The Ricci Operator and Shape Operator of Real Hypersurfaces in a Non-Flat 2-Dimensional Complex Space Form

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ABSTRACT

In this paper, we study a real hypersurface M in a non-flat 2-dimensional complex space form M₄(η) with η-parallel Ricci and shape operators. The characterizations of these real hypersurfaces are obtained.

Keywords: Real Hypersurface; η-Parallel Shape Operator; η-Parallel Ricci Operator; Hopf Hypersurface; Ruled Real Hypersurfaces

1. Introduction

A complex n-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by M₄(c). As is well-known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space Pn(C), a complex Euclidean space Cⁿ or a complex hyperbolic space Hⁿ(c), according to c > 0, c = 0 or c < 0.

In this paper we consider a real hypersurface M in a complex space form M₄(c), c ≠ 0. Then M has an almost contact metric structure φ, g, ξ, η induced from the Kaehler metric and complex structure J on M₄(c). The structure vector field ξ is said to be principal if Aξ = αξ is satisfied, where ξ is the shape operator of M and α = η(ξAξ). In this case, it is known that α is locally constant ([1]) and that M is called a Hopf hypersurface.

Typical examples of Hopf hypersurfaces in Pn(C) are homogeneous ones, R. Takagi [2] and M. Kimura [3] completely classified such hypersurfaces as six model spaces which are said to be A₁, A₂, B₁, B₂, C,D and E. On the other hand, real hypersurfaces in H₄(C) have been investigated by J. Berndt [4], S. Montiel and A. Romero [5] and so on. J. Berndt [4] classified all homogeneous Hopf hypersurfaces in H₄(C) as four model spaces which are said to be A₁, A₂, A₃ and B. Further, Hopf hypersurfaces with constant principal curvatures in a complex space form have been completely classified as follows:

Theorem 1.1. ([2]) Let M be a homogeneous real hypersurface of Pn(C). Then M is tube of radius r over one of the following Kaehlerian submanifolds:

(A) a hyperplane Pn−1(C), where 0 < r < π/√c;
(B) a totally geodesic P₁(C)(1 ≤ k ≤ n − 2), where 0 < r < π/√c;
(C) a hyperquadric Qn−1(C), where 0 < r < π/2√c and n ≥ 5 is odd;
(D) a complex Grassmann G₂₅C, where 0 < r < π/2√c and n = 9;
(E) a Hermitian symmetric space SO(10)/U(5), where 0 < r < π/2√c and n = 15.

Theorem 1.2. ([4]) Let M be a real hypersurface in H₄(C). Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the followings:

(A) a self-tube, that is, a horosphere;
(B) a geodesic hypersphere;
(C) a tube over a totally geodesic H₄(C)(1 ≤ k ≤ n − 1).
(B) a tube over a totally real hyperbolic space \( H^*_c(\mathbb{R}) \).

A real hypersurface of type \( A_i \) or \( A_2 \) in \( P_i(\mathbb{C}) \) or type \( A_i, A_j \) or \( A_3 \) in \( H^*_c(\mathbb{C}) \), then \( M \) is said to be of type \( A \) for simplicity. As a typical characterization of real hypersurfaces of type \( A \), in a complex space form \( M_c(\cdot,\cdot) \) was given under the condition

\[
g\left((A\phi - A\phi)X,Y\right) = 0, \tag{1.1}
\]

for any tangent vector fields \( X \) and \( Y \) on \( M \) by M. Okumura [5] for \( c > 0 \) and S. Montiel and A. Romero [6] for \( c < 0 \). Namely

**Theorem 1.3.** ([5,6]) Let \( M \) be a real hypersurface in \( M_n(c) \). It satisfies (1.1) on \( M \) if and only if \( M \) is locally congruent to one of the model spaces of type \( A \).

The holomorphic distribution \( T_0 \) of a real hypersurface \( M \) in \( M_n(c) \) is defined by

\[
T_0(p) = \{X \in T_p(M) | g(X,\xi)_p = 0\}. \tag{1.2}
\]

The following theorem characterizes ruled real hypersurfaces in \( M_n(c) \).

