfisz Theorem, see for example [1], shows that this estimate is true only if $q \leq L^4$, where and throughout this paper we denote $\log x$ by $L$. The Generalized Riemann Hypothesis for Dirichlet L-functions could give a much better result: non-trivial estimate holds for $q \leq x^2 L^{-2}$. Unfortunately the Generalized Riemann Hypothesis has withstood the attack of several generations of researchers and it is still out of reach. However number theorists still want to live a better life without the Generalized Riemann Hypothesis.

Therefore they try to find a satisfactory substitute. In this direction the famous Bombieri-Vinogradov theorem [2, 3], states that

**Theorem A.** For any $A > 0$ there exists a constant $B = B(A) > 0$ such that

$$\sum_{q \leq y^2} \max_{y \leq \frac{q}{2}} \left| \pi(y; q, a) - \frac{\text{Li}(y)}{\phi(q)} \right| \ll yL^{-A},$$

where $\phi(q)$ is the Euler totient function, $Q = x^2 L^{-2}$, and $\text{Li}(y) = \int_{2}^{y} \frac{du}{\log u}$.

Recently in order to study the arithmetic functions on shifted primes, Elliott [4] studied the distribution of primes in arithmetic progressions whose moduli can be divisible by high-powers of a given integer. More precisely, he showed that

**Theorem B.** Let $a$ be an integer, $a \geq 2$. If $A > 0$, then there is a $B = B(A) > 0$ such that

$$\sum_{d \leq x^{\frac{1}{2}L^{-1}Q^{-1}}} \max_{y \leq x^{\frac{1}{2}L^{-1}Q^{-1}}} \left| \pi(y; qd, r) - \frac{\text{Li}(y)}{\phi(qd)} \right| \ll \frac{x}{\phi(q)L^4}.$$
holds uniformly for moduli \( q \leq x^3 \exp \left( -\log \log x \right)^3 \) that are powers of \( a \).

When \( q = 1 \), his result recovers the Bombieri-Vinogradov theorem. And obviously his result gives a deep insight into the distribution of primes in arithmetic progressions.

The most important thing Elliott concerned in [4] is that in Theorem B the parameter \( q \) may reach a fixed power of \( x \). However we want to pursue the widest uniformity in \( q \) by using some new techniques established in the study of Waring-Goldbach problems.

We shall prove the following result.

**Theorem 1.1.** Let \( a \) be an integer, \( a \geq 2 \). If \( A > 0 \), then there is a \( B = B(A) > 0 \) such that

\[
\sum_{d \leq x^{\delta}q^{1/2} \leq n \atop (d,q)=1} \max_{y \leq x} \max_{(y,qd)=1} \left| \pi \left( x; qd, r \right) - \frac{L_i(y)}{\phi(qd)} \right| \ll \frac{x}{\phi(q)L^4},
\]

holds uniformly for moduli \( q \leq x^3 \exp \left( -\log \log x \right)^3 \) that are powers of \( a \).

When \( d = 1 \) and \( a \) an odd prime, our result gives that for these particular moduli \( q \) with the form \( q = p^n, (n = 1,2,3,\ldots) \)

\[
\pi \left( x; q, r \right) = \left\lfloor 1 + O \left( L^{-4} \right) \right\rfloor \frac{L_i(x)}{\phi(q)},
\]

holds uniformly for moduli \( q \leq x^3 \exp \left( -\log \log x \right)^3 \) and \( d \) be prime in these special progressions.

Then the special case of our result shows that the least prime \( P_{\min} (q, r) \) in these special progressions \( n \equiv r \pmod{q} \) satisfies

\[
P_{\min} (q, r) \ll q^{5/12}. \]

This result improves a former result given by Barban, Linnik and Tshudakov [5],

\[
P_{\min} (q, r) \ll q^{9/12},
\]

where \( q = p^n, (n = 1,2,3,\ldots) \).

If we focus our attention on the least prime in arithmetic progressions with special moduli, we can prove the following result.

**Theorem 1.2.** Let \( a \) be an integer, \( a \geq 2 \). If \( A > 0 \), then there is a \( B = B(A) > 0 \) such that

\[
\sum_{d \leq x^{\delta}q^{1/2} \leq n \atop (d,q)=1} \max_{y \leq x} \max_{(y,qd)=1} \left| \pi \left( x; qd, r \right) - \frac{L_i(y)}{\phi(qd)} \right| \ll \frac{x}{\phi(q)L^4},
\]

holds uniformly for moduli \( q \leq x^{3} \exp \left( -\log \log x \right)^3 \) that are powers of \( a \).

Then our result shows that the least prime \( P_{\min} (q, r) \) in these special progressions \( n \equiv r \pmod{q} \) satisfies

\[
P_{\min} (q, r) \ll q^{12/5}. \]

It should be remarked that the Generalized Riemann Hypothesis for Dirichlet L-functions would allow \( qd \leq x^4L^{1-1} \) with no further restriction upon the nature of \( q \). Therefore our Theorems 1.1 and 1.2 can be compared with the result under the Generalized Riemann Hypothesis.

