Equivalence Problem of the Painlevé Equations

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Received November 1, 2012; revised January 17, 2013; accepted January 26, 2013

ABSTRACT

The manuscript is devoted to the equivalence problem of the Painlevé equations. Conditions which are necessary and sufficient for second-order ordinary differential equations \( y'' = F(x, y, y') \) to be equivalent to the first and second Painlevé equation under a general point transformation are obtained. A procedure to check these conditions is found.

Keywords: Equivalence Problem; Painlevé Equations; Point Transformation

1. Introduction

Many physical phenomena are described by differential equations. Ordinary differential equations play a significant role in the theory of differential equations. In the 19th century an important problem in analysis was the classification of ordinary differential equations [1-4]. One type of classification problem is an equivalence problem: a system of equations is equivalent to another system of equations if there exists an invertible change of the independent and dependent variables (point transformations) which transforms one system into another.

The six Painlevé equations (PI-PVI) are nonlinear second-order ordinary differential equations which are studied in many fields of Physics. These equations and their solutions, the Painlevé transcendent, play an important role in many areas of mathematics.

The Painlevé equations belongs to the class of equations of the form

\[
y'' + a_1(x, y) y'^3 + 3a_2(x, y) y'^2 + 3a_3(x, y) y' + a_4(x, y) = 0.
\]

This form is conserved with respect to any change of the independent and dependent variables\(^1\)

\[
t = \varphi(x, y), u = \psi(x, y).
\]

In fact, since under the change of variable (3) derivatives are changed by the formulae

\[
\begin{align*}
PI : & y'' = 6y'^2 + x, \\
PII : & y'' = 6y'^3 + xy + \alpha, \\
PIII : & y'' = \frac{y'^2}{y} - \frac{y'}{x} + \frac{\left(\alpha y^2 + \beta\right)}{x} + y' + \frac{\delta}{y}, \\
PIV : & y'' = \frac{y'^2}{2y} - \frac{3y^3}{2} + 4xy^2 + 2\left(x^2 - \alpha\right)y + \frac{\beta}{y}, \\
PV : & y'' = \left(\frac{1}{2y} + \frac{1}{y - 1}\right)y'^2 - \frac{y'}{x} + \frac{(y - 1)^2}{x^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{x} + \frac{\delta y(y + 1)}{y - 1}, \\
PVI : & y'' = \left(\frac{1}{2y} + \frac{1}{y + 1} + \frac{1}{y - x}\right)y'^2 - \left(\frac{1}{x} + \frac{1}{x - 1} + \frac{1}{y - x}\right)y' + \frac{y(y - 1)(y - x)}{x^2(x - 1)^2}\left(\alpha + \beta \frac{x}{y^2} + \gamma \frac{x - 1}{(y - 1)^2} + \frac{1}{2}\frac{\delta x}{(x - 1)^2}\right)
\end{align*}
\]

\(^1\)Point transformations are weaker than contact transformations. S. Lie showed that all second-order equations are equivalent with respect to contact transformations.
\[ u' = g(x, y, y') = \frac{Dy'}{D\varphi} = \psi_x + y'\psi_y, \]

\[ u^* = P(x, y, y', y^*) = \frac{Dg}{Ds\varphi} \varphi_x + y'\varphi_y. \]

\[ u^* = P(x, y, y', y^*) = \frac{Dg}{Ds\varphi} \varphi_x + y'\varphi_y. \]

Here

\[ \Delta = \varphi_y - \varphi_x \neq 0, \]

subscript means a derivative, for example,

\[ \varphi_x = \partial\varphi/\partial x, \varphi_y = \partial\varphi/\partial y. \]

