The Integrals of Entwining Structure

Yuzhuo Yuan\textsuperscript{1,2}, Lihong Dong\textsuperscript{1}, Zhengming Jiao\textsuperscript{*}

\textsuperscript{1}Department of Mathematics, Henan Normal University, Xinxiang, China
\textsuperscript{2}Department of Mathematics, Nanyang Normal University, Nanyang, China
Email: *zmjiao@htu.cn

Received January 31, 2013; revised March 18, 2013; accepted April 27, 2013

Copyright © 2013 Yuzhuo Yuan et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

ABSTRACT

In this paper the integrals of entwining structure $(A, C, \psi)$ are discussed, where $A$ is a $k$-algebra, a $k$-coalgebra and $C$ a $k$-linear map. We prove that there exists a normalized integral $\gamma : C \to \text{Hom}(C, A)$ of $(A, C, \psi)$ if and only if any representation of $(A, C, \psi)$ is injective in a functorial way as a corepresentation of $C$. We give the dual results as well.

Keywords: Entwining Structure; Integral; Forgetful Functor; Natural Transformation

1. Introduction

The integrals for Hopf algebras were introduced by Larson and Sweedler [1,2]. Integrals have proven to be essential instruments in constructing invariants of surgically presented 3-manifolds or 3-dimensional topological quantum field theories [3-5]. The aim of this paper is to show that some results of recent paper [6] concerning integrals and its properties for Doi-Koppinen structure hold for the more general concept known as entwining structure [7,8]. It is a structure of an algebra, a coalgebra and a $k$-linear map such that several compatibility conditions are satisfied. Unlike Doi-Koppinen structure, there is no need for a background bialgebra, which is an indispensable part of the Doi-Koppinen construction. The bialgebra-free formulation also has a remarkable self-duality property, which essentially implies that for every statement involving coalgebra structure of an entwining structure there is a corresponding statement involving its algebra structure.

This paper is organized as follows. In Section 2, we recall definitions and give examples of entwining structures and entwined modules. In Section 3, we introduce the integrals of entwining structure and analyse its properties generalizing the results of [6]. Finally, in Section 4 we derive the dual form of the integrals of entwining structure and its properties.

2. Preliminaries

Throughout this paper, $k$ will be a field. Unless specified otherwise, all modules, algebras, coalgebras, bialgebras, tensor products and homomorphisms are over $k$. For a $k$-algebra $A$, $\mathcal{M}_A$ (resp. $\mathcal{M}_C$) will be the category of right(resp. left) $A$-modules and $A$-linear maps. $H$ will be a Hopf algebra over $k$. We omit Sweedler’s sigma-notion [9] extensively. For example, if $(C, \Delta)$ is a coalgebra, then for all $c \in C$ we write $\Delta(c) = c^{(1)} \otimes c^{(2)}$.

**Definition 2.1** An entwining structure on $k$ consists of a triple $(A, C, \psi)$, where $A$ is a $k$-algebra, $C$ a $k$-coalgebra and $\psi : C \otimes A \to A \otimes C$ a $k$-linear map satisfying the relations

\[
(ab) \otimes e^\nu = a_b b_{\psi} \otimes e^{\nu},
\]

\[
(1_A) \otimes e^\nu = 1_C \otimes c,
\]

\[
a_{\psi} \otimes \Delta_C(e^\nu) = a^{(1)}_{\psi} \otimes c^{(1)} \otimes c^{(2)},
\]

\[
e_C(e^\nu) a_{\psi} = e_C(c) a
\]

for all $a, b \in A, c \in C$, where

\[
\psi(c \otimes a) = a_{\psi} \otimes e^\nu = a_{\psi} \otimes e^\nu.
\]

**Remark 2.2** Generally, we call the entwining structure in Definition 2.1 a right-right entwining structure. Unless specified otherwise, all the entwining structures mentioned in this paper are right-right entwining structures.

**Definition 2.3** Let $(A, C, \psi)$ and $(\bar{A}, \bar{C}, \bar{\psi})$ be two entwining structures, $f : A \to \bar{A}$ be an algebra map and $g : C \to \bar{C}$ be a coalgebra map. We call...
\[(f, g): (A, C, \psi) \rightarrow (\tilde{A}, \tilde{C}, \tilde{\psi})\] is an entwining map if \((f \otimes g) \circ \psi = \psi \circ (g \otimes f)\).