### 2. Preliminary Reduction

Let \( \Lambda(n) \) denote von Mangoldt’s function, and for mutually prime integers \( w \) and \( r \), let

\[
\psi(y;w,r) = \sum_{n \equiv r \pmod{w}} \Lambda(n).
\]

For \( 2 \leq w \leq x^{3/4} \) and an integer \( q \geq 1 \), define

\[
G(w) = \sum_{d \leq w} \max_{(y,qd)=1} \left| \psi \left( y; qd, r \right) - \frac{1}{\phi(d)} \psi \left( y; q, r \right) \right|.
\]

Then

**Lemma 2.1.** For any \( K > 0, 1/4 < \delta \leq 1/2 \), we have

\[
G \left( x^\delta q^{-1} L^{-K} \right) \ll \left( \exp \left( \frac{1}{2} \left( \log \log x \right)^3 \right) \right) \log x
\]

\[
+ \tau(q) x^{\delta} \left( \log x \right)^6 - K.
\]

uniformly for positive integers \( q \leq x^\delta \exp \left( -\left( \log \log x \right)^3 \right), x \geq 3 \) where \( \theta = 2/5 \), if \( 9/20 < \delta \leq 1/2 \) and \( \theta = 5/12 \), if \( 1/4 < \delta \leq 9/20 \). Here \( \tau(q) = \sum_{\nu|q} 1 \).

For Dirichlet characters \( \chi \) and real \( y > 0 \) define

\[
\psi(y, \chi) = \sum_{n \equiv r \pmod{q}} \chi(n) \Lambda(n)
\]

**Lemma 2.2.** Let \( \psi(y, \chi) \) defined as in (2). Then

\[
\sum_{d \leq Q \pmod{D}} \max_{y \leq x} \left| \psi(y, \chi) \right| \ll \left( x + x^\delta Q^2 D + x^{4/5} QD^{1/2} \right) L^4,
\]

holds uniformly for all integers \( D \geq 1 \) and real numbers \( x \geq 2, Q \geq 1 \).

**Lemma 2.3.** Let \( \psi(y, \chi) \) defined as in (2). Then

\[
\sum_{d \leq Q \pmod{D}} \max_{y \leq x} \left| \psi(y, \chi) \right| \ll \left( x + x^{1/20} Q^2 D \right) L^4.
\]

holds uniformly for all integers \( D \geq 1 \) and real numbers \( x \geq 2, Q \geq 1 \). Here the inner sum is taken over all primitive Dirichlet characters \( \pmod{D} \).

### 3. Proof of Lemma 2.2

Let

\[
X^{2/3} < Y \leq X
\]
and $M_1, \ldots, M_{10}$ be positive real numbers such that
\[ Y \leq M_1 \cdots M_{10} \leq X \quad \text{and} \quad 2M_6, \ldots, 2M_{10} \leq X^{\frac{1}{2}}. \tag{5} \]

For $j = 1, \ldots, 10$ define
\[ a_j(m) = \begin{cases} \log m, & \text{if } j = 1, \\ 1, & \text{if } j = 2, \ldots, 5, \\ \mu(m), & \text{if } j = 6, \ldots, 10, \end{cases} \tag{6} \]
where $\mu(n)$ is the Möbius function. Then we define the functions
\[ f_j(s, \chi) = \sum_{m=1} a_j(m) \chi(m) m^{-s}, \]
and
\[ F(s, \chi) = f_1(s, \chi) \cdots f_{10}(s, \chi), \tag{7} \]
where $\chi$ is a Dirichlet character, $s$ a complex variable.

**Lemma 3.1.** Let $F(s, \chi)$ be as in (7), and $A \geq 1$ arbitrary. Then for any $1 \leq R \leq X^{1/4}$ and $0 < T \ll X^A$,
\[
\frac{1}{R} \sum_{r=R^{(mod\,r)}} \left| \sum_{r=R^{(mod\,r)}} \int_{-R/2}^{R/2} \left| F \left( \frac{1}{2} + it, \chi \right) \right| dt \right| \leq \left( \frac{T^2}{2} + \frac{R^2}{d^2} X^{3/2} + X^2 \right) \log^2 X, \tag{8}
\]
where $c > 0$ is an absolute constant independent of $A$, but the constant implied in $\ll$ depends on $A$.

**Proof of Lemma 3.1.** This lemma with $d = 1$ was established in [6], and in this general form [7]. We mention that in general the exponent 3/10 to $X$ in the second term on the right-hand side is the best possible on considering the lack of sixth power mean value of Dirichlet L-functions.

Now we complete the proof of Lemma 2.2.

