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Existence of Weak Solutions for a Class of Quasilinear Parabolic Problems in Weighted Sobolev Space*

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ABSTRACT

In this paper, we investigate the existence and uniqueness of weak solutions for a new class of initial/boundary-value parabolic problems with nonlinear perturbation term in weighted Sobolev space. By building up the compact imbedding in weighted Sobolev space and extending Galerkin's method to a new class of nonlinear problems, we drive out that there exists at least one weak solution of the nonlinear equations in the interval $[0,T]$ for the fixed time $T>0$.

Keywords: Weighted Sobolev Space; Energy Estimates; Compact Imbedding; Sobolev Interpolation Inequalities

1. Introduction

Now we consider the initial/boundary-value problem [1] as following

$$\begin{cases} u_t - \operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) \\ = \lambda |u|^{p-2}u + b(x)|u|^{\alpha-1}u, & \text{in } \Omega_T, \\ u(x,t)=0 & \text{on } \partial\Omega\times[0,T], \\ u(x,0)=g(x), & \text{on } \Omega\times(t=0). \end{cases} \tag{1.1}$$

where $2<p\le p^*\left(p^*=\dfrac{np}{n-p}\right)$, $1<\alpha<p-1$, λ is a real positive parameter and $\dfrac{\partial}{\partial_t}+\cdots$ is (uniformly) parabolic, $\Omega_T=\Omega\times[0,T]$ for some fixed time $T>0$, Ω is an open bounded subset with smooth boundary in R^N, $g\in L^2(\Omega):\Omega\to R$ is given, $u:\bar{\Omega}_T\to R$ is the unknown, $u=u(x,t)$, $a(x)$, $b(x)$ are functions satisfying some suitable conditions [2-4].

The main purpose of this paper is to establish the existence of weak solutions for the parabolic initial/boundary-value problem (1.1) in a weighted Sobolev space. For this purpose, we assume for now that

1) $a(x)$ is a positive measurable sufficiently smooth function,

2) $b(x):\bar{\Omega}\to R$ is a non-negative smooth function which may change sign,

3) $W_0^{1,p}(a(x),\Omega)$ is a weighted Sobolev space [5-8] with a weight function $a(x)$, its norm defined as

$$\|u\|_{W_0^{1,p}(a(x),\Omega)}=\left\{\int_\Omega\left(a(x)|\nabla u|^p\right)\mathrm{d}_x\right\}^{\frac{1}{p}}.$$

For convenience, we will denote $W_0^{1,p}(a(x),\Omega)$ by *X*, note $\|u\|_{W_0^{1,p}(a(x),\Omega)}$ by $\|u\|_X$, and unless otherwise stated, integrals are over Ω.

Similar problems have been studied by Evans [9], he investigated the solvability of the initial/bondary-value problem for the reaction-diffusion system

$$\begin{cases} u_t-\Delta u=f(u), & \text{in } \Omega_T, \\ u=0, & \text{on } \partial\Omega\times[0,T], \\ u=0, & \text{on } \Omega\times(t=0). \end{cases} \tag{1.2}$$

Here $u=(u^1,u^2,\cdots,u^m)$, $g=(g^1,g^2,\cdots,g^m)$, and as usual $\Omega_T=\Omega\times[0,T]$, $\Omega\in R^n$ is open and bounded with smooth boundary. Via the techniques of Banach's fixed point theorem method, he obtained the existence and uniqueness and some estimates of the weak solution under the assumer that the initial function $g(x)$ belongs to $H_0^1(\Omega,R^m)$ and $f:R^m\to R^m$ is Lipschitz continuous. He also studied the nonlinear heat equation with a simple quadratic nonlinearity

$$\begin{cases} u_t-\Delta u=u^2, & \text{in } \Omega_T, \\ u=0, & \text{on } \partial\Omega\times[0,T], \\ u=0, & \text{on } \Omega\times(t=0). \end{cases} \tag{1.3}$$

The Blow-up solution has been established under the assumer that $T>0$ and $g\ge 0$ are large enough in an

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appropriate sense.

The main results of this paper can be stated as follows,

Theorem 1.1. *There exists a unique weak solution of problem* (1.1) *on the interval* $[0,T]$ *for the fixed time* $T>0$.

For the further argument, we need the following Lemma.