**Example 2.4** Let \(H\) be a bialgebra, \(A\) a right \(H\)-comodule algebra, \(C\) a right \(H\)-module coalgebra. Then Doi-Koppinen structure \((H, A, C)\) [10] is an entwining structure. The entwining structure map is
\[\psi: C \otimes A \rightarrow A \otimes C, c \otimes a \rightarrow a_{[0]} \otimes c \cdot a_{[1]}\]
If \(H\) has a bijective antipode \(S\), the \(\psi\) is bijective with
\[\psi^{-1}: a \otimes c \rightarrow c \cdot S^{-1}(a_{[0]}) \otimes a_{[1]}\]

**Example 2.5** [3] Let \(C\) be a coalgebra, \(A\) an algebra and \(B\) a \(C\)-comodule. Let
\[B = \{b \in A | \forall a \in A, \rho_{A}(ba) = b \rho_{A}(a)\}\]
and assume that the canonical right \(A\)-module, right \(C\)-comodule map
\[can: A \otimes_B A \rightarrow A \otimes C, a \otimes a' \rightarrow a \rho_{A}(a')\]
is bijective. Let \(\psi: C \otimes A \rightarrow A \otimes C\) be a \(k\)-linear map given by
\[\psi(c \otimes a) = can^{-1}(1 \otimes c) \otimes a\]
Then \((A, C, \psi)\) is an entwining structure. The extension \(B \subset A\) is called a co-algebra-Galois extension and denoted by \(A(B)^C\). \((A, C, \psi)\) is the canonical entwining structure associated to \(A(B)^C\).

**Lemma 2.6** Let \((A, C, \psi)\) be a right-right entwining structure. where \(\psi\) is invertible, its inverse is \(\varphi: A \otimes C \rightarrow C \otimes A\), then \((A, C, \varphi)\) is a left-left entwining structure, i.e.
\[(ab)_{\varphi} \otimes c^\varphi = a_{\varphi} b_{\varphi} \otimes c^\varphi,\]
\[(1_a)_{\varphi} \otimes c^\varphi = 1_a \otimes c,\]
\[a_{\varphi} \otimes 1_c = a_{\varphi} \otimes c^{\varphi}(c^\varphi),\]
\[c^\varphi a_{\varphi} = c_{\varphi} a,\]
for all \(a, b \in A, c \in C\), where
\[\varphi(c \otimes a) = a_{\varphi} c^\varphi = a_{\varphi} \otimes c^\varphi.\]

**Definition 2.7** Let \((A, C, \psi)\) be an entwining structure. An \((A, C, \psi)\)-entwined module is a \(k\)-module \(M\) with a right \(A\)-action and a right \(C\)-coaction such that for all \(a \in A, m \in M\),
\[\rho_{M}(ma) = m_{[0]} \otimes a_{\varphi} \otimes m_{[1]}^{\varphi}.\]
A module morphism of entwining structure \((A, C, \psi)\) is a right \(A\)-module map and a right \(C\)-comodule map. Generally, we denote the module category of \((A, C, \psi)\) by \(M(\psi)^C_A\). Modules associated to the entwining structure in Example 2.4 are Doi-Hopf module. But entwined modules associated to the entwining structure in Example 2.5 are not of Doi-Hopf type.

Let \(\mathcal{F}_M: \mathcal{M}(\psi)^C_A \rightarrow \mathcal{M}_A\) be the forgetful functor which forgets the \(C\)-coaction and
\[\cdot \otimes C: M_A \rightarrow M(\psi)^C_A, M \rightarrow M \otimes C\]
its right adjoint, where the structure maps on \(M \otimes C\) are given by
\[(m \otimes c) \cdot a = ma_{\varphi} \otimes c^\varphi,\]
\[\rho_{M(\psi)^C_A}(m \otimes c) = m \otimes c_{(1)} \otimes c_{(2)}^{\varphi},\]
for any \(a \in A, c \in C, m \in M\). The unit of the adjoint pair \((\mathcal{F}_M, \cdot \otimes C)\) is
\[\rho: 1_{\mathcal{M}(\psi)^C_A} \rightarrow (\cdot \otimes C) \cdot \mathcal{F}_M\]
the \(C\)-coaction \(\rho_M: M \rightarrow M \otimes C\), therefor \(\rho_M\) is \(A\)-linear and \(C\)-colinear and can be viewed as a natural transformation between the functor \(1_{\mathcal{M}(\psi)^C_A}\) and \((\cdot \otimes C) \cdot \mathcal{F}_M\).

3. The Integrals of the Entwining Structure
In this section, we first present a point of view which is
Lemma 3.1 [6] Let $H$ be a finite dimension Hopf algebra over $k$, $H^*$ is its dual. There exists a right integral $\gamma \in H^*$ on $H$ such that $\gamma h = \langle h, 1_R \rangle$ for all $h \in H^*$ if and only if $\gamma : H \to k$ is right $H$-comodule, where $k$ has the trivial right $H$-co-module structure. □

Doi [11] generalizes this result as follows.

Definition 3.2 Let $A$ be a $H$-comodule algebra. A map $\gamma : H \to A$ is called an integral if $\gamma$ is right $H$-linear. $\gamma$ is called a total integral if additionally $\gamma(1_R) = 1_A$.

The criterion for the existence of a total integral is given by the theorem following.