**Proof of Lemma 2.2.** In (5), we take
\[ Y = \frac{2}{7}, \quad X = x. \]

Define $a_j(m), f_j(s, \chi)$ and $F(s, \chi)$ as above. To go further, we first recall Heath-Brown’s identity [8], which states that for any $n < 2x^{\frac{3}{4}}$ with $z \geq 1$ and $k \geq 1$,
\[ \Lambda(n) = \sum_{j=1}^{k} (1)^{-j} \left( \sum_{n_1 \cdots n_j = n} \mu(n_1) \cdots \mu(n_j) \right) \cdot \left( \sum_{n_{j+1} \cdots n_z} \log(n_{j+1}) \cdots \log(n_z) \right). \]

Then for $2Y = x^{\frac{3}{2}} < y \leq X = x,$
\[ \psi(y, \chi) \quad \text{is a linear combination of } O(L^3) \text{ terms, each of which is of the form} \]
\[ \mathcal{S}(\mathbb{M}) := \sum_{m_1 \cdots m_{10}} a_1(m_1) \chi(m_1) \cdots a_{10}(m_{10}) \chi(m_{10}), \]
where $\mathbb{M}$ denotes the vector $(M_1, M_2, \ldots, M_{10})$ with $M_j$ as in (5). Obviously some of the intervals $\mathbb{M}$ may contain only integer 1. By using Perron’s summation formula with $T = y$ (see Proposition 5.5 in [1]), and then shifting the contour to the left, we have
\[
\mathcal{S}(\mathbb{M}) = \frac{1}{2\pi i} \int_{y=1}^{y=1} F(s, \chi) \frac{y^{s} - (y/2)^{s}}{s} ds + O(L^2)
= \frac{1}{2\pi i} \int_{y=1}^{y=1} \left[ \left(\frac{y}{2}\right)^{s} + \int_{y=1/2}^{y=1} \frac{d}{dx} \right] + O(L^2).
\]

On using the trivial estimate
\[ F(\sigma \pm iy, \chi) \ll |f_1(\sigma \pm iy, \chi)| \cdots |f_{10}(\sigma \pm iy, \chi)| \ll \left( M_1 \cdots M_{10} \cdots M_{10} \right. \ll \chi^{1/4} L, \]
the integral on the two horizontal segments above can be estimated as
\[
\ll \max_{|\sigma| \leq \frac{1}{2} - \frac{1}{z+1/4}} |F(\sigma \pm iy, \chi)| y^{\sigma} - \frac{1}{2} \ll \chi^{1/4} L. \]

Then we have
\[
\mathcal{S}(\mathbb{M}) = \frac{1}{2\pi i} \int_{y=1}^{y=1} F\left( \frac{1}{2} + it, \chi \right) \frac{y^{it} - (y/2)^{it}}{1 + it} dt + O\left( \chi^{10} L \right)
= \left( \frac{1}{2\pi i} \int_{y=1}^{y=1} F\left( \frac{1}{2} + it, \chi \right) \frac{dt}{|t|+1} \right) + \chi^{10} L
\]
Noting that $F(s, \chi)$ does not depend on $Y$, we have
\[ \max_{2Y \leq x \leq x} |\psi(y, \chi)| \ll L^{10} \frac{1}{x} \int_{y=1}^{y=1} \left| F\left( \frac{1}{2} + it, \chi \right) \right| dt + \frac{1}{x} \chi^{10} L. \]

On the other hand we have
\[ \max_{2Y \leq x \leq x} |\psi(y, \chi)| \ll Y. \tag{10} \]

From (9) and (10), we have
\[ \sum_{d \leq Q \chi \left( mod \, D_1 \right)} \sum_{y \leq x} \max_{2Y \leq x \leq x} |\psi(y, \chi)| \ll \sum_{d \leq Q \chi \left( mod \, D_1 \right)} \sum_{y \leq x} \max_{2Y \leq x \leq x} |\psi(y, \chi)| + \sum_{d \leq Q \chi \left( mod \, D_1 \right)} \sum_{y \leq x} \max_{2Y \leq x \leq x} |\psi(y, \chi)|
\ll L^{10} x^{1/2} \sum_{d \leq Q \chi \left( mod \, D_1 \right)} \int_{y=1}^{y=1} \left| F\left( \frac{1}{2} + it, \chi \right) \right| dt + \frac{1}{x} \chi^{10} L. \]

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Further let $q = Dd$ and then we obtain
\[
\sum_{d \equiv Q (\text{mod } D)} \sum_{\gamma \leq T} \max_{y \leq x} \left| \mathcal{V}(y, \gamma) \right| \leq L x^2 \max_{0 \leq \gamma \leq T} \frac{1}{T+1} \left| \sum_{d \equiv Q (\text{mod } D)} \sum_{\gamma \leq T} \int_{-2\pi T}^{2\pi T} F \left( \frac{1}{2} + it, \gamma \right) dt \right| + O^2 D x^2.
\]
From Lemma 3.1, we have
\[
\sum_{d \equiv Q (\text{mod } D)} \sum_{\gamma \leq T} \max_{y \leq x} \left| \mathcal{V}(y, \gamma) \right| \leq L x^2 \max_{0 \leq \gamma \leq T} \frac{1}{T+1} \left| \sum_{d \equiv Q (\text{mod } D)} \sum_{\gamma \leq T} \int_{-2\pi T}^{2\pi T} F \left( \frac{1}{2} + it, \gamma \right) dt \right| + O^2 D x^2
\]
\[
\approx L x^2 \left( \frac{(OD)^2}{D} + \frac{OD}{D^2} (T+1) \frac{1}{2} x^\frac{3}{2} + \frac{1}{2} x^2 (T+1)^{\frac{1}{2}} \right) + O^2 D x^2
\]
\[
\approx \left( x + x^\frac{3}{2} Q D + x^2 Q D^2 \right) L^2.
\]
This completes the proof of Lemma 2.2.