Lemma 1.1. *If* $2<p\le p^*,\left(p^*=\dfrac{np}{n-p}\right)$, *then*,

1) $W_0^{1,p}(a(x),\Omega)\rightarrowtail H_0^1(a(x),\Omega)\rightarrowtail H_0^1(\Omega)\rightarrowtail L^2(\Omega)$ are the compact imbedding [6],

2) $W_0^{1,p}(a(x),\Omega)\rightarrowtail W_0^{1,p}(\Omega)\rightarrowtail L^p(\Omega)$ are also compact imbedding.

Proof. 1) Since $p>2$, and $a(x)$ is a positive sufficiently smooth function, there exists a positive constant C, such that $\|a(x)\|_{L^2(\Omega)}\ge\|a(x)\|_{L^1(\Omega)}\ge C$. Hence

$$\begin{aligned}\int_\Omega\left(a(x)|\nabla u|^p\right)\mathrm{d}_x&\ge\int_\Omega\left(a(x)|\nabla u|^2\right)\mathrm{d}_x\\&=\|a(x)\|_{L^1(\Omega)}\int_\Omega|\nabla u|^2\,\mathrm{d}_x\\&\ge C\int_\Omega|\nabla u|^2\,\mathrm{d}_x\ge C\int_\Omega|u|^2\,\mathrm{d}_x,\end{aligned}$$

for all $x\in\Omega$, and a.e. time $0\le t\le T$. We used the poincare's inequality in the last inequality above. Thus,

1) Holds and is compact.

2) The proof of 2) is almost the same as 1). This completes the proof of Lemma 1.1.

2. Weak Solutions

According to Lemma 1.1, it suffices to consider the initial/boundary-value problem (1.1) in spaces $H_0^1(\Omega)$ and $L^2(\Omega)$. We will employ the Galerkin's method to prove our results.

Definition 2.1. We say a function

$$u\in L^p\left(0,T;W_0^{1,p}\left(a(x),\Omega\right)\right)\subset L^2\left(0,T;H_0^1(\Omega)\right),$$

with $u'\in L^2\left(0,T;H^{-1}(\Omega)\right)$

is a weak solution of the parabolic initial/boundary-value problem (1.1) provided

1) $\langle u',v\rangle+B[u,v;t]=(f(u),v)$, for each $v\in H_0^1(\Omega)$, and a.e. time $0\le t\le T$, and

2) $u(0)=g$.

Here $B[u,v;t]$ denotes the time-dependent bilinear form

$$B[u,v;t]=\int_\Omega a(x)|\nabla u|^{p-2}(\nabla u,\nabla v)\,\mathrm{d}_x,$$

for each $u,v\in H_0^1(\Omega)$ and a.e. time $0\le t\le T$.

$f(u)=\lambda|u|^{p-2}u+b(x)|u|^{\alpha-1}u$ is the nonlinearity term. the pairing $(,)$ denoting inner product in $L^2(\Omega)$, $\langle,\rangle$ being the pairing of $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

By the Definition 2.1, we see $u\in C\left(0,T;L^2(\Omega)\right)$, and thus the equality 2) makes sense.

We now switch our view point, by associating with u a mapping

$$u:[0,T]\mapsto H_0^1(\Omega),$$

defined by

$$[u(t)](x)=u(x,t),(x\in\Omega,0\le t\le T).$$

More precisely, assume that the functions $w_i=w_i(x)$ $(i=1,2,\cdots,m)$ are smooth,

1) $\{w_i\}_{i=1}^\infty$ is an orthogonal basis of $H_0^1(\Omega)$, and

2) $\{w_i\}_{i=1}^\infty$ is an orthogonal basis of $L^2(\Omega)$, ($0\le t\le T$, $i=1,2,\cdots,m$) taken with the inner product $(u,v)=\int_\Omega\nabla u\cdot\nabla v\mathrm{d}_x$. S_m is the finite dimensional subspace spanned by $\{w_i\}_{i=1}^\infty$. Fix a positive integer m, we will look for a function $u_m:[0,T]\mapsto H_0^1(\Omega)$ of the form

$$u_m(t)=\sum_{i=1}^m d_m^i(t)w_i,(i=1,2,\cdots,m),\tag{2.1}$$

Here we hope to select the coefficients $d_m^i(t)$, ($0\le t\le T$, $i=1,2,\cdots,m$) such that

$$\begin{cases}(u_m',w_i)+B[u_m,w_i;t]=(f(u_m),w_i),\\ d_m^i(0)=(g,w_i),(i=1,2,\cdots,m).\end{cases}\tag{2.2}$$

That is

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}_t}(u_m,v)+B[u_m,v;t]=(f(u_m),v),\text{ for }\forall v\in S_m,\\ (u_m,v)\big|_{t=0}=(g,v).\end{cases}\tag{2.3}$$

This amounts to our requiring that u_m solves the "projection" of problem (1.1) onto the finite dimensional subspace S_m.