Theorem 3.3 [11] Let $A$ be a right $H$-comodule algebra. The following are equivalent

1) There exists a total integral $\gamma : H \to A$;
2) Any Hopf module $M \in \mathcal{M}^H$ injective as right $H$-comodule, i.e., the right $H$-coaction $\rho : M \to M \otimes H$ splits in the category $\mathcal{M}^H$;
3) $\rho_\gamma : A \to A \otimes H$ splits in the category $\mathcal{M}^H$.

The theorem of Doi can be restated as follows.

Theorem 3.4 [6] Let $A$ be a right $H$-comodule algebra. The following are equivalent

1) There exists a total integral $\gamma : H \to A$;
2) There exists a natural transformation $\lambda : F_A \circ (\otimes H) \circ F^H \to F_A \circ 1_{\mathcal{M}^H}$ that splits
$$\rho : F_A \circ 1_{\mathcal{M}^H} \to F_A \circ (\otimes H) \circ F^H;$$
3) $\rho_\gamma : A \to A \otimes H$ splits in the category $\mathcal{M}^H$.

Remark 3.5 1) The above theorem is still valid leaving aside the normalizing condition $\gamma(1_R) = 1_A$.

More exactly, there exists an integral $\gamma : H \to A$ if and only if there exists $\lambda : F_A \circ (\otimes H) \circ F^H \to F_A \circ 1_{\mathcal{M}^H}$. In particular, if $A = k$, we obtain that there exists a right integral $\gamma : H \to k$ on $H$ if and only if there exists a natural transformation $\lambda : (\otimes H) \circ F^H \to 1_{\mathcal{M}^H}$. Furthermore, $\gamma(1_H) = 1_k$ if and only if $\lambda$ splits $\rho : 1_{\mathcal{M}^H} \to (\otimes H) \circ F^H$. This is equivalent to the fact that the forgetful functor $F^H : \mathcal{M}^H \to \mathcal{M}_k$ is separable.

2) Let $A$ be a right $H$-comodule algebra. The version of Theorem 3.4 for the category $\mathcal{M}^H$ is still true. In this case the $H$-co-linear split of $\rho_M : M \to M \otimes H$ associated to a right total integral $\gamma : H \to A$ is given by the formula
$$\lambda_M : M \otimes H \to M,$$
$$\lambda_M(m \otimes h) = m_0 \gamma(S(m_0)h).$$

We will now give the definition of integral of entwine-
(śu ◦ f)(m)
= u\left(f(m)_{(0)}\right) γ\left(f(m)_{(1)}\right)u\left(f(m)_{(0)}\right)_{(0)}
= m_{(0)} γ\left(m_{(1)}\right)_{(0)}
= m_{(0)} γ\left(m_{(1)}\right)_{(0)} = m.

hence $u$ is a right $C$-colinear retraction of $f$.

On the other hand, if $u: N \to M$ is a section of $f$, then for $n \in N$
\( (f \circ u)(n) = f\left(u(n_{(0)})\right) γ\left(u(n_{(1)})\right) = f\left(u(n_{(0)})\right) γ\left(u(n_{(1)})\right) = n. \)

i.e. $u$ is a right $C$-colinear section of $f$. □

**Definition 3.9** [11] A right $C$-comodule $M$ is called injective, if for any $k$-split monomorphism $i: U \to V$ in $M^C$, and for any $C$-colinear map $f : U \to M$, there exists a $C$-colinear map $g : V \to M$ such that $g \circ i = f$.

**Lemma 3.10** [11] A right $C$-comodule $M$ is injective, if $ρ_M : M \to M \otimes C$ splits in $M^C$, i.e. there exists a $C$-colinear map $λ_M : M \otimes C \to M$ such that $λ_M \circ ρ_M = id_M$. □

We will prove now the version of Theorem 3.4 for entwining modules which have inverse entwining structure map. Parts of the following theorem are closely related to the ideas presented in [6].

**Theorem 3.11** Let $(A, C, ψ)$ be an entwining structure, where $ψ$ is invertible, its inverse is $ϕ$. The following statements are equivalent:

1) There exists a normalized integral
\( γ : C \to \text{Hom}(C, A); \)

2) The natural transformation
\( ρ : F_g \circ 1_{M(ψ)_{(-1)}} \to F_g \circ (\otimes C) \circ F^C \) splits;

3) The right $C$-coaction on $A \otimes C$,
\( ρ^C_{\otimes C} : A \otimes C \to A \otimes C \otimes C \) splits in $\hat{C} M^C$.

Consequently, if one of the equivalent conditions holds, any entwining module is injective as a right $C$-comodule.