4. Proof of Lemma 2.3

Firstly we recall one result of Choi and Kumchev [9] about mean value of Dirichlet polynomials. Let $m \geq 1, r \geq 1$, and $Q \geq r$, Let $\mathcal{H}(m, r, Q)$ denote the set of character $\chi = \xi \psi$ modulo $mq$, where $\xi$ is a character modulo $m$ and $\psi$ is a primitive character modulo $q$ with $r \leq q \leq Q$, $\eta \| q$ and $(q, m) = 1$. Then the result of Choi and Kumchev states as follows.

**Lemma 4.1.** Let $m \geq 1, r \geq 1, T \geq 2, N \geq 2$, and $\mathcal{H}(m, r, Q)$ be a set of characters as described as above, Then
\[
\sum_{\chi \in \mathcal{H}(m, r, Q)} \int_{-T}^{T} \sum_{N = \chi \equiv \gamma \chi} \Lambda(n) \chi(n) n^{-s} \, dt \ll \left( N + HN^{\frac{11}{20}} \right) L^2,
\]
where $c$ is an absolute constant, $H = mr^{-1}Q^2T$ and $L = \log HN$. Now we complete the proof of Lemma 2.3.

**Proof of Lemma 2.3.** Let $Y = x^2$ and $X = x$. We define
\[
F(s, \gamma) = \sum_{\gamma \in \mathcal{H}(m, r, Q)} \Lambda(n) \chi(n) n^{-s}.
\]
If $y$ satisfies
\[
Y < y \leq X,
\]
we apply Perron’s summation formula with $T = y$ (see Proposition 5.5 in [1]), and then obtain
\[
\psi(y, \gamma) = \frac{1}{2\pi i} \int_{b-\gamma}^{b+\gamma} F(s, \gamma) \frac{y^n}{s} \, ds + O \left( yx^{-1} L^2 \right)
\]
\[
= \frac{1}{2\pi i} \int_{b-\gamma}^{b+\gamma} F(s, \gamma) \frac{y^n}{s} \, ds + O \left( \frac{1}{x^2} L^2 \right),
\]
where $0 < b < L^{-1}$. If we let $b \to 0$, we have
\[
\psi(y, \gamma) \ll \int_{-\gamma}^{\gamma} F(it, \gamma) \frac{1}{|t|^4} \, dt + O \left( yx^{-1} L^2 \right).
\]
Noting that $F(s, \gamma)$ does not depend on $y$, we have
\[
\max_{y \leq x} \left| \psi(y, \gamma) \right| \ll \int_{-\gamma}^{\gamma} F(it, \gamma) \frac{1}{|t|^4} \, dt + O \left( x^2 L^2 \right). \tag{12}
\]
On the other hand we have
\[
\max_{y \geq 2Y} \left| \psi(y, \gamma) \right| \ll Y = x^2. \tag{13}
\]
From (12) and (13), we have
\[
\sum_{d \equiv Q (\text{mod } D)} \sum_{\gamma \leq T} \max_{y \leq x} \left| \psi(y, \gamma) \right| \ll \sum_{d \equiv Q (\text{mod } D)} \sum_{\gamma \leq T} \max_{y \leq x} \left| \psi(y, \gamma) \right| + \sum_{d \equiv Q (\text{mod } D)} \sum_{\gamma \geq 2Y} \max_{y \leq x} \left| \psi(y, \gamma) \right|
\]
\[
\ll \sum_{d \equiv Q (\text{mod } D)} \int_{-\gamma}^{\gamma} F(it, \gamma) \frac{1}{|t|^4} \, dt + O^2 D x^2 L^2.
\]
Further let $q = Dd$ and then we obtain
\[
\sum_{d \equiv Q (\text{mod } D)} \sum_{\gamma \leq T} \max_{y \leq x} \left| \psi(y, \gamma) \right| \ll \max_{0 \leq \gamma \leq T} \frac{1}{T+1} \sum_{d \equiv Q (\text{mod } D)} \sum_{\gamma \leq T} \int_{-2\pi T}^{2\pi T} F(it, \gamma) \, dt + O^2 D x^2 L^2.
\]
**Lemma 4.1 with $m = 1$** gives that
\[
\sum_{d \equiv Q (\text{mod } D)} \sum_{\gamma \leq T} \int_{-\gamma}^{\gamma} F(it, \gamma) \frac{1}{|t|^4} \, dt \ll \left( x + \frac{R^2 T}{D} \frac{11}{30} \right) L^2. \tag{14}
\]
From (14), we have
\[
\sum_{d \equiv Q (\text{mod } D)} \sum_{\gamma \leq T} \max_{y \leq x} \left| \psi(y, \gamma) \right| \ll \max_{0 \leq \gamma \leq T} \frac{1}{T+1} \left( x + \frac{R^2 T}{D} \frac{11}{30} \right) L^2 + O^2 D x^2 L^2
\]
\[
\ll L^2 \left( \frac{(OD)^2}{D} x^{\frac{11}{30}} + x(T+1)^{-1} \right) + O^2 D x^2 L^2
\]
\[
\ll \left( x + x^{\frac{11}{30}} Q D \right) L^2.
\]
This completes the proof of Lemma 2.3.
5. Proof of Lemma 2.1