**Theorem 1.4.** ([7]) Let \( M \) be a real hypersurface in \( M_n(c) \). Then \( M \) is a ruled real hypersurfaces if and only if \( \phi A\phi = 0 \), or equivalently

\[
g(A\phi Y, Z) = 0, \quad \text{for any } X, Y, Z \in T_0.
\]

A \((1,1)\) type tensor field \( T \) of \( M \) is said to be \( \eta \)-parallel if

\[
g\left((\nabla_X T)Y, Z\right) = 0 \tag{1.3}
\]

for any vector fields \( X, Y \) and \( Z \) in \( T_0 \). Real hypersurfaces with \( \eta \)-parallel shape operator or Ricci operator have been studied by many authors (see [13]). Nevertheless, the classification of real hypersurfaces with \( \eta \)-parallel shape operator or Ricci operator in \( M_n(c) \) remains open up to this point. Recently, S.H. Kon and T.H. Loo ([9]) investigated the conditions \( \eta \)-parallel shape operator.

**Theorem 1.5.** ([9]) Let \( M \) be a real hypersurface of \( M_n(c) \). Then the shape operator \( A \) is \( \eta \)-parallel if and only if \( M \) is locally congruent to a ruled real hypersurface, or a real hypersurface of type \( A \) or \( B \).

Also, M. Kimura and S. Maeda ([10]) and Y.J. Suh ([11]) investigated the conditions \( \eta \)-parallel Ricci operator.

**Theorem 1.6.** ([10,11]) Let \( M \) be a real hypersurface in a complex space form \( M_n(c) \). Then the Ricci operator of \( M \) is \( \eta \)-parallel and the structure vector field \( \xi \) is principal if and only if \( M \) is locally congruent to one of the model spaces of type \( A \) or type \( B \).

As for the structure tensor field \( \phi \), shape operator \( A \) and \( \eta \)-parallel, I.-B. Kim, K. H. Kim and one of the present authors ([12]) have proved the following.

**Theorem 1.7.** ([12]) Let \( M \) be a real hypersurface in a complex space form \( M_n(c) \). If \( M \) has the cyclic \( \eta \)-parallel shape operator (resp. Ricci operator) and satisfies

\[
g\left((A\phi - A\phi)X,Y\right) = 0 \tag{1.4}
\]

for any \( X \) and \( Y \) in \( T_0 \), then \( M \) is locally congruent to either a real hypersurface of type \( A \) or a ruled real hypersurface (resp. \( M \) is locally congruent to a real hypersurface of type \( A \)).

The purpose of this paper is to give some characterizations of real hypersurfaces satisfying (1.4) and having the \( \eta \)-parallel shape operator or Ricci operator in \( M_n(c) \). We shall prove the following.

**Theorem 1.8.** Let \( M \) be a real hypersurface in a complex space form \( M_n(c) \). If \( M \) has the \( \eta \)-parallel shape operator and satisfies (1.4), then \( M \) is locally congruent to a ruled real hypersurface.

**Theorem 1.9.** Let \( M \) be a real hypersurface in a complex space form \( M_n(c) \). If \( M \) has the \( \eta \)-parallel Ricci operator and satisfies (1.4), then \( M \) is locally congruent to a real hypersurface of type \( A \).

All manifolds in the present paper are assumed to be connected and of class \( C^1 \) and the real hypersurfaces are supposed to be orientable.