**Proof.** (1) ⇒ (2) Let $γ : C \to \text{Hom}(C, A)$ be a normalized integral. We have to construct a natural transformation $λ$ that splits $ρ$. Let $M \in M(ψ)_{(-1)}$ and $u_M : M \otimes C \to M$, $u_M (m \otimes c) = c (m)$ be the $k$-linear retraction of $ρ_M : M \to M \otimes C$ given by $u_M (m \otimes c) = c (m)$ for all $c \in C, m \in M$. We define $λ_M = u_M$, i.e. for all $c \in C, m \in M$

\( λ_M (m \otimes c) = m_{(0)} γ (c) (m_{(1)}). \)

It follows from Proposition 3.8 that the map $λ_M$ is a right $C$-colinear retraction of $ρ_M$.

It remains to prove that $λ = λ(g)$ is a natural transformation. Let $f : M \to N$ be a morphism in $M(ψ)^C_A$. We have to prove that

\( f \circ λ_M = λ_N \circ (f \otimes id_A). \)

Since $f$ is right $A$-linear, we have

\( (f \circ λ_M)(m \otimes c) = f (m_{(0)} γ (c) (m_{(1)}) \)
\( = f (m_{(0)} γ (c) (m_{(1)}) = f (m_{(0)} γ (c) (m_{(1)}), \)
\( λ_N (f \otimes id_A) (m \otimes c) = λ_N (f \otimes id_A) (m \otimes c) \)
\( = f (m_{(0)} γ (c) (m_{(1)}). \)

i.e. $λ$ is a natural transformation that splits $ρ$.

(2) ⇒ (3) Assume that for any $M \in M(ψ)_{(-1)}$, the $C$-coaction splits in $M^C$. In particular,
\( ρ^C_{\otimes C} : A \otimes C \to A \otimes C \otimes C \) splits in $M^C$. Let $λ = λ_{ψ_C} : A \otimes C \otimes C \to A \otimes C$ be a right $C$-colinear retraction of $ρ^C_{\otimes C}$. Using the naturality of $λ_C$, we will prove that $λ_C$ is also left $C$-colinear, where $A \otimes C$ and $A \otimes C \otimes C$ are left $C$-comodules via:

\( ρ^C (a \otimes c) = c^{(1)} \otimes a_{(0)} \otimes c^{(2)} \)
\( ρ^C (a \otimes c \otimes d) = c^{(1)} \otimes a_{(0)} \otimes c^{(2)} \otimes d \).

Let $V$ be a $k$-module and $M \in M(ψ)_{(-1)}$. Then $V \otimes M \in M(ψ)^C_A$ via the structures arising from the ones of $M$ as follows
\( (v \otimes m) = v \otimes ma, \quad ρ_{V \otimes M} = id_V \otimes ρ_M. \)

Using the naturality of $λ$, we shall prove that
\( λ_{V \otimes M} = id_V \otimes λ_M. \)

Let $v \in V$ and $g_M : M \to V \otimes M, g_M (m) = v \otimes m$. From the naturality of $λ$, we obtain that
\( g_M \circ λ_M = λ_{V \otimes M} \circ (g_M \otimes id_C). \)

Hence
\( λ_{V \otimes M} (v \otimes m \otimes c) \)
\( = g_M \circ λ_M (m \otimes c) = v \otimes λ_M (m \otimes c) \)
\( = (id_V \otimes λ_M) (v \otimes m \otimes c). \)