Theorem 2.1. (*construction of approximate solutions*)

For each integer $m=1,\cdots,$ there exists a function u_m of the form (2.1) satisfying the identities (2.3).

Proof. Taking $v=\sum_{i=1}^m v_iw_i(x)$ arbitrary, then

$$(u_m,v)=\sum_{i=1}^m d_m^i(t)v_i,$$

$$B[u_m,v;t]=B\left[\sum_{i=1}^m d_m^i(t)w_i(x),\sum_{j=1}^m v_jw_j(x);t\right],$$

note that $B\left[w_i,w_j;t\right]=k_{ij}$.

Thus, $B[u_m,v;t]=\sum_{i,j=1}^m k_{ij}d_m^i(t)v_i$, and

$$\begin{aligned}(f(u_m),v)&=\left(f\left(\sum_{i=1}^m d_m^i(t)w_i(x)\right),\sum_{j=1}^m v_jw_j(x)\right)\\&=\left(f\left(\sum_{i=1}^m d_m^i(t)w_i(x)\right),w_j(x)\right)v_j,\end{aligned}$$

$$\left(u_m(0),v\right)=\left(\sum_{i=1}^{m}d_m^i(0)w_i(x),\sum_{j=1}^{m}v_jw_j(x)\right)=\left(\sum_{i=1}^{m}d_m^i(0),\sum_{j=1}^{m}v_j\right)=\sum_{i=1}^{m}gv_i,$$

$i=1,2,\cdots,m.$ Hence,

$$\begin{cases}\left(\sum_{i=1}^{m}\frac{\mathrm{d}}{\mathrm{d}_t}d_m^i(t)+\sum_{i,j=1}^{m}k_{ij}d_m^i(t)\right)v_i\\ =\left(f\left(\sum_{i=1}^{m}d_m^i(t)w_i(x)\right),w_j(x)\right)v_i,\text{ for }\forall v_i,\\ \left(u_m(0),v\right)=(g,v)=\sum_{i=1}^{m}gv_i,(i=1,2,\cdots,m).\end{cases}\tag{2.4}$$

Since v_i is random, therefore, system (2.4) becomes

$$\begin{cases}\left(\sum_{i=1}^{m}\frac{\mathrm{d}}{\mathrm{d}_t}d_m^i(t)+\sum_{i,j=1}^{m}k_{ij}d_m^i(t)\right)\\ =\left(f\left(\sum_{i=1}^{m}d_m^i(t)w_i(x)\right),w_j(x)\right),\ i,j=1,2,\cdots,m,\\ \sum_{i=1}^{m}d_m^i(0)=\sum_{i=1}^{m}gw_i.\end{cases}\tag{2.5}$$

This is a nonlinear system of ordinary differential equation, according to the existence theory for nonlinear ODE, there exists a unique local solution on interval $[0,T]$ for fixed time $T>0$. That is, the initial/boundary-value problem (1.1) has a unique local weak solution on the interval $[0,T]$.

3. Energy Estimates

Theorem 3.1. *There exists a constant C, depending only on* Ω,T *and* $\lambda,\sup_{x\in\Omega}|b(x)|$, *such that*

$$\max_{0\le t\le T}\|u_m(t)\|_{L^2(\Omega)}^2+\|u_m\|_{L^p\left(0,T;W_0^{1,p}(a(x),\Omega)\right)}+\|u_m'\|_{L^2\left(0,T;H^{-1}(\Omega)\right)}\le C\|g\|_{L^2(\Omega)}^2,\tag{3.1}$$

for $m=1,2,\cdots$.

Proof. We separate this proof into 3 steps.