We partition the moduli $qd$ as $qd, d_2$, where the prime factor of $d_2$ not exceed $L^k$ and those of $d_1$ do not. If $\omega(n)$ denotes the number of distinct prime divisors of the integer $n$, and $t = 2K \log \log x / \log \log \log x$, with estimate $\psi(x; q, r) \ll x/\phi(q)$, for $1/4 < \delta \leq 1/2$, we have

$$
\sum_{d \leq \sqrt{x}} \psi(x; q, d) \ll \frac{x}{\phi(q)} \sum_{d \leq \sqrt{x}} \phi(d) \sum_{d_2 \leq x} \frac{1}{d_2}.
$$

Noting that $\phi(n)$ is not principal. We denote $\phi(d)$ but $\psi(x; q, r) \ll x/\phi(q)$, induce $\psi(x; q, d_2)$ is also $\psi(x; q, d_2) \ll x/\phi(q)$. We collect together those moduli with a fixed $\psi(x; q, d_2)$ is also $\psi(x; q, d_2) \ll x/\phi(q)$. Moreover the corresponding sum, taken over those $d_2$ for which $d_2$ is divisible by the $v^m$ power of some prime, $v \geq 8$, is

$$
\ll \frac{x}{\phi(q)} \sum_{d \leq \sqrt{x}} \phi(d_2) \sum_{p \leq \sqrt{x}} \phi(p^m) \sum_{m \geq \log \log x} \frac{1}{m!}.
$$

Hence $\ll \psi(x; q, d_2) \ll x/\phi(q)$, too.

We denote $\exp(\frac{1}{2} \log \log x)$ by $\Delta$. Arguing similarly for $\phi(d)^{-1} \psi(x; q, r)$, we have

$$
\sum_{d \leq \sqrt{x}} \max_{d \leq \Delta} \max_{y \leq \Delta} \left| \psi(y; q, d, r) - \frac{1}{\phi(d)} \psi(y; q, r) \right| \ll \frac{x}{\phi(q) \Delta^2}.
$$

We collect together those moduli $qd$ with a fixed value of $d_1$ not exceeding $\Delta$ and set $D = qd_1$. Noting that $\psi(y; q, r) = \sum_{n \leq y} \Lambda(n) + O(\log y d_2)$, we see from the orthogonality of Dirichlet characters that

$$
\psi(y; qd_1, r) - \frac{1}{\phi(d_2)} \psi(y; q, r) \ll \frac{1}{\phi(d_2)} \left| \psi(y; qd_1, r) - \psi(y; q, r) \right| \ll \psi(y; qd_2, \Delta) \ll \frac{1}{\phi(q)} \psi(y; q, d_2) \ll x/\phi(q).
$$

For a fixed value of $D(qd_1, \Delta)$, we collect together those terms involving the characters $\chi$ induced by a particular primitive character $\chi^*(\log D, \rho)$, where $D \mid D$ and $\rho \mid d_2$. Since $\chi$ and $\chi^*$ differ on at most the integers $n$ for which $(n, D, \rho) = 1$ but $(n, D_2) > 1$,

$$
\psi(y, \chi) = \psi(y, \chi^*) + O(\log D dy).
$$

Interchanging summations,

$$
\ll \sum_{d_2 \leq L^K} \sum_{v^m \leq D} \max_{y \leq \Delta} \left| \sum_{d \equiv (\log D, \rho) \pmod{D}} \frac{1}{\phi(d)} \psi(y, \chi) \right| \ll \frac{1}{\phi(D_2)} \psi(y, \chi).
$$