In particular, let $M = A \otimes C, V = C$, then

Copyright © 2013 SciRes.
\( C \otimes A \otimes C \in \mathcal{M}(\psi)^C \) via the structures arising from the ones of \( A \otimes C \), i.e. for all \( a, b \in A, c, d \in C \),
\[
(c \otimes b \otimes d) \cdot a = c \otimes ba \otimes d^\psi,
\]
\[
\rho^\psi_{C,ABC}(c \otimes b \otimes d) = c \otimes b \otimes d_{(1)} \otimes d_{(2)}.
\]
With these structures the map
\[
f = \rho^\psi_{ABC} : A \otimes C \to C \otimes A \otimes C,
\]
\[
f(a \otimes c) = c^l \otimes a \otimes c_{(2)}
\]
is a morphism in \( \mathcal{M}(\psi)^C \). From the naturality of \( \lambda \), the following diagram is commutative.
\[
\begin{array}{c}
A \otimes C \otimes C \xrightarrow{\lambda_{ABC}} A \otimes C \\
\rho^\psi_{ABC} \otimes id_c \downarrow \quad \downarrow \rho^\psi_{ABC} \\
C \otimes A \otimes C \otimes C \xrightarrow{id_c \otimes \lambda_{ABC}} C \otimes A \otimes C
\end{array}
\]
i.e. \( \lambda = \lambda_{ABC} \) is also left \( C \)-colinear.
\[(3) \Rightarrow (1) \text{ The right } C \text{-coaction}
\rho^\psi_{ABC} : A \otimes C \to A \otimes C \otimes C \text{ is a } C\text{-bicomodule map. Let}
\lambda = \lambda_{ABC} : A \otimes C \otimes C \to A \otimes C \text{ be a split of } \rho^\psi_{ABC} \text{ in } C \mathcal{M}^C. \text{ In particular, for all } a \in A, c \in C,
\lambda \left(a \otimes c_{(1)} \otimes c_{(2)}\right) = a \otimes c.
\]
For all \( c, d \in C \), define
\[
g : C \to \text{Hom}(C, A),
\]
\[
g(c)(d) = (id_A \otimes c) \lambda(1_A \otimes d \otimes c).
\]
We will prove that \( g \) is a normalized integral.
Because the right \( C \)-coaction \( \rho^\psi_{ABC} \) is a \( C \)-bicomodule map, on the one hand
\[
g(c)(d) \otimes c_{(2)} = (id_A \otimes c) \lambda(1_A \otimes d \otimes c) \otimes c_{(2)}
\]
\[
= (id_A \otimes c) \left(1_A \otimes d \otimes c\right) \otimes c_{(2)}
\]
\[
= (id_A \otimes c) \left(\lambda \otimes id_c\right)
\]
\[
= \rho^\psi_{ABC}(1_A \otimes d \otimes c)
\]
\[
= (id_A \otimes c) \rho^\psi_{ABC} \lambda(1_A \otimes d \otimes c)
\]
\[
= \lambda(1_A \otimes d \otimes c);
\]
on the other hand,
\[
g(c)(d_{(2)})_\psi \otimes d^\psi_{(1)}
\]
\[
= \psi(d_{(1)} \otimes g(c)(d_{(2)}))
\]
\[
= \psi(d_{(1)} \otimes (id_A \otimes c) \lambda(1_A \otimes d_{(2)} \otimes c)).
\]
We adopt the temporary notion
\[
\lambda(1_A \otimes d \otimes c) = \sum_i a_i \otimes c_i \in A \otimes C,
\]
then
\[
\rho^\psi_{ABC} \lambda_{ABC}(1_A \otimes d \otimes c)
\]
\[
= \rho^\psi_{ABC} \left(\sum_i a_i \otimes c_i\right) = \sum_i \left(a_i \otimes c_i \otimes c_i\right)
\]
\[
= (id_c \otimes \lambda_{ABC}) \rho^\psi_{ABC}(1_A \otimes d \otimes c)
\]
\[
= (id_c \otimes \lambda_{ABC})(d_{(1)} \otimes 1_A \otimes d_{(2)} \otimes c)
\]
\[
= d_{(1)} \otimes \lambda(1_A \otimes d_{(2)} \otimes c).
\]
For \( \lambda_{ABC} \) is left \( C \)-colinear,
\[
\sum d_{(1)} \otimes \lambda(1_A \otimes d_{(2)} \otimes c) = \sum c_i \otimes a_i \otimes c_i\).
\]
Hence
\[
\sum d_{(1)} \otimes (id_A \otimes c) \lambda(1_A \otimes d_{(2)} \otimes c) = \sum c_i \otimes a_i,
\]
i.e.
\[
\sum \psi(d_{(1)} \otimes (id_A \otimes c) \lambda(1_A \otimes d_{(2)} \otimes c)) = \sum a_i \otimes c_i.
\]
We have proved that \( g \) is a normalized integral of \( (A, C, \psi) \) at last. \( \Box \)

Leaving aside the normalizing condition, we obtain the following corollary.

**Corollary 3.12** Let \( (A, C, \psi) \) be an entwining structure, where \( \psi \) is invertible. The following statements are equivalent

1) There exists an integral \( \gamma : C \to \text{Hom}(C, A) \) of \( (A, C, \psi) \).
2) There exists \( \lambda : F_d \circ (\cdot \otimes C) \circ F^C \to F_d \circ F_{\lambda(\psi)^C} \) a natural transformation;
3) There exists \( \lambda' : A \otimes C \otimes C \to A \otimes C \) a \( C \)-bicomodule map. \( \Box \)

If the entwining structure map is invertible, the object \( A \otimes C \) take an important role in \( \mathcal{M}(\psi)^C \). We shall prove the main application of the existence of a normalized integral in this paper.

**Theorem 3.13** Let \( (A, C, \psi) \) be an entwining structure, where \( \psi \) is invertible, its inverse is \( \phi \). Suppose there exists a normalized integral of \( (A, C, \psi) \) \( \gamma : C \to \text{Hom}(C, A) \), for any \( M \in \mathcal{M}(\psi)^C \), the map
\[
f : M \otimes A \otimes C \to M,
\]
\[
f(m \otimes a \otimes c) = m_{(0)} \gamma\left(c'\right)\left(m_{(1)}\right)a_{(2)}
\]
is a \( \phi \)-split epimorphism in \( \mathcal{M}(\psi)^C \). In particular, \( A \otimes C \) is a generator in the category \( \mathcal{M}(\psi)^C \).