Step 1. Multiply equality (2.2) by $d_m^i(t)$ and sum for $i=1,2,\cdots,m$, and then recall to (2.1) to find

$$\left(u_m',u_m\right)+B[u_m,u_m;t]=\left(f(u_m),u_m\right),\quad\text{for a.e. }0\le t\le T.\tag{3.2}$$

Whereas,

$$\left(u_m',u_m\right)=\frac{\mathrm{d}}{\mathrm{d}_t}\left(\frac{1}{2}\|u_m\|_{L^2(\Omega)}^2\right)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}_t}\|u_m\|_{L^2(\Omega)}^2,$$

$$B[u_m,u_m;t]=\int_\Omega a(x)|\nabla u_m|^{p-2}(\nabla u_m,\nabla u_m)\mathrm{d}_x=\int_\Omega\left(a(x)|\nabla u_m|^p\right)d_x=\|u_m\|_{W_0^{1,p}(a(x),\Omega)}^p=\|u_m\|_X^p,$$

and

$$\left(f(u_m),u_m\right)=\lambda\int_\Omega|u_m|^p\,\mathrm{d}_x+\int_\Omega b(x)|u_m|^{\alpha+1}\mathrm{d}_x\le\lambda\|u_m\|_{L^p(\Omega)}^p+\bar{b}\|u_m\|_{L^{\alpha+1}(\Omega)}^{\alpha+1},$$

for a.e. time $0\le t\le T$, here, $\bar{b}=\sup_{x\in\Omega}|b(x)|$, since $b(x)$ is a smooth function.

Consequently (3.2) yields the inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}_t}\|u_m\|_{L^2(\Omega)}^2+\|u_m\|_X^p\le\lambda\|u_m\|_{L^p(\Omega)}^p+\bar{b}\|u_m\|_{L^{\alpha+1}(\Omega)}^{\alpha+1}.\tag{3.3}$$

Since $1<\alpha<p-1$, that is $\alpha+1<p$, then by Sobolev imbedding theorem, we obtain $L^p(\Omega)\rightarrowtail L^{\alpha+1}(\Omega)$, and moreover,

$$\|u_m\|_{L^{\alpha+1}(\Omega)}^{\alpha+1}\le K\|u_m\|_{L^p(\Omega)}^{\alpha+1}\le K\|u_m\|_{L^p(\Omega)}^p,$$

here k is the best Sobolev constant [10-13].

Thus, we can write inequality (3.3) as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}_t}\|u_m\|_{L^2(\Omega)}^2+\|u_m\|_X^p\le C_1\|u_m\|_{L^p(\Omega)}^p\tag{3.4}$$

For a.e. time $0\le t\le T$, and appropriate constant C_1.

In addition, since $2<p\le p^*$, $\left(p^*=\frac{np}{n-p}\right)$, by Sobolev interpolation inequality, we find

$$\|u_m\|_{L^p(\Omega)}\le\|u_m\|_{L^{p^*}(\Omega)}^\theta\|u_m\|_{L^2(\Omega)}^{1-\theta}\le\varepsilon\theta\|u_m\|_{L^{p^*}(\Omega)}+\frac{C_2}{\varepsilon}(1-\theta)\|u_m\|_{L^2(\Omega)}\le\varepsilon\|u_m\|_{L^{p^*}(\Omega)}+C_\varepsilon\|u_m\|_{L^2(\Omega)},$$

here $\frac{1}{p}=\frac{\theta}{p^*}+\frac{1-\theta}{2}$, $0<\theta<1$, and we have used the Young's inequality with ϵ in the last inequality. Thus

$$\|u_m\|_{L^p(\Omega)}^p\le\|u_m\|_{L^{p^*}(\Omega)}^{p\theta}\|u_m\|_{L^2(\Omega)}^{p(1-\theta)}\le\varepsilon\|u_m\|_{L^{p^*}(\Omega)}^p+C_\varepsilon\|u_m\|_{L^2(\Omega)}^2.\tag{3.5}$$

By Lemma 1.1 2) and Sobolev's inequality, we have $\|u_m\|_{L^{p^*}(\Omega)}\le K_1\|u_m\|_X$, K_1 is the best Sobolev imbedding constant, insert the inequality above and (3.5) into inequality (3.4) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}_t}\|u_m\|_{L^2(\Omega)}^2+\|u_m\|_X^p\le C_3\|u_m\|_X^p+C_4\|u_m\|_{L^2(\Omega)}^2,\tag{3.6}$$

for a.e. time $0 \le t \le T$, and appropriate constants C_3 and C_4.