Since the Jacobian of the change of variables \( \Delta \neq 0 \), the equation

\[ u^* + b_1(t, u)u'^* + 3b_2(t, u)u'^{**} + 3b_3(t, u)u' + b_4(t, u) = 0, \]

becomes (2), where

\[ a_1 = \Delta^{-1} \left( \varphi_y - \varphi_x + \varphi^3_y b_4 + 3\varphi^2_y \varphi_x b_3 + 3\varphi_y \varphi_x^2 b_2 + \varphi^3_x b_1 \right), \]

\[ a_2 = \Delta^{-1} \left[ 3^{-1} \left( \varphi_y - \varphi_x + 2\varphi_y \varphi_x \right) + \varphi^3_y b_4 \right. \]

\[ + \varphi_x \left( 2\varphi_y \varphi_x + \varphi_x \right) b_3 + \left( \varphi^2_y + 2\varphi_y \varphi_x \right) b_2 + \varphi_x^2 b_1 \right], \]

\[ a_3 = \Delta^{-1} \left[ 3^{-1} \left( \varphi_y - \varphi_x + 2\varphi_y \varphi_x \right) + \varphi^3_y b_4 \right. \]

\[ + \left( \varphi^3_y + 2\varphi_y \varphi_x \right) b_3 + \left( 2\varphi_y \varphi_x + \varphi^2_y \right) b_2 + \varphi_x^2 b_1 \right], \]

\[ a_4 = \Delta^{-1} \left( \varphi_y - \varphi_x + \varphi^3_y b_4 + 3\varphi^2_y \varphi_x b_3 + 3\varphi_y \varphi_x^2 b_2 + \varphi^3_x b_1 \right). \]

Two quantities play a major role in the study of Equation (5):

\[ L_1 = \frac{\partial\Pi_{11}}{\partial t} - b_4 \Pi_{22} - b_2 \Pi_{11} + 2b_3 \Pi_{12}, \]

\[ L_2 = \frac{\partial\Pi_{12}}{\partial t} + \frac{\partial\Pi_{22}}{\partial t} - b_1 \Pi_{11} - b_2 \Pi_{22} + 2b_3 \Pi_{12}, \]

where

\[ \Pi_{11} = 2\left( b_1^2 + b_4 b_6 \right) + b_4 - b_{2w}, \]

\[ \Pi_{22} = 2\left( b_1^2 + 3b_1 b_3 \right) + b_3 - b_{2w}, \]

\[ \Pi_{12} = b_2 b_3 - b_1 b_4 + b_2 - b_{2w}. \]

Under a point transformation (3) these components are transformed as follows [2]:

\[ \tilde{L}_1 = \Delta \left( L_1 \varphi_x + L_2 \varphi_y \right), \]

\[ \tilde{L}_2 = \Delta \left( L_1 \varphi_y + L_2 \varphi_x \right). \]

Here tilde means that a value corresponds to system (2): the coefficients \( b_i \) are exchanged with \( a_i \), the variables \( t \) and \( u \) are exchanged with \( x \) and \( y \), respectively.

S. Lie showed that any equation with \( L_1 = 0 \) and \( L_2 = 0 \) is equivalent to the equation \( u'^* = 0 \). For the Painlevé equations \( L_1 \neq 0 \) and \( L_2 = 0 \).

R. Liouville [2] also found other relative invariants, for example,

\[ v_3 = L_2 \left( L_1 L_{21} - L_2 L_{11} \right) + L_1 \left( L_1 L_{11} - L_2 L_{21} \right) - b_1 L_1^2 \]

\[ + 3b_1 L_1 L_2 - 3b_1 L_2^2 + b_4 L_2^2, \]

and

\[ w_1 = L_1^4 \left( -L_1^2 \left( \Pi_{12} \Pi_{11} - \Pi_{12} \Pi_{11} \right) + R_1 \left( L_1^2 \right) \right) \]

\[ - L_1^2 R_1 + L_1 R_1 \left( b_1 L_1 - b_2 L_2 \right) \],

where

\[ R_1 = L_1 L_{21} - L_2 L_{11} + b_2 L_1^2 - 2b_1 L_1 L_2 + b_4 L_2^2. \]

For the Painlevé equations \( v_3 = 0 \) and \( w_1 = 0 \) [5].

Up to now, the equivalence problem has been solved in a more convenient form for testing only for (PI) and (PII) equations, by using an explicit point change of variables was given in [6].

The manuscript is devoted to solving the problem of describing all second-order differential equations.
\[ y'' = F(x, y, y') \] which are equivalent with respect to point transformations (3) to the first and second Painlevé equation (PI) and (PII). Example of the first Painlevé equation (PI) is presented.