## 2. Preliminaries

Let \( M \) be a real hypersurface immersed in a complex space form \( M_n(c) \), and \( N \) be a unit normal vector field of \( M \). By \( \tilde{\nabla} \) we denote the Levi-Civita connection with respect to the Fubini-Study metric tensor \( \tilde{g} \) of \( M_n(c) \). Then the Gauss and Weingarten formulas are given respectively by

\[
\tilde{\nabla}_XY = \nabla_XY + g(A\phi Y, Z)N, \tilde{\nabla}_XN = -AX
\]

for any vector fields \( X, Y \) tangent to \( M \), where \( g \) denotes the Riemannian metric tensor of \( M \) induced from \( \tilde{g} \), and \( A \) is the shape operator of \( M \) in \( M_n(c) \). For any vector field \( X \) on \( M \) we put

\[
JX = \phi X + \eta(X)N, JN = -\xi,
\]

where \( J \) is the almost complex structure of \( M_n(c) \). Then we see that \( M \) induces an almost contact metric structure \((\phi, g, \xi, \eta)\), that is,

\[
\phi^2X = -X + \eta(X)\xi, \phi\xi = 0, \eta(\xi) = 1,
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \eta(X) = g(X, \xi)
\]

for any vector fields \( X, Y \) on \( M \). Since the almost complex structure \( J \) is parallel, we can verify from the Gauss and Weingarten formulas the followings:
\[ \nabla_X \xi = \phi AX, \quad (2.2) \]
\[ (\nabla_X \phi) Y = \eta(Y) AX - g(AX, Y) \xi. \quad (2.3) \]

Since the ambient manifold is of constant holomorphic sectional curvature \( c \), we have the following Gauss and Codazzi equations respectively:

\[ R(X, Y, Z) = \frac{c}{4} (g(Y, Z) X - g(X, Z) Y + g(\phi Y, Z) \phi X - g(\phi X, Z) \phi Y - 2g(\phi X, Y) \phi Z) \quad (2.4) \]
\[ + g(AX, Z) AX - g(AX, X) AZ, \]
\[ (\nabla_X A) Y - (\nabla_Y A) X = \frac{c}{4} \{ \eta(X) \phi Y - \eta(Y) \phi X - 2g(\phi X, Y) \phi \xi \} \quad (2.5) \]

for any vector fields \( X, Y \) and \( Z \) on \( M \), where \( R \) denotes the Riemannian curvature tensor of \( M \). From (1.3), the Ricci operator \( S \) of \( M \) is expressed by

\[ SX = \frac{c}{4} \{ (2n+1) \eta(X) \xi + mX \} \]
\[ + m(\nabla_X A) Y - (\nabla_Y A) Y - A(\nabla_X A) Y, \quad (2.6) \]

where \( m = \text{trace} A \) is the mean curvature of \( M \), and the covariant derivative of (2.5) is given by

\[ (\nabla_X S) Y = \frac{3c}{4} \{ g(\phi AX, Y) \xi + \eta(Y) \phi AX \} + (Xm) Y \]
\[ + m(\nabla_X A) Y - (\nabla_Y A) Y - A(\nabla_X A) Y. \quad (2.7) \]

Let \( U \) be a unit vector field on \( M \) with the same length of the vector field \( -\phi \nabla_X \xi \xi \) if it does not vanish, and zero (constant function) if it vanishes. Then it is easily seen from (1.1) that

\[ AU = \alpha \xi + \beta U, \quad (2.8) \]

where \( \alpha = \eta(AX) \xi \). We notice here that \( U \) is orthogonal to \( \xi \). We put

\[ \Omega = \{ p \in M | \beta(p) \neq 0 \}. \quad (2.9) \]

Then \( \Omega \) is an open subset of \( M \).

### 3. Some Lemmas

In this section, we assume that \( \Omega \) is not empty, then there are scalar fields \( \gamma, \varepsilon, \alpha, \beta, \gamma, \delta \) and a unit vector field \( U \) and \( \phi U \) orthogonal to \( \xi \) such that

\[ AU = \beta \xi + \gamma U + \varepsilon \phi U, \quad A\phi U = \varepsilon U + \delta \phi U \quad (3.1) \]

and

\[ \beta = \text{trace} A = \alpha + \gamma + \delta \quad (3.2) \]

in \( M_2(c) \). We shall prove the following Lemmas.