**Proof**\( M \otimes A \otimes C \) is viewed as an object in \( \mathcal{M}(\psi)^C \), with the structures arising from the ones of \( A \otimes C \), i.e.
Taking a $k$-free presentation of $M$ in the category of $k$-modules

$$k^{(i)} \xrightarrow{\phi} M \to 0,$$

we obtain an epimorphism in $\mathcal{M}(\psi)^C_{A}$

$$(A \otimes C)^{(i)} \cong k^{(i)} \otimes A \otimes C \xrightarrow{\rho} M \to 0$$

where $g = f \circ (\pi \otimes I_{c} \otimes I_{C}).$ Hence $A \otimes C$ is a generator in $\mathcal{M}(\psi)^C_{A}$. \hfill $\Box$

**Remark 3.14** An important application of integrals in finite dimension Hopf algebra is Maschke theorem. It finds the condition of finite dimension Hopf algebras to be semisimple. [12-14] have studied the relation between the integral of Doi-Koppinen structure and Maschke theorem. The integrals of entwining structure we study here also have a tend relation with Maschke theorem, the readers can refer to the reference [3,15].

### 4. The Cointegrals of Entwining Structure

Because the entwining structure has the property of self-duality, we will get some dual results of Section 3. In order to give the dual definition of integrals of the entwining structure conveniently, we have the lemma as follows

**Lemma 4.1** Let $(A,C,\psi)$ be an entwining structure. The following are equivalent:

1) There exists a normalized integral $\gamma: C \to \text{Hom}(C,A)$;

2) There exists a $k$-linear map $\theta: C \otimes C \to A$ such that for all $c,d \in C$.

$$\theta(c \otimes d) = \theta(c) \otimes \theta(d),$$

$$\theta(c \otimes d) = \gamma(c,d) = \epsilon(c) \cdot 1_{A}.$$  

**Proof.** Let $\gamma(c,d) = \theta(c \otimes d).$ The proof of the lemma is obvious. \hfill $\Box$

**Definition 4.2** Let $(A,C,\psi)$ be an entwining structure. A $k$-linear map $\delta: C \to A \otimes A$ is called an cointegral of $(A,C,\psi)$ if for any

$$c \in C, \delta(c) = \delta^{1}(c) \otimes \delta^{2}(c) \in A \otimes A,$$

$$\delta^{1}(c) \otimes \delta^{2}(c) a = a_{\mu} \delta^{1}(c) \otimes \delta^{2}(c).$$

A cointegral $\delta$ is called normalized if

$$\delta^{1}(c) \otimes \delta^{2}(c) = \epsilon(c) \cdot 1_{A},$$

We shall have the dual results of Proposition 3.8, Theorem 3.11 and Theorem 3.13.

**Proposition 4.3** Let $(A,C,\psi)$ be an entwining structure. $M \in \mathcal{M}(\psi)^C_{A}$, $N \in \mathcal{M}_{A}$, $u: N \to M$ a $k$-linear map. Suppose that there exists $\delta: C \to A \otimes A$, then:
1) For all \( n \in \mathbb{N} \), the map 
\[
\tilde{u} : N \to M, \quad \tilde{u} = u \left( n_{(0)} \delta^1 (n_{(1)}) \right) \delta^2 (n_{(2)})
\]
is right \( A \)-linear;

2) Let \( \delta : C \to A \otimes A \) is a normalized cointegral and \( f : M \to N \) is a morphism in \( \mathcal{M}(\psi)^C_A \) which is a \( k \)-split injection (resp. a \( k \)-split surjection), then \( f \) has a \( \mu \)-linear retraction (resp. a section).

**Proof.** It is just the dual of Proposition 3.8. \( \square \)

**Theorem 4.4** Let \( (A, C, \psi) \) be an entwining structure, where \( \psi \) is invertible, its inverse is \( \phi \). The following statements are equivalent:

1) There exists a normalized cointegral \( \delta : C \to A \otimes A \);

2) The natural transformation \( \mu : G^C \circ (\cdot A) \circ G_A \to G^C \circ \alpha_{\mathcal{M}(\psi)^C_A} \) splits;

3) The right \( A \)-action on \( C \otimes A \), \( \mu_{C,\otimes A} : C \otimes A \otimes A \to C \otimes A \) splits in \( \mathcal{M}_A \).

Consequently, if one of the equivalent conditions holds, any entwined module is projective as a right \( C \)-comodule.