Necessary and sufficient conditions for an equation \[ y'' = F(x, y, y') \] to be equivalent to (PI) and (PII) are obtained. As was noted, some of the necessary conditions are [5]:

\[
\frac{\partial^4 F}{\partial y'^4} = 0, v_s = 0 \text{ and } w_1 = 0.
\]

Other conditions are also expressed in terms of relations for the coefficients of Equation (5).

The method of the study is similar to [7-9]. It uses analysis of compatibility of an overdetermined system of partial differential equations.

2. Equations Equivalent to the Painlevé Equations

This section studies Equation (5) which are equivalent to the first and second Painlevé equation (PI) and (PII). Since any equation of (1) belongs to the type of equation (2), the necessary condition for an equation \[ y'' = F(x, y, y') \] to be equivalent to the first and second Painlevé equation (PI) and (PII) are that it has to be of the same type. Since \[ v_s = 0 \] and \[ w_1 = 0 \] are relative invariants with respect to (3), they are also necessary condition.

2.1. The First Painlevé Equation (PI)

For obtaining sufficient conditions one has to find conditions for the coefficients \[ b_1(t, u), b_2(t, u), b_3(t, u) \] which guarantee existence of the functions \( \phi(x, y), \Psi(x, y) \) transforming the coefficient of Equation (6) into the coefficients of equations (PI).

Also note that the first Painlevé equation has the coefficients are

\[
a_1(x, y) = 0, a_2(x, y) = 0, \quad a_1(x, y) = 0, a_4(x, y) = -6y^2 - x.
\]

Without loss of generality it is assumed that \( L_1 \neq 0 \). Since for Equation (8), the value \( L_2 = 0 \), and hence, the functions \( \phi(x, y) \) and \( \Psi(x, y) \) satisfy the equation

\[
\phi_x L_1 + \Psi_y L_1 = 0.
\]

Substituting these coefficients into (6), one obtains overdetermined system of partial differential equations.

\[
\psi_y L_1^2 + \psi_x^2 \left( 3b_1 L_2^2 - 6b_1 L_1 L_2 + 3b_2 L_2^2 - 2L_1 L_2 + 2L_1 L_1 \right) = 0,
\]

\[
2\psi_y L_1^2 - \Delta_{1x} \psi_x \Delta_{11} L_1^2 + \psi_y \Delta_1 \left( L_1 - 3b_2 L_2 + 3b_3 L_1 \right) + \psi_x \left( 6b_1 L_2^2 - 12b_1 L_1 L_2 + 6b_2 L_1^2 - 4L_1 L_2 + L_1 L_3 + 3L_2 L_1 \right) = 0,
\]

\[
\psi_x L_1^2 - \Delta_{1x} \psi_x \Delta_{11} L_1^2 + b_1 \Delta_1^2 + \psi_x \Delta_1 \left( L_1 - 3b_2 L_2 + 3b_3 L_1 \right) + \psi_y \left( 3b_1 L_2^2 - 6b_3 L_1 L_2 + 3b_2 L_2^2 - 2L_1 L_2 + L_1 L_1 + L_2 L_1 \right) + \psi_y \left( 6y^2 + x \right) = 0,
\]

where \( \Delta_1 = \phi_x L_1 + \psi_x L_2 \). Notice that

\[
L_1 \Lambda_{1x} = \psi_x \Delta_1 \left( L_1 - L_2 \right).
\]

From Equations (10)-(12) one can find the derivatives

\[
\psi_{xy} = \psi_x^2 \left( 2L_1 L_2 - 2L_2 L_1 - 3b_1 L_2^2 + 6b_1 L_1 L_2 - 3b_2 L_2^2 \right),
\]

\[
L_1^2 \Lambda_{1x} \psi_{xy} = 2\psi_y \psi_x \psi_x^{-1} L_1^3 - b_1 \Delta_1^2 - \psi_y L_1^3 \left( 6y^2 + x \right) + \psi_x^2 \left( 3b_1 L_2^2 - 6b_1 L_1 L_2 + 3b_1 L_1^2 - 2L_1 L_2 + 2L_1 L_1 \right),
\]

\[
L_1^2 \Lambda_{1x} = 2\psi_y \psi_x \psi_x^{-1} \Delta_{11} L_1^2 + \psi_x \left( 3b_1 L_2^2 - 6b_3 L_1 L_2 + 3b_2 L_2^2 - 2L_1 L_2 + L_1 L_1 + L_2 L_1 \right) + \psi_x \Delta_1 \left( L_1 - 3b_2 L_2 + 3b_3 L_1 \right) + \psi_x \left( 3b_1 L_2^2 - 6b_3 L_1 L_2 + 3b_2 L_2^2 - 2L_1 L_2 + L_1 L_1 + L_2 L_1 \right) + \psi_x \left( 6y^2 + x \right).
\]