Here $\rho \leq L^K D^{-1} x^{1/2}$, and the innermost bounding sum is $\ll \phi(D_2)^{-1} x \log x$. We cover the range of $\rho$ with adjoining intervals $U < \rho < 2U$, subject to $L^K \leq U < L^K D^{-1} x^{1/2}$. When $\delta = 1/2$, by Lemma 2.2 a typical interval contributes

$$
\ll \frac{x}{U + x^2 U D^4 + x^5 D^8} \log x \log \log x.
$$

Since $D^{1/2} \leq D^{1/2} \leq (qd_1)^{1/2} \leq x^{1/5} \exp(-\frac{1}{4} \log \log x)$, the whole sum over $\rho$ is

$$
\ll D^{-1} x (\log x)^{5/2} \log \log x.
$$

Arguing similarly for $\delta = 9/20$, by Lemma 2.3 the whole sum over $\rho$ is also $\ll D^{-1} x (\log x)^{5/2} \log \log x$. Noting that
\[
\sum_{d_1 \leq A} \frac{\tau(D)}{D} \ll \tau(q) q^{-1} \sum_{d_1 \leq A} \frac{\tau(d_1)}{d_1} \\
= \tau(q) q^{-1} \prod_{p \leq A} \left(1 + \frac{\tau(p) + \tau(p^2)}{p^2} + \cdots \right) \\
= \tau(q) q^{-1} \prod_{p \leq A} \left(1 - \frac{1}{p}\right)^{-2} \ll \tau(q) q^{-1} (\log \Delta)^2 \\
= \tau(q) q^{-1} (\log \log q)^\xi.
\]

This completes the proof of Lemma 2.1.

6. Zeros of Dirichlet L-Functions

**Lemma 4.1.** Let \( L(s, \chi), s = \sigma + it, \) denote an L-function formed with a Dirichlet character \( \chi \mod q, q \geq 3, h = \prod p^k \). With \( \ell = \log(q(\ell + 3)), \)

\[
\theta^{-1} = 4.10^{-4} \left( \log h + (\log 2) \right)^{3/4}. 
\]

Then there can be at most one non-principal character \( \chi \mod q \) for which the corresponding L-function has a zero in the region \( \sigma > 1 - \theta \). Moreover such a character would be real and the zero would be real and simple.

**Proof of Lemma 4.1.** This is Theorem 2 of Iwaniec [10].

**Lemma 4.2.** Let \( \chi_i \mod D_i, i = 1, 2 \) be distinct primitive real characters. There is a positive real \( c_1 \) so that at most one of the functions \( L(s, \chi) \) formed with these characters can vanish on the line segment

\[
1 - c_1 (\log D_1 D_2)^{-1} \leq \sigma \leq 1, \quad t = 0.
\]

**Proof of Lemma 4.2.** This is result of Landau, which can be found at Satz 6.4. p. 127, of Prachar [11].

**Lemma 4.3.** For any modulus \( D, 0 < \alpha \leq 1, T \geq 0, \) let \( N(\alpha, T, D) \) denote the number of zeros, counted with multiplicity, of all functions \( L(s, \chi) \) formed with a character \( \chi \mod D \), that lie in the rectangle \( \alpha \leq \text{Re} s \leq 1, |\text{Im} s| \leq T \). Then we have

\[
N(\alpha, T, D) \ll (DT)^{4(1-\alpha)},
\]

uniformly for \( 0 \leq \alpha \leq 1, T \geq 2 \).

**Proof of Lemma 4.3.** This is Theorem of Heath-Brown [12], on p. 249.

7. Proof of Theorems 1.1 and 1.2

We shall first provide a version of the theorem with \( \psi(y; q, d, r) \) in place of \( \pi(y; q, d, r) \). After Lemma 2.1 it will suffice to establish the bound

\[
G(\Delta) \ll x \left( \phi(q)(\log x)^\xi \right)^{-1},
\]

for any fixed positive \( A \).

We employ the representation

\[
\sum_{n \leq y} \psi^{*}(n) \Phi(n) = E_{y} \gamma - \sum_{\rho \in \xi} \frac{\gamma^{\rho}}{\rho} + O \left( \frac{(\log Dy)^{2}}{T} + \frac{y^{1/4} \log Dy}{T} \right),
\]

valid for all characters \( \chi \mod D \), where \( y \geq T \geq 2 \); \( E_{y} \) is 1 if \( \chi \) is principal, zero otherwise; \( \rho = \beta + iy \) runs through all the zeros of \( L(s, \chi) \) in the rectangle \( 0 \leq \text{Re} s < 1, |\text{Im} s| \leq T \) with a half disc \( |1 - \beta| < c_{1}(\log D)^{-1} > 0, \text{Re} s \geq 0 \) removed. This representation is a slightly modified version of that given in Satz 4.6, pp. 232-234 of Prachar [11].

Since \( L(s, \chi) \) has \( \ll \log DT \) zeros in the strip \( 0 \leq \text{Re} s < 1, T \leq |\text{Im} s| \leq T + 1 \), cf. Prachar [11], Satz 3.3, p. 220,

\[
\sum_{\rho \in \xi} \frac{\gamma^{\rho}}{\rho} \ll y^{1/2} \left( \log 2D + \sum_{m \leq T} \sum_{m < \text{Im} z \leq m + 1} \frac{1}{|\rho|} \right) \\
\ll y^{1/2} (\log Dy) \sum_{m \leq T} \frac{1}{m-1} \ll y^{1/2} (\log Dy)^{2},
\]

and at the expense of raising \( y^{1/4} \log Dy \) to \( y^{1/2} (\log Dy)^{2} \) we may confine the zeros \( \rho \) to the half-plane \( \text{Re} s > 1/2 \).