**Lemma 3.1.** Let \( M \) be a real hypersurface in a complex space form \( M_2(c), c \neq 0 \). If \( M \) satisfies (1.4), then we have \( AU = \beta \xi + \gamma U, \quad A\phi U = \delta \phi U \) and \( \delta = \gamma \).

**Proof.** If we put \( X = Y = U \), or \( X = U \) and \( Y = \phi U \) into (1.4) and make use of (3.1), then we have

\[ \varepsilon = 0 \quad \text{and} \quad \delta = \gamma. \quad (3.3) \]

Therefore, it follows that \( AU \) is expressed in terms of \( \xi \) and \( U \) only and \( A\phi U \) given by \( \phi U \). \( \square \)

It follows from (2.6), (2.8) and Lemma 3.1 that

\[ S\xi = \left( \frac{c}{2} + 2\alpha \gamma - \beta^2 \right) \xi + \beta \gamma U, \]
\[ SU = \beta \gamma \xi + \left( \frac{5c}{4} + \alpha \gamma - \beta^2 + \gamma^2 \right) U, \quad (3.4) \]
\[ S\phi U = \left( \frac{5c}{4} + \alpha \gamma + \gamma^2 \right) \phi U. \]

**Lemma 3.2.** Under the assumptions of Lemma 3.1. If \( M \) has the \( \eta \)-parallel Ricci operator \( S \) then we have \( \beta U = 0 \) and \( (\phi U) \beta = -\gamma^2 \).

**Proof.** Differentiating the second of (3.4) covariantly along vector field \( X \) in \( T_0 \), we obtain

\[ (\nabla_X S) U = \left( \frac{5c}{4} + \alpha \gamma - \beta^2 + \gamma^2 \right) I - S \right] \nabla_X U + \beta \gamma \phi AX \]
\[ + X(\beta \gamma) \xi + X \left( \frac{5c}{4} + \alpha \gamma - \beta^2 + \gamma^2 \right) U. \quad (3.5) \]

Taking inner product of (3.5) with \( U \) and \( \phi U \) and making use of (3.5) and Lemma 3.1, we have

\[ 2\beta \gamma^2 g(\phi U, X) = X \left( \frac{5c}{4} + \alpha \gamma - \beta^2 + \gamma^2 \right) \]
\[ \beta g(\nabla_U U, \phi U) = \gamma^2 g(U, X). \quad (3.6) \]

If we put \( X = U \) and \( Y = \phi U \) into (3.6) then we have

\[ (\alpha + 2\gamma) U + \gamma U \alpha - 2\beta U \beta = 0 \quad (3.8) \]

and

\[ 2\beta \gamma^2 = (\alpha + 2\gamma) (\phi U) \gamma + \gamma (\phi U) \alpha - 2\beta (\phi U) \beta. \quad (3.9) \]

Putting \( X = U \) and \( Y = \phi U \) into (3.7), then we obtain

\[ \beta g(\nabla_U U, \phi U) = \gamma^2 \quad \text{and} \quad \beta g(\nabla_U U, \phi U) = 0. \quad (3.10) \]

If we differentiate the third of (3.4) covariantly along vector field \( X \) in \( T_0 \), we obtain
\[(\nabla_x S)\phi U = \left(\frac{5c}{4} + \alpha\gamma + \gamma^2\right)I - S\nabla_x \phi U + X\left(\frac{5c}{4} + \alpha\gamma + \gamma^2\right)\phi U.\] (3.11)

If we take inner product of \(\phi U\) and using (3.4), then we have
\[X\left(\frac{5c}{4} + \alpha\gamma + \gamma^2\right)\phi U = 0.\] (3.12)

Substituting \(X = U\) and \(\phi U\) into (3.12), we obtain
\[(\alpha + 2\gamma)U\gamma + \gamma\alpha = 0\quad \text{and} \quad (\alpha + 2\gamma)(\phi U)\gamma + \gamma(\phi U)\alpha = 0.\] (3.13)

By comparing (3.8) and (3.9) with (3.13), we have \(U\beta = 0\) and \(\phi U\beta = -\gamma^2\). \(\Box\)

**Lemma 3.3.** Under the assumptions of Lemma 3.2, we have \(\nabla_x U = \gamma g(\phi U , X)\xi + \frac{\gamma^2}{\beta}g(U , X)\phi U\).