Furthermore, we rewrite inequality (3.6) as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}_t}\|u_m\|_{L^2(\Omega)}^2 + \beta\|u_m\|_X^p \le C_4\|u_m\|_{L^2(\Omega)}^2, \tag{3.7}$$

for a.e. time $0 \le t \le T$, and appropriate constants β and C_4.

By Gronwall's inequality, (3.7) yields the estimate

$$\max_{0\le t\le T}\|u_m(t)\|_{L^2(\Omega)}^2 \le C\|u_m(0)\|_{L^2(\Omega)}^2 = C\|g\|_{L^2(\Omega)}^2, \tag{3.8}$$

for a.e. time $0 \le t \le T$, and appropriate constant C.

Step 2. Returning once more to inequality (3.7), we integrate from 0 to *T* and employ the inequality (3.8) to obtain

$$\|u_m\|_{L^p(0,T;X)}^p = \int_0^T \|u_m\|_X^p \,\mathrm{d}_t \le C\|g\|_{L^2(\Omega)}^2, \tag{3.9}$$

for a.e. time $0 \le t \le T$, and appropriate constant *C*.

Step 3. Fix any $v \in H_0^1(\Omega)$, with $\|v\|_{H_0^1(\Omega)} \le 1$, and write $v = v^1 + v^2$, where $v^1 \in S_m = \operatorname{span}\{w_i\}_{i=1}^m$ and $(v^2, w_i) = 0, (i = 1,2,\cdots,m)$. Since functions $\{w_i\}_{i=1}^\infty$ are orthogonal in $H_0^1(\Omega)$, $\|v^1\|_{H_0^1(\Omega)} \le \|v\|_{H_0^1(\Omega)} \le 1$. Utilizing (2.2) we deduce for a.e. time $0 \le t \le T$, that

$$(u_m', v^1) + B[u_m, v^1; t] = (f(u_m), v^1).$$

Then (2.1) implies

$$\begin{aligned}\langle u_m', v\rangle &= (u_m', v) = (u_m', v^1)\\ &= (f(u_m), v^1) - B[u_m, v^1; t].\end{aligned}$$

Consequently,

$$|\langle u_m', v\rangle| \le C\left(\|f(u_m)\|_{L^2(\Omega)} + \|u_m\|_{H_0^1(\Omega)}\right),$$

since $\|v^1\|_{H_0^1(\Omega)} \le 1$. Thus

$$\begin{aligned}\|u_m'\|_{H^{-1}(\Omega)} &\le C\left(\|f(u_m)\|_{L^2(\Omega)} + \|u_m\|_{H_0^1(\Omega)}\right)\\ &\le C\left(\|f(u_m)\|_{L^p(\Omega)} + \|u_m\|_X\right),\end{aligned}$$

$$\begin{aligned}\int_0^T \|u_m'\|_{H^{-1}(\Omega)}^2 \,\mathrm{d}_t &\le C\int_0^T \left(\|f(u_m)\|_{L^p(\Omega)}^2 + \|u_m\|_X^2\right)\mathrm{d}_t\\ &\le C\|g\|_{L^2(\Omega)}^2,\end{aligned} \tag{3.10}$$

for a.e. time $0 \le t \le T$, and appropriate constant *C*.

Combing (3.8), (3.9) and (3.10) we complete the proof of Theorem 3.1.

4. Existence of Weak Solutions

Next we pass to limits as $m \to \infty$, to build a weak solution of our initial/boundary-value problem (1.1).

Theorem 4.1. *There exists a local weak solution of problem* (1.1).

Proof. According to the energy estimates (3.1), we see that the sequence $\{u_m\}_{m=1}^\infty$ is bounded in $L^p\left(0,T;W_0^{1,p}(a(x),\Omega)\right)$, and $\{u_m'\}_{m=1}^\infty$ is bounded in $L^2\left(0,T;H^{-1}(\Omega)\right)$. Consequently there exists a subsequence $\{u_{m_l}\}_{l=1}^\infty \subset \{u_m\}_{m=1}^\infty$ and a function

$$u \in L^p\left(0,T;W_0^{1,p}(a(x),\Omega)\right) \subset L^2\left(0,T;H^{-1}(\Omega)\right)$$

with $u' \in L^2\left(0,T;H^{-1}(\Omega)\right)$, such that

1) $u_{m_l} \to u$ weakly in $L^p\left(0,T;W_0^{1,p}(a(x),\Omega)\right)$, and $u_{m_l} \to u$ strongly in $L^p(\Omega)$.