Taking the mixed derivatives \( \psi_{xy} = \left( \psi_{xy} \right)_y \), one obtains

\[
\psi_y \Delta_1^2 + 12L_1 = 0.
\]

Differentiating this equation with respect to \( x \) and \( y \), and substituting \( \psi_y \) found from Equation (17), one gets

\[
5\psi_{xy} \Delta_1^2 L_1 - 12\psi_y \left( 12 \left( b_1 L_2^2 - 2b_1 L_1 L_2 + b_2 L_1^2 \right) - 7L_1 L_2 + L_1 L_3 + 6L_2 L_1 \right) - 12\Delta_1 \left( L_1 - 6b_2 L_2 + 6b_3 L_1 \right) = 0,
\]

\[
3b_1 L_2^2 - 6b_3 L_1 L_2 + 3b_2 L_2^2 - 3L_1 L_2 - L_1 L_1 + 4L_2 L_1 = 0.
\]

Finding the derivatives: \( L_{2x} \) from the equation \( v_s = 0 \), \( L_{2y} \) from the equation \( w_1 = 0 \), and \( L_2 \) from (19), and composing
the equations

\[(L_{2v})_t - L_{2w r} = 0, (L_{2v})_y - (L_{2u})_t = 0,\]

one can find the derivatives

\[L_{2u} = \left(4b_1^2 L_1^2 - 18b_1 b_2 L_1 + 60b_2 b_2 L_1^2 + 80b_1 b_2 L_1^2 - 3b_2 L_1 L_2^2 - 36b_2^2 L_2 L_1 - 90b_2 b_3 L_2 - 12b_3 L_2 L_2^2 + 30b_1 L_1 L_2^2 - 15b_2 L_1^2 + 20b_3 b_2 L_1^2 + 80b_3 L_2 - 100b_3 b_2 L_1^2 - 2 L_1 L_2^2 + 25L_1 L_2 L_2^2 + 2KL_1^2 \right),\]

\[L_{2v} = \left(-b_1^2 L_1^2 + 12b_1 b_2 L_1 + 40b_2 b_2 L_1^2 + 2b_3 L_1 L_2^2 - 36b_2^2 L_2 L_1^2 + 120b_2 b_3 L_2 - 12b_3 L_1 L_2^2 - 135b_2 L_2^2 + 30b_2 L_1 L_1^2 - 20b_3 L_1 L_2^2 + 40b_2 b_2 L_1^2 - 20b_2 L_1 L_2^2 + 60b_2 L_1 L_2^2 + 80b_2 b_3 L_2 - 120b_2 L_1 L_2^2 + 25L_1 L_2 L_2^2 + 2KL_1^2 \right),\]

where

\[K = 36b_2 L_1^2 - 6b_2 b_2 L_1 L_2 - 105b_2 b_2 L_2 L_2 - 15b_2 L_1 L_1 + 108b_2^2 L_1^2 + 6b_2 b_2 L_1 L_2 - 10b_2 L_2 L_2 - 50b_2 L_1 L_1^2 + 60b_2 L_1 L_2 + 10L_1 L_2 - 12L_2^2.\]

Since of (14), (15) and (18) all second order derivatives of the function \(\Psi(x, y)\) can be found, then one can compose the equations \((\Psi_y)_y - (\Psi_x)_x = 0\), which are reduced to the only equation

\[\Delta_t^2 K - 60L_1^4 y = 0.\]  

The equation \((L_{2u})_t - (L_{2u})_x = 0\) gives

\[2L_1 (K - L_2) + 3K (6b_2 L_1^2 - 2b_2 L_2 L_2 + b_2 L_2^2) + 5KL_1 L_2 L_2 = 0.\]

Differentiating Equation (23) with respect to \(x\), one obtains

\[250\Psi_y L_1^2 y + \Delta_t L_1^2 y (6b_2 KL_2 - 6b_2 KL_1 - 5K L_1 + 14K L_2) = 0.\]