**Proof.** Since we have \(A\phi U = \gamma\phi U\) and using (3.7), we get
\[a(X) = g(\nabla_x U , \xi) = \gamma(\phi U , X)\quad \text{and} \quad c(X) = g(\nabla_x U , \phi U) = \frac{\gamma^2}{\beta}g(U , X).\] (3.14)

Thus, it follows from (3.14) that
\[\nabla_x U = \gamma g(\phi U , X)\xi + \frac{\gamma^2}{\beta}g(U , X)\phi U.\] \(\Box\)

**Lemma 3.4.** Under the assumptions of Lemma 3.2, we have \(\xi\alpha = \xi\beta = \xi\gamma = 0\) and \(U\alpha = U\gamma = 0\).

**Proof.** Differentiating the smooth function \(\alpha = g(A\xi , \xi)\) along any vector field \(X\) on \(\Omega\) and using (2.2) and (2.5) and Lemma 3.1, we have
\[X\alpha = g((\nabla_x A)\xi - 2\beta\gamma\phi U , X).\] (3.15)

Since we have \(\nabla_x (\nabla_x A)\xi = \nabla_x (\alpha\xi + \beta U) - \nabla_x \xi\), we see from the equation above that the gradient vector field \(\nabla\alpha\) of \(\alpha\) is given by
\[\nabla\alpha = \beta\nabla U + (\xi\alpha)\xi + (\xi\beta)U + (\alpha\beta - 3\beta\gamma)\phi U.\]

If we put \(X = \xi\) into Lemma 3.3, then we have
\[\nabla U = 0.\] (3.16)

Thus, the above equation is reduced to
\[\nabla\alpha = (\xi\alpha)\xi + (\xi\beta)U + (\alpha\beta - 3\beta\gamma)\phi U.\] (3.17)

Taking inner product of this equation with \(U\) and \(\phi U\) respectively, we obtain
\[U\alpha = \xi\beta \quad \text{and} \quad (\phi U)\alpha = \alpha\beta - 3\beta\gamma.\] (3.18)

If we differentiate the smooth function \(\beta = g(AU , \xi)\) along any vector field \(X\) on \(M\) and using (2.2), (2.5) and (2.8) and Lemma 3.2, we have
\[\nabla\beta = \beta\nabla U + (U\alpha)\xi + (U\beta)U + \left(\frac{c}{2} + 2(\alpha\gamma - \gamma^2)\right)\phi U.\] (3.19)

Putting \(X = U\) into Lemma 3.3, then we have
\[\nabla U = \frac{\gamma^2}{\beta}\phi U.\] (3.20)

If we substitute (3.20) into (3.19), then we obtain
\[\nabla\beta = (U\alpha)\xi + (U\beta)U + \left(\frac{c}{2} + 2\alpha\gamma - \gamma^2\right)\phi U.\] (3.21)

If we take inner product of this equation with \(\phi U\) and using \(\phi U\beta = -\gamma^2\) in Lemma 3.2, then we have
\[\alpha\gamma + \frac{c}{4} = 0.\] (3.22)

As a similar argument as the above, we can verify that the gradient vector fields of the smooth function
\(\gamma = g(AU , U) = g(A\phi U , \phi U\) is given respectively by
\[\nabla\gamma = -(A - \gamma I)\nabla U + \left(U\beta\right)\xi + (U\gamma)U + 3\beta\gamma\phi U\] (3.23)
and
\[\nabla\gamma = \left((\phi U)\gamma\phi U\right.\] (3.24)
by virtue of (2.3) and Lemma 3.2.