2) $u_{m_l}' \to u'$ weakly in $L^2\left(0,T;H^{-1}(\Omega)\right)$.

Now we fix an integer N and choose a function $v \in C^1\left(0,T;H^{-1}(\Omega)\right)$ having the form

$$v(t) = \sum_{i=1}^N d^i(t) w_i, \tag{4.1}$$

here $\{d^i(t)\}_{i=1}^N$ are given smooth functions. We choose $m \ge N$, multiply (2.2) by $d^i(t)$, sum $i = 1,2,\cdots,N$, and then integrate with respect to *t*, we find

$$\int_0^T \langle u_m', v\rangle + B[u_m, v; t]\mathrm{d}_t = \int_0^T (f(u_m), v)\mathrm{d}_t. \tag{4.2}$$

We set $m = m_l$, and recall 1), 2) to find upon passing to weak limits that

$$\int_0^T \langle u', v\rangle + B[u, v; t]\mathrm{d}_t = \int_0^T (f(u), v)\mathrm{d}_t. \tag{4.3}$$

This equality then holds for all functions $v \in L^2\left(0,T;H_0^1(\Omega)\right)$, as functions of the form (4.1) are dense in this space. Hence in particular

$$\langle u', v\rangle + B[u, v; t] = (f(u), v). \tag{4.4}$$

for each $v \in H_0^1(\Omega)$ and a.e. time $0 \le t \le T$.

In order to prove $u(0) = g$, we first note from (4.3) that

$$\begin{aligned}&\int_0^T -\langle v', u\rangle + B[u, v; t]\mathrm{d}_t\\ &= \int_0^T (f(u), v)\mathrm{d}_t + (u(0), v(0)).\end{aligned} \tag{4.5}$$

for each $v \in C^1\left(0,T;H_0^1(\Omega)\right)$ with $v(T) = 0$. Similarly, from (4.2) we deduce

$$\begin{aligned}&\int_0^T -\langle v', u_m\rangle + B[u_m, v; t]\mathrm{d}_t\\ &= \int_0^T (f(u_m), v)\mathrm{d}_t + (u_m(0), v(0)).\end{aligned} \tag{4.6}$$

We set $m = m_l$ and once again employ 1), 2), we obtain

$$\int_0^T -\langle v', u\rangle + B[u,v;t]\,\mathrm{d}_t = \int_0^T (f(u),v)\,\mathrm{d}_t + (g, v(0)), \tag{4.7}$$

since $u_{m_l}(0) \to g$ in $L^2(\Omega)$. As $v(0)$ is arbitrary, comparing (4.5) and (4.7), we conclude $u(0) = g$. This completes the proof of theorem 4.1.

5. Uniqueness of Weak Solutions

In this part, we will prove Theorem 1.1.

Proof. Let u_1 and u_2 are two weak solutions for the initial/boundary-value problem, put $u = u_1 - u_2$, and insert it into the origin equation, we discover

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}_t}(u,v) + B[u,v;t] = 0, \forall v \in H_0^1(\Omega), \\ u|_{t=0} = 0 \end{cases}$$

Taking $v = u$, we obtain the energy estimates inequality

$$\|u\|_{L^2(\Omega)}^2 + \int_0^T \|u\|_X^p \,\mathrm{d}_t \le 0.$$

Since $\|u\|_{L^2(\Omega)}^2 \ge 0$, $\int_0^T \|u\|_X^p \,\mathrm{d}_t \ge 0$. So we have $u \equiv 0$ for a.e. time $0 \le t \le T$. This completes the proof of Theorem 1.1.

6. Conclusion

In this paper, we established the existence and uniqueness of weak solutions for initial/boundary-value parabolic problems with nonlinear perturbation term in weighted Sobolev space. First, we investigated the compact imbedding in weighted Sobolev space, which can be imbedded compactly into $H_0^1(\Omega)$ and $L^2(\Omega)$ spaces. By exploiting Sobolev interpolation inequalities and extending Galerkin's method to a new class of nonlinear problems, we proofed the energy estimates of the equations and furthermore obtained the unique weak solution of the problem.

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