From this equation one can find the derivative \(\nu\),

\[\nu = (\Delta_t (6b_2 KL_2 - 6b_2 KL_1 + 5K L_1 - 14K L_2) + 4KL_1 L_2 = 0.\]

Notice that the equations \((\Psi_y)_y - (\Psi_x)_x = 0\) and \(\Psi_y - (\Psi_x)_x = 0\) are satisfied, and the equation \(\Psi_{xx} - (\Psi_y)_y = 0\) becomes

\[Q \Delta^2 + x = 0,\]

where

\[Q = 6K^2 (390b_2^2 KL_2^2 + 780b_2 b_2 KL_1 L_2 + 2850b_2 KL_1^2 + 300b_1 KL_1^2 + 1710b_2 L_2^3 + 1400b_2 KL_1^2 - 3240b_2^2 KL_1^2 - 300b_1 KL_1^2 + 720b_1 KL_1 L_2 + 400b_1 KL_1 L_2 + 840 KL_1 L_1 - 240 KL_2 K^2 + 12 KL_1^2 - 8 KL_1 L_2 + 4KL_1^2 + 60 KL_1 L_1 - 840 KL_1 K + 2KL_1^2).\]

Because of (25), the function \(Q(t, u) \neq 0\). Differentiating (25) with respect to \(x\) and \(y\), one gets

\[\Delta_t L_1^2 y - 5K L_1^2 = 0,\]

\[K (L_1 Q_u - Q L_1) - 100QL_1^2 = 0,\]

where

\[R = K (4QK (3b_2 L_2 - 3b_2 L_1 + 7L_1) - 5L_1 (Q K + 2KL_1^2)).\]

Differentiating Equation (27) with respect to \(x\) and \(y\) one obtains the only equation

\[R L_t - R (7L_1 - 3b_2 L_2 - 3b_2 L_1) = 0.\]

Finding the function \(A_t\) from (27), and substituting it into (23), (16), (13) one gets

\[24L_1^2 R^2 - K^2 = 0.\]

Notice that

\[4L_1 (L_1 R - L_2) + 25R (L_1 L_2 - L_1 L_2 + 10R (b_2 L_2 - 2b_2 L_1 - b_1^2 L_1^2) = 0.\]

Thus, the necessary and sufficient conditions for equation \(y'' = F(x, y, y')\) to be equivalent to the first Painlevé equation are: the equation has to be of the form (5) with the coefficients \(b_i(t, u), (i = 1, 2, 3, 4)\) satisfying the conditions \(v_3 = 0, (19)-(21), (24), (28) and (32),\) where the functions \(K(t, u), R(t, u)\) and \(Q(t, u)\) are defined by Equations (22), (26), (29). The transformation is defined by (25) and (31).

### 2.2. The Second Painlevé Equation (PII)

Similar to the first Painlevé equation one can study the second Painlevé equation. Painlevé equation (PII) has the coefficients are

\[a_1 (x, y) = 0, a_2 (x, y) = 0,\]

\[a_3 (x, y) = 0, a_4 (x, y) = -(2y^3 + xy + \alpha).\]

Substituting these coefficients into (6), one obtains over determined system of partial differential equations.
\[ \psi_{yy} L_1^2 + \psi_y^2 (3b_y L_2^2 - 6b_y L_y L_2 + 3b_y L_2^2 - 2L_y L_2 + 2L_y L_1) = 0, \]  
(34)

\[ 2\psi_{yy} L_1^2 - \Delta_y \psi_y \Delta_y L_1^2 + \psi_y \Delta_y (L_{yy} - 3b_y L_y + 3b_y L_y) + \psi_{yy} (6b_y L_2^2 - 12b_y L_y L_2 + 6b_y L_2^2 - 4L_y L_2 + L_{uu} L_1 + 3L_y L_1) = 0, \]  
(35)

\[ \psi_{xx} L_1^2 - \Delta_x \psi_x \Delta_x L_1^2 + b_x \Delta_x^2 + \psi_x \Delta_x (L_{xx} - 3b_x L_x + 3b_x L_x) + \psi_{xx} (3b_x L_2^2 - 6b_x L_x L_2 + 3b_x L_2^2 - 2L_x L_2 + L_{uu} L_1 + L_{uu} L_1) + \psi_y L_1^2 (2y^3 + xy + \alpha) = 0, \]  
(36)

where \( \Delta_y = \varphi_{L_y} + \psi_y L_y \). Notice that  
(37)