**Proof.** (1) \( \Rightarrow \) (2) Let \( \delta : C \to A \otimes A \) be a normalized cointegral. We have to construct a natural transformation \( \eta \) that splits \( \mu \). Let \( M \in \mathcal{M}(\psi)^C_A \) and for any \( m \in M, c \in C, u_\mu : M \to M \otimes A, u_\mu (m) = m \otimes 1 \), be the \( k \)-linear section of \( \mu^M : M \otimes A \to M \). We define \( \eta_M = \tilde{u} \), i.e.

\[
\eta_M = \eta_M (\delta) : M \to M \otimes A,
\]

\[
\eta_M (m) = m_{(0)} \delta^1 (m_{(1)}) \otimes \delta^2 (m_{(2)})
\]

where \( m \in M \). It follows from Proposition 4.3 that the map \( \eta_M \) is a right \( A \)-linear section of \( \mu_M \). It remains to prove that \( \eta = \eta (\delta) \) is a natural transformation.

Let \( f : M \to N \) be a morphism in \( \mathcal{M}(\psi)^C_A \). We have to prove that

\[
(f \otimes \text{id}_A) \circ \eta_M = \eta_N \circ f.
\]

For any \( m \in M \), using that \( f \) is right \( C \)-colinear, we have

\[
(f \otimes \text{id}_A) \eta_M (m) = (f \otimes \text{id}_A) \left( m_{(0)} \delta^1 (m_{(1)}) \otimes \delta^2 (m_{(2)}) \right)
\]

\[
= f (m_{(0)} \delta^1 (m_{(1)}) \otimes \delta^2 (m_{(2)})
\]

\[
= f (m_{(0)} \delta^1 (m_{(1)}) \otimes \delta^2 (m_{(2)})
\]

\[
= f \left( m_{(0)} \delta^1 (m_{(1)}) \otimes \delta^2 (m_{(2)}) \right)
\]

\[
\eta_N f (m)
\]

\[
= f \left( m_{(0)} \delta^1 (f (m_{(1)}) \otimes \delta^2 (f (m_{(1)}))
\]

\[
= f \left( m_{(0)} \delta^1 (m_{(1)}) \otimes \delta^2 (m_{(2)}) \right)
\]

\[
i.e. \ \eta \text{ is a natural transformation that splits } \mu.
\]

(2) \( \Rightarrow \) (3) Assume that for any \( M \in \mathcal{M}(\psi)^C_A \), right \( A \)-action splits in \( \mathcal{M}_A \). In particular, \( \mu_{C,\otimes A} : C \otimes A \otimes A \to C \otimes A \) splits in \( \mathcal{M}_A \). Let \( \eta = \eta_{C,\otimes A} : C \otimes A \to C \otimes A \) be a right \( A \)-linear section of \( \mu_{C,\otimes A} \). Using the naturality of \( \eta_{C,\otimes A} \), we will prove that \( \eta_{C,\otimes A} \) is also left \( A \)-linear, where \( C \otimes A \) and \( C \otimes A \otimes A \) are left \( A \)-modules via:

\[
a \cdot (c \otimes b) = c^\psi \otimes a, b,
\]

\[
a \cdot (c \otimes b \otimes b') = c^\psi \otimes a, b \otimes b',
\]

where \( a, b, b' \in A, c \in C \).

First let \( V \) be a \( k \)-module and \( \mu \in \mathcal{M}(\psi)^C_A \). Then \( V \otimes M \in \mathcal{M}(\psi)^C_A \) via the structures arising from the ones of \( M \), i.e.

\[
(v \otimes m) = v \otimes ma, \rho_{V \otimes M} = \text{id}_v \otimes \rho_M.
\]

Using the naturality of \( \eta \), we prove that

\[
\eta_{V \otimes M} = \text{id}_v \otimes \eta_M.
\]

Hence

\[
\eta_{V \otimes M} (v \otimes m) = \eta_{V \otimes M} \circ g_V (m) = (g_V \otimes \text{id}_A) \circ \eta_M (m)
\]

\[
= v \otimes \eta_M (m) = (\text{id}_v \otimes \eta_M) (m).
\]

In particular, let \( M = C \otimes A, V = A \), then \( A \otimes C \otimes A \in \mathcal{M}(\psi)^C_A \) via the structures arising from the ones of

\[
(b' \otimes c \otimes b) \cdot a = b' \otimes c \otimes ba,
\]

\[
\rho_{A \otimes C \otimes A} (b' \otimes c \otimes b) = b' \otimes c_{(1)} \otimes b \cdot c_{(2)},
\]

for all \( a, b, b' \in A, c \in C \). With these structures the map

\[
f = \mu_{C,\otimes A} : A \otimes C \otimes A \to C \otimes A,
\]

\[
a \cdot (c \otimes b) = c^\psi \otimes a, b
\]
is a morphism in \( \mathcal{M}(\psi)^C_A \). From the naturality of \( \eta \), the following diagram is commutative.

\[
\begin{array}{ccc}
A \otimes C \otimes A & \xrightarrow{id_A \otimes \eta_{C,\otimes A}} & A \otimes C \otimes A \otimes A \\
\mu_{C,\otimes A} & & \mu_{C,\otimes A} \otimes \text{id}_A \\
\downarrow & & \downarrow \\
C \otimes A & \xrightarrow{\eta_{C,\otimes A}} & C \otimes A \otimes A
\end{array}
\]