From the orthogonality of Dirichlet characters

\[
\phi(D)\psi(y; D, r) = \gamma \\
= \sum_{\chi \mod D} \overline{\chi}(r) \sum_{n \leq y} \chi(n) \Lambda(n) - E_{y} \gamma \\
\ll \sum_{\chi \mod D} \sum_{\nu \neq \rho} \frac{\gamma^{\nu}}{\nu} + \left( \frac{y}{T} + y^{1/2} \right) (\log Dy)^{2},
\]

where it is understood that the \( \rho (= \beta + iy) \) are the zeros of the L-function formed with the character \( \chi \) of the outer summation.

We replace \( \gamma \) by \( z \) and average over the interval \( y \leq z \leq y + w \) with \( w = y(\log y)^{-2/3} \) to obtain

\[
\frac{1}{w} \int_{y}^{y+w} (\phi(D)\psi(z; D, r) - z) \, dz \\
\ll \frac{w}{y} \sum_{\chi \mod D} \sum_{n \leq y} \frac{\gamma^{\nu}}{\nu} \left( \frac{y}{T} + y^{1/2} \right) (\log Dy)^{2}.
\]

Replacing \( z \) in the integrand by \( y \) introduces an error of

\[
\ll w + \phi(D) \sum_{\chi \mod D} \log y \ll w + \phi(D) \left( \frac{w}{D} + 1 \right) \log y,
\]

and we may remove the integral averaging:
\[ \phi(D)\psi(y;D,r) - y \ll (\log y)^{4\beta} \sum_{\xi} \sum_{|\beta|^2} \frac{y^\beta}{|\beta|^2} \]

\[ + \left( \frac{y}{T} + \frac{y}{\sqrt{2}} \right) D(\log y)^2 + \frac{y}{(\log y)^{\frac{1}{12}}} \]

This bound will be satisfactory for \( y > x(\log x)^{-\frac{1}{2}} \).

Otherwise, we shall employ the crude bound
\[ \psi(y;D,r) - \frac{y}{\phi(D)} \ll \frac{x(x(\log x)^{-\frac{1}{2}}}{D} + 1, \]

which is valid for all positive \( y \). With these bounds
\[ R_D = \max_{y \leq x} \psi(y;D,r) - \frac{y}{\phi(D)} \ll \sum_{\xi} \sum_{\beta_{12} / \gamma \in \mathbb{Z}} \frac{x^\beta (\log x)^{1/2}}{\beta + iy} \]

\[ + \left( \frac{x}{T} + \frac{x}{\sqrt{2}} \right) D(\log x)^2 + \frac{x}{(\log x)^{1/12}} \]

holds uniformly for \( 2 \leq T \leq x(\log x)^{-\frac{1}{2}}, D \leq x^{3/4} \). We set \( T = x^{1/2} \).

The double-sum does not exceed
\[ 4 \sum_{1 \leq \delta x T} 2^{-\delta x} \sum_{\xi} \sum_{\beta_{12} / \gamma \in \mathbb{Z}} \frac{x^\beta}{\beta + iy} \]
\[ = -4 \sum_{1 \leq \delta x T} 2^{-\delta x} \int_{y/2}^{1/\delta} x^\beta dN(u, \tau, D) \]

where \( 1-\theta \) is the largest value of \( \beta \) taken over all the zeros \( \beta + iy \) in the rectangle
\[ 0 < Re(s) < 1, \lim(s) \leq 2T. \]

Supposing for the moment that \( D = qd \) and that there is no zero that is exceptional in the sense of Lemma 4.1, then we may take
\[ \theta = c \left( \log d + (\log 2q(T + 3) \log 2q(T + 3) \right)^{1/2} \]

In view of Lemma 4.3, typically
\[ \int_{y/2}^{1/\delta} x^\beta dN(u, \tau, D) \]
\[ = -x^\beta N(u, \tau, D) \int_{y/2}^{1/\delta} N(u, \tau, D) x^\beta \log x \, dx \]

\[ \ll x^{1/2} N(1/2, \tau, D) + c \int_{y/2}^{1/\delta} N(u, \tau, D) x^\beta \log x \, dx \]

with restriction \( q \leq x^{5/12} \Delta^{-2} \), we have
\[ D^{12/5} \leq x(\log x)^{1/2}, \]

then the integral is
\[ \ll x(\log x)^{-1/2} \] uniformly for \( r \leq 2T \) and \( d \leq \Delta \). Moreover, \( N(1/2, \tau, D) \ll D(\tau + 2) \log D(\tau + 2) \), Prachar [11], Satz 3.3, p. 220, as earlier. Altogether
\[ R_{qd} \ll x(\log x)^{-3/12} \]

with the same uniformity in \( d \).