From Equations (34)-(36) one can find the derivatives  
(38)

\[ L_1^2 \psi_{xx} = 2\psi_y \psi_y^{-1} L_1^2 - 6b_y \Delta_x^2 - \psi_y L_1^2 (2y^3 + xy + \alpha) + \psi_{xx} (3b_y L_2^2 - 6b_y L_x L_2 + 3b_y L_2^2 - 2L_x L_2 + 2L_y L_1), \]  
(39)

\[ L_1^2 \Delta_x = 2\psi_y \psi_y^{-1} \Delta_x L_1^2 + \Delta_x^2 (L_{xx} - 3b_x L_x + 3b_x L_x) + \psi_x L_1 (L_{uu} L_1 - 4L_y L_2 + 3L_y L_1 + 6b_y L_2 - 12b_y L_x + 6b_x L_2^2). \]  
(40)

Taking the mixed derivatives \( \psi_{xy} = \psi_{yx} \), one obtains  
(41)

Differentiating this equation with respect to \( x \) and \( y \), and substituting \( \psi_y \) from Equation (41), one gets  
(42)

\[ 5\psi_{xy} \Delta_x L_1 - 12y (\Delta_x (L_{uu} - 6b_y L_x + 6b_x L_2) + \psi_y (12b_y L_2^2 - 24b_y L_x L_2 + 12b_y L_2^2 - 7L_y L_2 + L_{uu} L_1 + 6L_y L_1)) = 0, \]  
(43)

where the function \( K(t,u) \) is defined by the formula  
(44)

Since \( \Delta_y \neq 0 \), then \( K \neq 0 \). Hence, Equations (42) and (43) define \( \Delta_y = 12KY \) and the derivative \( \psi_{xy} \). Thus, all second-order derivatives \( \psi_{x}, \psi_{yy}, \psi_{xy} \) and the derivative \( \psi_{yx} \) of the function \( \psi(x,y) \) are defined.

Substituting the expression of \( \Delta_y \) into Equations (37) and (40), one obtains  
(45)

\[ \begin{align*}
4L_1 (K_{u} L_1 - K_{L_2}) - 3K ((L_{uu} L_1 - L_{uu} L_2 + 12K^2 L_1 + b_y L_2^2 - 2b_y L_y L_2 + b_y L_2^2) &= 0, \\
3KL \psi_y &= y (L_{uu} - b_y L_2 + b_y L_2) - 5K^{-1} L_y L_1. 
\end{align*} \]  
(46)

Equations (41) and (46) define all first-order derivatives \( \psi_{x}, \psi_{y} \) of the function \( \psi(x,y) \). Since the second-order derivatives \( \psi_{xx}, \psi_{yy}, \psi_{xy} \) have been found, one needs to check the conditions  
(47)

\[ \begin{align*}
\psi_{x} &= \psi_{xx}, \\
\psi_{y} &= \psi_{yy}, \\
\psi_{xy} &= \psi_{yx}, \\
\psi_{yx} &= \psi_{yy}. 
\end{align*} \]  
(48)

All these conditions are satisfied except the first one, which becomes  
(49)

\[ 2y^2 Q - x = 0. \]  
(49)
where
\[ Q = L_1^4 \left( 8L_1 \left( 3K_5 KL_5 - 4K_1^2 L_1 - 3b_2 K, KL_2 + 3b_1 K, KL_4 \right) \right. \]
\[ \left. -18K^2 \left( b_1^2 L_2^2 - 2b_1 b_3 L_3 L_1 + 9b_2 b_3 L_3^2 - b_4 b_3 L_2 + b_4 L_4 L_1 + 12b_4 K^2 L_1 - 8b_2^3 L_2^2 + 4b_3 L_3^3 - 4b_3 L_4 \right) \right) - 3. \]