Copyright © 2013 SciRes.
be an entwining structure, i.e. we have:

\[ \delta : C \otimes A \rightarrow A \otimes C \]

In particular, for \( \delta \in \mathcal{M}_A \), we have:

\[ (\text{id} \otimes m) \eta (c \otimes a) = c \otimes a. \]

For all \( c \in C, a \in A \), we define

\[ \delta : C \rightarrow A \otimes A, \quad \delta(c) = (\epsilon \otimes \text{id}_A) \eta (c \otimes 1_A). \]

We will prove that \( \delta \) is a normalized cointegral.

\[ \delta^1(c) \delta^2(c) = m(\epsilon \otimes \text{id}_A \otimes \text{id}_A) \eta (c \otimes 1_A) = (\epsilon \otimes \text{id}) (\text{id} \otimes m) \eta (c \otimes 1_A) = \epsilon(c)1_A. \]

For any \( c \in C, a \in A \), on the one hand

\[ a_\phi \delta^1(c^\psi) \otimes \delta^2(c^\psi) = (m \otimes \text{id}_A)(\text{id}_A \otimes \delta) \psi (c \otimes a) = (m \otimes \text{id}_A)(\text{id}_A \otimes (\epsilon \otimes \text{id}_A \otimes \text{id}_A) \eta) \psi (c \otimes 1_A) = (\psi \otimes \text{id}_A)(c \otimes 1_A) = (m \otimes \text{id}_A)(a_\phi \otimes (\epsilon \otimes \text{id}_A \otimes \text{id}_A) \eta (c^\psi \otimes 1_A)). \]

Let \( \eta (c^\psi \otimes 1_A) = \sum c_j \otimes a_j \otimes b_j \), then

\[ a_\phi \delta^1(c^\psi) \otimes \delta^2(c^\psi) = (m \otimes \text{id}_A)(\sum_j c_j \otimes a_j \otimes b_j). \]

on the other hand,

\[ \delta^1(c) \otimes \delta^1(c) a = (\text{id} \otimes m)(\delta \otimes \text{id}_A)(c \otimes a) = (\text{id} \otimes m)(\epsilon \otimes \text{id}_A \otimes \text{id}_A)(\eta \otimes \text{id}_A)(c \otimes 1_A \otimes a) = (\epsilon \otimes \text{id}_A \otimes \text{id}_A) \eta (c \otimes a) = (\epsilon \otimes \text{id}_A \otimes \text{id}_A)(a^\psi \eta (c^\psi \otimes 1_A)) = (\epsilon \otimes \text{id}_A \otimes \text{id}_A)(a^\psi \eta (c^\psi \otimes 1_A)) = (\epsilon \otimes \text{id}_A \otimes \text{id}_A)(a^\psi \eta (c^\psi \otimes 1_A)) = (\epsilon \otimes \text{id}_A \otimes \text{id}_A)(\sum c_j \otimes a_j \otimes b_j). \]

we have used \( \eta \) is a \( A \)-bimodule map. Hence

\[ \delta^1(c) \otimes \delta^2(c) a = a_\phi \delta^1(c^\psi) \otimes \delta^2(c^\psi). \]

Then we obtained that \( \delta \) is a normalized cointegral of \( (A, \psi) \) finally.

Corollary 4.5 Let \( (A, \psi) \) be an entwining structure, where \( \psi \) is invertible. The following statements are equivalent:

1) There exists a cointegral \( \delta : C \rightarrow A \otimes A \); 2) There exists \( \eta : G^C \circ \text{id}_{M_A} \psi C^G \circ (\otimes A) \circ G_1 A \) a natural transformation;

3) There exists \( \eta' : C \otimes A \rightarrow C \otimes A \otimes A \) an \( A \)-bi-module map.

Theorem 4.6 Let \( (A, \psi) \) be an entwining structure, where \( \psi \) is invertible, its inverse is \( \phi \). Suppose there exists a normalized cointegral of \( (A, \psi) \)

\[ \delta : C \rightarrow A \otimes A, \quad \text{for any } M \in \mathcal{M}_A, \text{ the map} \]

\[ f : M \rightarrow M \otimes C \otimes A, \]

\[ f(m) = m_\phi \delta^1(m_\psi) \otimes m_\psi \delta^2(m_\psi). \]

for all \( a \in A, c \in C, m \in M \) is a \( k \)-split monomorphism in \( \mathcal{M}(\psi)_A \). In particular, \( C \otimes A \) is a cogenerator in the category \( \mathcal{M}(\psi)_A \).

Proof: It is just the dual of Theorem 3.13.

5. Acknowledgements

This work was partially supported by the Natural Science Foundation of Henan Province (102300410409).

REFERENCES