If there is an exceptional zero \( (mod qd) \), for which \( \beta > 1 - C_1 (2 \log 4a\Delta)^{-1} \), and the corresponding function \( L(s, \chi) \) is attached to a real character induced by a primitive character \( \chi' (mod D') \), then \( D' \) is a divisor of some \( 4ad \) with \( d \leq \Delta \), and an application of Lemma 4.2 shows that there is no further L-function formed with a real character \( (mod D), D \leq 4a\Delta \), that has a real zero on the line-segment
\[ 1 - C_1 (2 \log 4a\Delta)^{-1} \leq \Re(s) < 1, \Im(s) = 0 \]

unless that character is also induced by \( \chi' (mod D') \). In particular, \( D \) will be divisible by \( D' \). For those moduli \( qd \) for which \( 4ad \) is not a multiple of \( D' \) we may choose the same \( \theta \) as before and recover the above estimate for \( R_{qd} \).

Hence
\[ \sum_{d \leq \Delta} \max_{y \leq x} \left\| \psi(y;qd,r) - \frac{y}{\phi(qd)} \right\| \ll \sum_{d \leq \Delta} \frac{1}{\phi(qd)} \ll \frac{x}{(\log x)^{1/12}}, \]

where " indicates that the moduli are not divisible by the (possibly non-existent) modulus \( D' \).

A theorem of Siegel shows that for any \( \epsilon > 0 \) there is a positive constant \( c(\epsilon) \) so that an L-function formed with a real character \( (mod D) \) has no zero on the line-segment
\[ 1 - c(\epsilon) D^{-\epsilon} \leq \Re(s) < 1, \Im(s) = 0 \; \text{cf. Prachar [11], Satz 8.2, p.144. Unless} \]

\[ D' \geq c(\epsilon)(\log x)^{1/2} \]

this again allows the argument to proceed. We may therefore assume that \( D' > (\log x)^{1/2} \) and remove the restriction " from the above summation over \( d \) at an expense of
\[ \ll \sum_{4ad = \theta (mod D')} \frac{x \log x}{qd} \ll \frac{x(\log x)^{1/2}}{q D'} \ll \frac{x}{q(\log x)^{1/2}} \]

A modified version of this argument delivers the bound
\[ \max_{y \leq x} \left\| \psi(y;qd,r) - \frac{y}{\phi(q)} \right\| \ll \frac{x}{\phi(q)(\log x)^{3/12}} \]

and in this case there is no exceptional zero.

By subtraction we see that
\[ G(\Delta) \ll x(\phi(q)(\log x)^{1/2})^{-1} \]

indeed holds for every fixed \( A > 0 \).

Since \( \tau(q) \ll \log q \), an application of Lemma 2.1 shows that with \( B = A + 6 \),

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\[
\sum_{qd \leq x^\delta} \max_{y \leq x} \left| \psi(y; qd, r) - \frac{y}{\phi(qd)} \right| \leq \frac{x}{\phi(q)} \left( \log x \right)^{\frac{\delta}{1 - \delta - \epsilon}},
\]

uniformly for moduli \( q \leq x^\delta \exp\left(-\left(\log \log x\right)^3\right) \) that are powers of \( a \) where \( \theta = 2/5 \), if \( \delta = 1/2 \) and \( \theta = 5/12 \), if \( \delta = 9/20 \).

Replacing \( \psi(y; qd, r) \) in this bound by
\[
\theta(y; qd, r) = \sum_{\text{prime } p \leq x(qd)} \log p
\]
introduces an error
\[
\ll \sum_{qd \leq x^{\frac{2}{3}}} \sum_{\text{prime } p \leq x} \log p \leq x^{\frac{2}{3}} \ll xq^{-1} \left( \log x \right)^{-\delta - 6},
\]
the congruence condition \( p^m \equiv r \pmod{qd} \) having been ignored.

Employing the Brun-Titchmarsh bound
\[
\pi(y; D, r) \ll y \left( \phi(D) \log y \right)^{-1},
\]
valid uniformly for \( 1 \leq D \leq y^{1/4}, (r, D) = 1 \). We see that the contribution to the sum in the theorem that arises from maxima that occur in the range \( 0 < y \leq y_0 = x \left( \log x \right)^{-2} \) is
\[
\ll \sum_{d \leq x^{\frac{2}{3}}} y_0 \left( \phi(qd) \log y_0 \right)^{-1} \ll y_0 \phi(q)^{-1}
\]
\[
\ll x \left( \phi(q)(\log x)^{-1} \right)^{-1}.
\]

We may therefore confine our attention to maxima over the range \( y_0 \leq y \leq x \).

Integration by parts shows that
\[
\max_{y_0 \leq y \leq x} \left| \psi(y; D, r) - \frac{y}{\phi(D)} \right| \ll \left| \pi(y_0; D, r) - \frac{Li(y)}{\phi(D)} \right| + \frac{1}{\log x} \max_{y_0 \leq y \leq x} \left| \theta(y; D, r) - \frac{y}{\phi(D)} \right|.
\]

The theorems hold with \( B = A + 6 \).

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REFERENCES