If we substitute (3.24) into (3.23) and make use of (3.20) and Lemma 3.1, then we obtain
\[\left(U\beta\right)\xi + (U\gamma)U + (\phi U)\gamma - 3\beta\gamma\phi U = 0.\] (3.25)

If we take inner product of this equation with \(U\) and \(\phi U\) respectively, then we have
\[U\gamma = 0 \quad \text{and} \quad (\phi U)\gamma = 3\beta\gamma.\] (3.26)

If we substitute (3.26) into (3.14) and take account of (3.21), then we have \(U\alpha = 0\). Also, if we differentiate (3.21) along any vector field \(\xi\), then we have
\[\alpha\xi\gamma + \xi\alpha\gamma = 0.\] (3.27)

Taking inner product of (3.23) with \(\xi\) and using (3.18), we get \(\xi\gamma = U\beta\). Since \(U\alpha = 0\), we see from (3.27) and the first of (3.18) that \(\xi\gamma = 0, \xi\alpha = 0\) and \(\xi\beta = 0\). \(\Box\)

4. Proofs of Theorems

**Proof Theorem 1.8.** If (1.4) is given in \(M\), then we see that Lemma 3.1 holds on \(M\). If we differentiate (1.3) along any vector field \(X\) in \(T_0\) and using (2.3) and (2.8), then we have
for any vector fields $X, Y$ and $Z$ on $T_0$. Putting $X = Y = Z = U$ into (4.1) and using Lemma 3.1 and 3.3, then we have

$$\beta \gamma = 0. \quad (4.2)$$

Since $\Omega$ is not empty, we have $\gamma = 0$ hold on $\Omega$. It follows from (2.8) and Lemma 3.1 that

$$A\xi = \alpha \xi + \beta U, \quad AU = \beta \xi \quad \text{and} \quad A\phi U = 0.$$  \hspace{1cm} (4.3)

Thus $M$ is locally congruent to real hypersurface (see [7]). □

**Proof Theorem 1.9.** Assume that the open set $\Omega = \{ p \in M \mid \beta (p) \neq 0 \}$ is not empty. Then we consider from Lemma 3.2 and 3.3 that $\beta (U) = -\gamma^2$ and

$c(U) = \frac{\gamma^2}{\beta}$. If we differentiate the smooth function

$\beta = g(A\xi, U)$ along vector field $X$ on $M$ and (2.2), (2.5) and (2.8), we have

$$X \beta = g\left( \nabla_\xi A, U + \left[ \frac{c}{4\alpha} + \alpha \gamma - \gamma^2 \right] \phi U, X \right). \quad (4.4)$$

Since we have $\left( \nabla_\xi A \right) U = \nabla_\xi (\beta \xi + \gamma U) - A\nabla_\xi U$, we see from this equation above that gradient vector field $\nabla \beta$ of $\beta$ is given by

$$\nabla \beta = -((A - \gamma I) \nabla_\xi U + (\xi \beta) \xi + (\xi \gamma) U$$

$$+ \beta \xi + \frac{c}{4} \alpha \gamma - \gamma^2 \phi U, \quad (4.5)$$

where $I$ indicates the identity transformation on $M$. If we substitute (3.16) into (4.4) and using Lemma 3.4, then we obtain

$$\nabla \beta = \left( \beta^2 + \frac{c}{4} \alpha \gamma - \gamma^2 \right) \phi U. \quad (4.6)$$

Since we have $(\phi U) \beta = -\gamma^2$, we get

$$\beta^2 + \frac{c}{4} \alpha \gamma = 0. \quad (4.7)$$

By (4.6) and (3.22), we have $\beta = 0$ and hence it is a contradiction. Thus the set $\Omega = \{ p \in M \mid \beta (p) \neq 0 \}$ is empty, and hence $M$ is a Hopf hypersurface. Since $M$ is a Hopf hypersurface, we see from (2.1) and (2.8) that $(A\phi - \phi A) \xi = 0$, which together with our assumption (1.4) implies (1.1), that is $A\phi = \phi A$ on $M$. Thus, Theorem 1.9 shows that $M$ is locally congruent to a real hypersurface of type $A$. □

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