Differentiating (49) with respect to \( x \) and \( y \), one gets, respectively,
\[ 2y^3 \left( (Q_L L_2 - Q L_1) \left( 3b_4 K L_2 - 3b_5 KL_1 + 5K_1 L_1 - 3L_0 K \right) + 36Q K^3 L_1 \right) - 3K^2 L_1^4 = 0, \]
\[ Q L_2 - Q L_1 + 24Q K^2 = 0. \]

Since \( KL_1 \neq 0 \), the coefficient with \( y^3 \) in (51) is not equal to zero. Hence, Equations (49) and (51) define the variable \( x \) and \( y \). Equation (48) becomes
\[ 18K^2 \left( 2L_{1u} - 2b_3 b_4 L_1 - 3b_4 L_{1u} L_1 - 3b_4 L_{1u} + 24b_3 L_{1u} L_1 - 12b_4 K^2 - 2b_4 K L_1 - 10b_4 L_1 + 12b_4 L_1 + 3b_4 L_{1u} \right) \]
\[ + 6K \left( 12K_1 + \alpha Q L_1 \right) \left( b_1 L_2 - b_1 L_1 - L_{1u} \right) + 120K_1^2 L_1 - 6Q \alpha K L_1^2 + Q L_1^2 \left( 20K_1 \alpha - 3L_1 \right) - 8L_1^4 = 0. \]

Remaining equations are obtained by differentiating (51) with respect to \( x \) and \( y \). Excluding from them \( x \) and \( y \) these equations are reduced to the equation
\[ -36Q K^2 L_1^2 + 12Q K L_1 \left( 22K_1 L_1 - 15b_4 K L_1 + 15b_4 K L_1 - 12L_0 K - \alpha Q L_1 \right) - 2Q \left( 6K_1 (L_{1u} - b_1 L_1 + b_1 L_1) - 10K, L_1 \right) \]
\[ - 4\alpha Q^2 L_1^2 \left( 6K \left( b_1 L_2 - b_1 L_2 + L_{1u} \right) - 10K, L_1 \right) + 864b_3 Q K^4 L_1 - 4Q^2 L_1^2 - Q_1^2 L_1^4 = 0. \]

Thus, the necessary and sufficient conditions for an equation \( y^s = F(x, y, y^r) \) which can be transformed to the second Painlevé equations are: this equation has to be of the form (5), where the coefficients satisfy the equations \( v_5 = 0, w_1 = 0 \), (45), (52)-(54), where the functions \( K(t, u) \) and \( Q(t, u) \) are defined by Equations (44) and (50). The transformation of the Equation (5) into the second Painlevé equation (PII) is defined by Equations (49) and (51).

3. Example of the Results

Example. The following equation is equivalent to the first Painlevé equation (PI)
\[ u^s + (1/2)u' - u^2/2 - 392/(625u^4) = 0. \]

This equation has to be of the form (5) with the coefficients
\[ b_1 = 0, b_2 = 0, b_3 = 1/(3t), b_4 = -u - u^2/2 - 392/(625u^4). \]

satisfying the conditions
\[ L_1 = -1, L_2 = 0, K = 2s/3, Q = 6 \times 5^4 t^3 s^{-2}, R = 6 \times 10^4 t^{-1}, \]
where \( s = 25t^2 (u + 1) - 4 \). Equations (19)-(21), (24), (28) and (32) are satisfied and Equations (25) and (31) become
\[ x = -Q v^s, y = 3 \times 10^3 t^3 s. \]

The changes of variable are the following:
\[ t = (1/10)(1/1944x)^{3/4}, u = \sqrt{6/xy} + 16 \times (3)^{10} \sqrt{6x} - 1. \]

4. Conclusion

The necessary and sufficient conditions that an equation of the form \( y^s = F(x, y, y^r) \) to be equivalent to the first and second Painlevé equation under a general point transformation are obtained. As was noted some of the necessary conditions are \( v_5 = 0 \) and \( w_1 = 0 \). Other conditions are also expressed in terms of relations for the coefficients of Equation (2). A procedure to check these conditions is found. Since intermediate calculations in the equivalence problem are cumbersome, computer algebra system have become an important computational tool.

5. Acknowledgements

This research is supported by Commission on Higher Education and the Thailand Research Fund under Grant. No. MRG 4980154, Naresuan University and Suranaree University of Technology.

REFERENCES

l’Équation Différentielle Ordinaire du Second Ordre

\[ y'' = \omega(x, y, y') \]


